Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 32 (2013), 389–409
DOI: 10.4171/ZAA/1491

Sobolev Theorems for Cusp Manifolds

Jürgen Eichhorn and Chunpeng Wang

Abstract. In the past, we established a module structure theorem for Sobolev spaces on open manifolds with bounded curvature and positive injectivity radius $r_{\rm inj}(M) = \inf_{x\in M} r_{\rm inj}(x) > 0$. The assumption $r_{\rm inj}(M) > 0$ was essential in the proof. But, manifolds (M^n, g) with $\mathrm{vol}(M^n, g) < \infty$ have been excluded. An extension of our former results to the case $\mathrm{vol}(M^n, g) < \infty$ seems to be hopeless. In this paper, we show that certain Sobolev embedding theorems and a (generalized) module structure theorem are valid in weighted spaces with the weight $\xi(x) = r_{\rm inj}(x)^{-n}$ or $\xi(x) = \mathrm{vol}(B_1(x))^{-1}$.

Keywords. Weighted Sobolev spaces, open manifolds, injectivity radius, finite volume, embedding theorems

Mathematics Subject Classification (2010). Primary 53C20, secondary 53C22, 58C35

1. Introduction

For open manifolds, we developed in [2–5] Sobolev uniform structures to introduce canonical intrinsic Sobolev topologies for geometric objects like Riemannian metrics, connections or spaces of mappings. To the generalized arc components we attached features like spectral properties or in certain cases characteristic numbers. All these constructions have been performed under the assumption of bounded geometry, i.e. bounded curvature up to a certain degree and injectivity radius $r_{\rm inj} = \inf_x r_{\rm inj}(x) > 0$. The reason for this permanent assumption was the frequent use of certain Sobolev embedding theorems, and, as an absolutely essential tool, module structure theorems for Sobolev spaces. To prove these theorems, we used bounded geometry.

Unfortunately, a large class of manifolds is excluded, namely manifolds with finite volume or $r_{\rm inj} = \inf_x r_{\rm inj}(x) = 0$, which appear, e.g., as locally symmetric

J. Eichhorn: Institut für Mathematik und Informatik, Universität Greifswald, D-17487 Greifswald, Germany; eichhorn@uni-greifswald.de
C. Wang: School of Mathematics, Jilin University, Changchun 130012, P. R. China; wangcp@jlu.edu.cn

spaces of finite volume in a very natural manner. The first main task to extend our theory of Sobolev uniform structures to open manifolds with $\mathrm{vol}(M) < \infty$ or $r_{\mathrm{inj}} = 0$, is to reestablish the corresponding Sobolev embedding theorems and module structure theorems for these classes of manifolds. But we see no chance to do that. The assumption $r_{\mathrm{inj}} > 0$ was essential in all proofs. As already indicated in [5, p. 18], a suitable approach could be to work with weighted Sobolev spaces. This is the content of this paper.

The paper is organized as follows. In Section 2, we start with a brief discussion of weighted Sobolev spaces and the fundamental behaviour of the injectivity radius at the ends. Then we establish our first embedding theorem concerning the embedding of sections of weighted spaces into the space of bounded C^m-sections. This is Theorem 2.11. Here we assume bounded sectional curvature and use the weight $\xi(x) = \max\{1, r_{\mathrm{inj}}(x)^{-n}\}$. In concrete cases, it could be difficult to calculate the continuous function $x \to r_{\mathrm{inj}}(x)$. Therefore we propose to work also with general estimates for the injectivity radius, e.g. $r_{\mathrm{inj}}(x) \geq C(p)\mathrm{e}^{-(n-1)\sqrt{K}d(x,p)}$ and $\xi(x) = \mathrm{e}^{\delta\rho_y}$, $\delta = n(n-1)\sqrt{K}$, $\rho_y(x) = \mathrm{dist}(x,y)$. Fortunately, an embedding theorem "weighted Sobolev space $\longrightarrow C^m$-sections" remains valid. This is Theorem 2.12. The key for module structure theorems for weighted Sobolev spaces is the so called scale of embeddings. We establish this scale for arbitrary weights, excluded the initial step. This are Theorems 2.16–2.18. According to [7, 8], the initial step in the unweighted case is valid if and only if $\inf_{x\in M} \mathrm{vol}(B_1(x)) > 0$.

We see that in our case $\mathrm{vol}(M) < \infty$ or $r_{\mathrm{inj}}(M) = 0$ for at least one end of M the weighted scale of embeddings fails. Hence we must work with weighted Sobolev spaces. The final Section 3 is devoted to this task. Fortunately, the initial step of the scale of embeddings has been established already in [10], and in an elaborated version by Hebey in [8]. The used weight is $\xi(x) = v(x) = \frac{1}{\mathrm{vol}(B_1(x))}$. After that we are able to establish our module structure theorem for the weight $\frac{1}{\mathrm{vol}(B_1(x))}$. Using our scale of embeddings and Hölder inequality for the weighted measure $d\mu_x = \frac{1}{\mathrm{vol}(B_1(x))} d\mathrm{vol}_x(g)$, the proof of our main Theorem 3.6 reduces to the solution of the system (38). This main theorem is not a module structure theorem as in [6] or in a pure algebraic sense. The multiplication of a Sobolev function with Sobolev sections of a vector bundle transforms these sections into a weighted Sobolev space of lower Sobolev index. This is expressed by the last two equations in (38). Important, our system (38) has many nontrivial solutions. Unfortunately, the weights $\xi(x) = r_{\mathrm{inj}}(x)^{-n}$ and $v(x) = \frac{1}{\mathrm{vol}(B_1(x))}$ do not give equivalent weighted Sobolev spaces.

In a forthcoming paper, we apply our results to the introduction, discussion and application of uniform Sobolev structures as in [2–5].

2. Weighted Sobolev spaces

We consider a Riemannian vector bundle $E = ((E, h, \nabla) \to (M^n, g))$, (M^n, g) open and complete. Then we set for $p \geq 1$, $r \geq 0$, $m \geq 0$,

$$\Omega^p_r(E) \equiv \left\{ \varphi \in C^\infty(E) : |\varphi|_{p,r} := \Big(\int_M \sum_{i=0}^{r} |\nabla^i \varphi|^p_x d\mathrm{vol}_x(g) \Big)^{\frac{1}{p}} < \infty \right\},$$

$$\overline{\Omega}^{p,r}(E) = \overline{\Omega^p_r(E)}^{|\,|_{p,r}}, \quad \mathring{\Omega}^{p,r}(E) = \overline{C^\infty_c(E)}^{|\,|_{p,r}},$$

$$\Omega^{p,r}(E) = \big\{ \varphi \text{ a distributional section of } E \text{ with } |\varphi|_{p,r} < \infty \big\},$$

$${}^b_m\Omega(E) = \left\{ \varphi \in C^\infty(E) : {}^{b,m}|\varphi| = \sum_{i=0}^{m} \sup_{x \in M} |\nabla^i \varphi|_x < \infty \right\},$$

$${}^{b,m}\Omega(E) = \overline{{}^b_m\Omega(E)}^{b,m|\,|} = \big\{ \varphi \in C^m(E) : {}^{b,m}|\varphi| < \infty \big\}.$$

Then, $\mathring{\Omega}^{p,r}(E) \subseteq \overline{\Omega}^{p,r}(E) \subseteq \Omega^{p,r}(E)$, ${}^{b,m}\Omega(E)$ are Banach spaces, for $p = 2$ Hilbert spaces. If (M^n, g) satisfies the conditions (I) and (B_0),

$$r_{\mathrm{inj}}(M^n, g) = \inf_{x \in M} r_{\mathrm{inj}}(x) > 0, \tag{I}$$

$$|\text{sectional curvature } K| \leq C, \tag{B_0}$$

then one has for $r > \frac{n}{p} + m$ a continuous embedding

$$\overline{\Omega}^{p,r}(E) \hookrightarrow {}^{b,m}\Omega(E). \tag{1}$$

If (M^n, g) and E satisfy additionally

$$|\nabla^i R^g| \leq C_i, \quad 0 \leq i \leq k, \tag{$\mathrm{B}_k(M,g)$}$$

$$|\nabla^i R^E| \leq D_i, \quad 0 \leq i \leq k, \tag{$\mathrm{B}_k(E)$}$$

then

$$\mathring{\Omega}^{p,r}(E) = \overline{\Omega}^{p,r}(E) = \Omega^{p,r}(E) \quad \text{for } r \leq k + 2. \tag{2}$$

A second Sobolev embedding theorem is given by

Proposition 2.1. *Assume*

$$k \geq r, \quad r - \frac{n}{p} \geq s - \frac{n}{q}, \quad r \geq s, \quad q \geq p \tag{3}$$

and (I), $(\mathrm{B}_k(M, g))$, $(\mathrm{B}_k(E))$. *Then*

$$\Omega^{p,r}(E) \hookrightarrow \Omega^{q,s}(E). \tag{4}$$

Proof. We refer to [5,6] for the proof. □

With the help of (1), (3) we proved in [6] our module structure theorem for Sobolev spaces which asserts that under certain (lengthy) conditions for p_1, r_1, p_2, r_2, p, r, k, $(\mathrm{B}_k(M,g))$, $(\mathrm{B}_k(E))$ and (I), the tensor product of sections defines a continuous bilinear map

$$\Omega^{p_1,r_1}(E_1,\nabla_1)\times\Omega^{p_2,r_2}(E_2,\nabla_2)\to\Omega^{p,r}(E_1\otimes E-2,\nabla_1\otimes\nabla_2). \tag{5}$$

Unfortunately, (5) is for cusp manifolds with (highly enough) bounded curvature not available since (1) and (4) fail. But the constant C in (1) for $m=0$,

$$|\varphi(x)|\le C|\varphi|_{p,r}, \tag{6}$$

behaves like

$$C=C\left(K,n,p,r,r_{\mathrm{inj}}(x)^{-\frac{n}{p}}\right)=O\left(r_{\mathrm{inj}}(x)^{-\frac{n}{p}}\right) \tag{7}$$

which can be really considered as a constant if $r_{\mathrm{inj}}(M)>0$. For cusp manifolds,

$$r_{\mathrm{inj}}(x)\to 0\quad \text{as } x\to\infty. \tag{8}$$

Hence the constant in (6) grows and grows if x approaches infinity, and we cannot have a continuous embedding theorem (1). The way out could be weighted Sobolev spaces with a weight $\sim r_{\mathrm{inj}}(x)^{-n}$. This is our main idea. Before considering special geometric weights, we briefly introduce general weighted spaces. Let $E=((E,h,\nabla)\to(M^n,g))$ be a Hermitean vector bundle, ξ a measurable section of $\mathrm{End}(E)$ with values in the pointwise endomorphisms such that ξ is locally bounded from above and is locally strictly positive. ξ extends to $(T^*)^{\otimes i}\otimes\otimes^j E$. Set

$$L_{p,\xi}(E)=\left\{\varphi\in L_{p,\mathrm{loc}}(E):\xi^{\frac{1}{p}}\varphi\in L_p(E)\right\}$$

with

$$|\varphi|_{p,\xi}=|\xi^{\frac{1}{p}}\varphi|_{L_p}\equiv|\xi^{\frac{1}{p}}\varphi|_p.$$

We restrict here to weights ξ, $\xi(x)(e)=\xi(x)\cdot e$. We now define for an arbitrary weight $\xi(x)=\xi(x)\cdot\in L_{1,\mathrm{loc}}$, $\xi>0$,

$$\begin{aligned}
&\Omega^p_{r,\xi}(E)=\left\{\varphi\in C^\infty(E):|\varphi|_{p,r,\xi}:=\Big(\int_M\sum_{i=0}^r\xi(x)|\nabla^i\varphi|_x^p d\mathrm{vol}_x(g)\Big)^{\frac{1}{p}}<\infty\right\},\\
&\overline{\Omega}^{p,r}_\xi(E)=\overline{\Omega^p_{r,\xi}(E)}^{|\,|_{p,r,\xi}},\quad \mathring{\Omega}^{p,r}_\xi(E)=\overline{C^\infty_c(E)}^{|\,|_{p,r,\xi}},\\
&\Omega^{p,r}(E)=\left\{\varphi:\varphi \text{ a distributional section with} |\varphi|_{p,r,\xi}<\infty\right\}.
\end{aligned}$$

In our applications below, $\xi(x)$ will be a negative power of $r_{\mathrm{inj}}(x)$ or a certain estimate for this. Therefore we must discuss same properties of the function $r_{\mathrm{inj}}(x)$.

We consider open complete manifolds (M^n, g) with bounded curvature K_M, $|K_M| \leq K$, with finitely many ends $\varepsilon_1, \ldots, \varepsilon_s$. Each of these is automatically isolated, i.e. there exists for any ε_σ, a neighborhood $U(\varepsilon_\sigma)$ which is not a neighborhood of any other end. Consider such an $U(\varepsilon_\sigma)$. Then either $\inf_{x \in U(\varepsilon_\sigma)} r_{\mathrm{inj}}(x) = 0$ or $\inf_{x \in U(\varepsilon_\sigma)} r_{\mathrm{inj}}(x) > 0$. This property is independent of $U(\varepsilon_\sigma)$ and a property of ε_σ alone, i.e. either $r_{\mathrm{inj}}(\varepsilon_\sigma) = 0$ or $r_{\mathrm{inj}}(\varepsilon_\sigma) > 0$. Analogous considerations lead to the alternative $r_{\mathrm{inj}}(\varepsilon_\sigma) < \infty$ or $r_{\mathrm{inj}}(\varepsilon_\sigma) = \infty$, which means that for $U(\varepsilon_\sigma)$ sufficiently near to infinity for all $x \in U(\varepsilon_\sigma)$ there holds $r_{inj}(x) = \infty$. An example is the hyperbolic space. Finally, $\sup_{x \in U(\varepsilon_\sigma)} r_{\mathrm{inj}}(x) = \infty$ is an invariant of ε_σ too, and in this case we write $\sup r_{\mathrm{inj}}(\varepsilon_\sigma) = \infty$.

Remark 2.2. It can happen, even in the case (B_0), that $r_{\mathrm{inj}}(\varepsilon_\sigma) = 0$ and $\sup_{x \in U(\varepsilon)} r_{\mathrm{inj}}(x) = \infty$.

Lemma 2.3. *Suppose* (M^n, g) *open, complete, satisfying* (B_0) *and let* ε *be an isolated end of* M, $U(\varepsilon)$ *a neighborhood of* ε *alone.*

1. *If* $\mathrm{vol}(E) < \infty$ *then* $r_{\mathrm{inj}}(\varepsilon) = 0$.
2. $\inf_{x \in U(\varepsilon)} \mathrm{vol}(B_1(x)) = 0$ *if and only if* $r_{\mathrm{inj}}(\varepsilon) = 0$.

Proof. The Case 1 is absolutely trivial.

For the Case 2 we start with the standard Bishop-Guenther-Gromov volume estimates. Let $x \in M$, $r \leq \min\left\{r_{\mathrm{inj}}(x), \frac{\pi}{12\sqrt{K}}\right\}$. Then

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sin t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt \leq \mathrm{vol}(B_r(x)) \leq \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt.$$

Suppose $\inf_{x \in U(\varepsilon)} \mathrm{vol}(B_1(x)) = 0$ and $\inf_{x \in U(\varepsilon)} r_{\mathrm{inj}}(x) = r_{\mathrm{inj}}(\varepsilon) = i_0 > 0$. If $i_0 \leq 1$, then

$$0 < \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^{i_0} \left(\frac{\sin t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt \leq \mathrm{vol}(B_{i_0}(x)) \leq \mathrm{vol}(B_1(x)) \tag{9}$$

for all $x \in U(\varepsilon)$. If $i_0 > 1$, choose $0 < r_0 \leq \min\left\{1, i_0, \frac{\pi}{12\sqrt{K}}\right\}$ and one gets

$$0 < \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^{r_0} \left(\frac{\sin t\sqrt{K}}{\sqrt{K}}\right)^{n-1} dt \leq \mathrm{vol}(B_{r_0}(x)) \leq \mathrm{vol}(B_1(x)) \tag{10}$$

for all $x \in U(\varepsilon)$, i.e. in any case a contradiction. For the converse direction we recall [1, Theorem 4.7]. $\square$

Proposition 2.4. *Let (M^n, g) be open, complete with $H \leq K_M \leq K$. Let $\rho = \operatorname{dist}(p, x)$ and fix r, r_0, s with $r_0 + 2s < \frac{\pi}{\sqrt{K}}$, $r_0 \leq \frac{\pi}{4\sqrt{K}}$, where in the case $K \leq 0$, $r_0 + 2s$, r_0 can be arbitrary, i.e. we set in the case $K \leq 0$, $\frac{1}{\sqrt{K}} = \infty$. Then*

$$r_{\mathrm{inj}}(x) \geq \frac{r_0}{2} \frac{1}{1 + \frac{V^H_{r_0+s}}{\mathrm{vol}(B_r(p))} \frac{V^H_{\rho+r}}{V^H_s)}}. \tag{11}$$

Moreover, if $r + s < \rho$, then

$$r_{\mathrm{inj}}(x) \geq \frac{r_0}{2} \frac{1}{1 + \frac{V^H_{r_0+s}}{\mathrm{vol}(B_r(p))}(V^H_{\rho+r} - V^H_{\rho-r})}. \tag{12}$$

Here $V^H_r = \mathrm{vol}(B^H_r(z))$, $B^H_r(z) \subset H$-space form. If now $\lim_{x_\nu \to \infty} r_{\mathrm{inj}}(x_\nu)$ then necessarily for fixed r, r_0, s, ρ

$$\frac{V^H_{r_0+s}}{\mathrm{vol}(B_r(p_\nu))}(V^H_{\rho+r} - V^H_{\rho-r}) \to \infty \quad \text{as } p_\nu \to \infty,$$

i.e. $\mathrm{vol}(B_r(p_\nu)) \to 0^+$ as $p_\nu \to \infty$. $\square$

Remark 2.5. It is very easy to construct ends ε with bounded curvature, $\mathrm{vol}(\varepsilon) = \infty$, $r_{\mathrm{inj}}(\varepsilon) = 0$.

Preparing our first embedding theorem, we must still introduce some notions and estimates. Set

$$\tilde{r}_{\mathrm{inj}}(x) := \min\left\{r_{\mathrm{inj}}(x), \frac{\pi}{12\sqrt{K}}\right\}. \tag{13}$$

Lemma 2.6. *There exists a constant $C > 0$, depending on K, such that for all $x, p \in M$*

$$\tilde{r}_{\mathrm{inj}}(x) \geq C\tilde{r}_{\mathrm{inj}}(p)^n \mathrm{e}^{-(n-1)\sqrt{K}d(x,p)}.$$

Corollary 2.7. *Given $p \in M$, there exists a constant $C = C(p) > 0$ such that*

$$\tilde{r}_{\mathrm{inj}}(x) \geq C\mathrm{e}^{-(n-1)\sqrt{K}d(x,p)}.$$

Lemma 2.8. *There exists a constant $C = C(K)$ such that for each $x, y \in M$ we have the inequality*

$$\tilde{r}_{\mathrm{inj}}(y) \geq C\tilde{r}_{\mathrm{inj}}(x)\exp\left\{\frac{-(n-1)\pi}{12} \frac{d(x,y)}{\tilde{r}_{\mathrm{inj}}(x)}\right\}. \tag{14}$$

Lemmas 2.6, 2.8 and Corollary 2.7 are [9, Lemmas 2.3–2.5] and we refer to [9] for the proofs. The key to the proofs are the standard volume comparison theorems and (9), (10).

The main point of these lemmas is the maximal decrease/increase of the injectivity radius.

Proposition 2.9. *Let (M^n, g) be open, complete with $H \le K_M \le K$. ε an end of M^n with $r_{\mathrm{inj}}(\varepsilon) = 0$, $U(\varepsilon)$ a neighborhood of ε. Denote*

$$I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right) = \left\{z \in U(\varepsilon) : r_{\mathrm{inj}}(z) < \frac{\pi}{12\sqrt{K}}\right\}.$$

Then there exists $a > 0$ such that for all $z \in I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right)$

$$\sup_{x \in B_{r_{\mathrm{inj}}(z)}(z)} r_{\mathrm{inj}}(x) \ge a r_{\mathrm{inj}}(z). \tag{15}$$

Proof. $r_{\mathrm{inj}}(\varepsilon) = 0$ implies $I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right)$ is not empty and $\tilde{r}_{\mathrm{inj}}(x) = r_{\mathrm{inj}}(x)$ for $x \in I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right)$. Suppose (15) is wrong. Then there exist sequences (x_ν), $(z_\nu) \to \infty$ in $I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right)$,

$$x_\nu \in B_{r_{\mathrm{inj}}(z_\nu)}(z_\nu) \quad \text{and} \quad r_{\mathrm{inj}}(x_\nu) > \nu r_{\mathrm{inj}}(z_\nu). \tag{16}$$

We set $y = z_\nu$, $x = x_\nu$ in (16) and obtain

$$\tilde{r}_{\mathrm{inj}}(z_\nu) \ge r_{\mathrm{inj}}(x_\nu) \exp\left\{\frac{-(n-1)\pi}{12} \frac{d(z_\nu, x_\nu)}{\tilde{r}_{\mathrm{inj}}(x_\nu)}\right\}.$$

But

$$\frac{d(z_\nu, x_\nu)}{r_{\mathrm{inj}}(x_\nu)} \le \frac{r_{\mathrm{inj}}(z_\nu)}{r_{\mathrm{inj}}(x_\nu)} < \frac{1}{\nu} \to 0,$$

$$1 > \exp\left\{\frac{-(n-1)\pi}{12} \frac{d(z_\nu, x_\nu)}{r_{\mathrm{inj}}(x_\nu)}\right\} > \frac{3}{4}, \quad \frac{4}{3} r_{\mathrm{inj}}(z_\nu) \ge r_{\mathrm{inj}}(x_\nu), \quad \nu \ge \nu_1,$$

which contradicts (16). □

Remark 2.10.

1. It is possible (and standard) to prove Proposition 2.9 by means of convexity considerations.
2. If $\mathrm{vol}(\varepsilon) < \infty$ then one can assume $I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right) = U(\varepsilon)$ and $r_{\mathrm{inj}}(z) < \frac{\pi}{12\sqrt{K}}$ for all $z \in U(\varepsilon)$.

Now we are able to prove our first embedding theorem.

Theorem 2.11 (First embedding theorem). *Let $E = ((E, h, \nabla) \to (M^n, g))$ be a Riemannian vector bundle, (M^n, g) being open, complete, $H \leq K_M \leq K$, with finitely many ends $\varepsilon_1, \varepsilon_2, \ldots$. Suppose, at least for one end ε, $r_{\mathrm{inj}}(\varepsilon) = 0$. Let*

$$\xi(x) = \max\left\{1, r_{\mathrm{inj}}(x)^{-n}\right\}. \tag{17}$$

Then, for $r > \frac{n}{p} + m$, there exists a continuous embedding

$$\Omega^{p,r}_{\xi}(E) \hookrightarrow {}^{b,m}\Omega(E). \tag{18}$$

Proof. According to our choice of ξ, $\Omega^{p,r}_{\xi}(E) \subset \Omega^{p,r}(E)$. We start with $r > \frac{n}{p}$ and suppose

$$\varphi \in \Omega^{p,r}_{\xi}(E) \cap C^{\infty}.$$

Then, according to [5, p. 16–18],

$$\begin{aligned}
|\varphi(z)| &\leq C_2 \Big(\sum_{i=0}^{r} \int_{B_{r_{\mathrm{inj}}(z)}(z)} |\nabla^i \varphi(x)|_x^p d\mathrm{vol}_x(g) \Big)^{\frac{1}{p}} \\
&= C_2 \Big(\sum_{i=0}^{r} \int_{B_{r_{\mathrm{inj}}(z)}(z)} \xi(x)^{-1} \xi(x) |\nabla^i \varphi(x)|_x^p d\mathrm{vol}_x(g) \Big)^{\frac{1}{p}} \\
&\leq C_2 \sup_{x \in B_{r_{\mathrm{inj}}(z)}(z)} \xi(x)^{-\frac{1}{p}} \Big(\sum_{i=0}^{r} \int_{B_{r_{\mathrm{inj}}(z)}(z)} \xi(x) |\nabla^i \varphi(x)|_x^p d\mathrm{vol}_x(g) \Big)^{\frac{1}{p}} \\
&= C_2 \sup_{x \in B_{r_{\mathrm{inj}}(z)}(z)} \max\left\{1, r_{\mathrm{inj}}(x)^{-n}\right\}^{-\frac{1}{p}} \\
&\quad \cdot \Big(\sum_{i=0}^{r} \int_{B_{r_{\mathrm{inj}}(z)}(z)} \xi(x) |\nabla^i \varphi(x)|_x^p d\mathrm{vol}_x(g) \Big)^{\frac{1}{p}} \\
&= C_2 \sup_{x \in B_{r_{\mathrm{inj}}(z)}(z)} \min\left\{1, r_{\mathrm{inj}}(x)^{\frac{n}{p}}\right\} \Big(\sum_{i=0}^{r} \int_{B_{r_{\mathrm{inj}}(z)}(z)} \xi(x) |\nabla^i \varphi(x)|_x^p d\mathrm{vol}_x(g) \Big)^{\frac{1}{p}}.
\end{aligned}$$

Let $z \in M^n \setminus \bigcup\limits_{\mathrm{vol}(\varepsilon_\sigma) < \infty} U(\varepsilon_\sigma)$. Then

$$C_2 = C_2\left(n, p, H, K, \left(r_{\mathrm{inj}}\Big|_{M \setminus \bigcup\limits_{\mathrm{vol}(\varepsilon_\sigma)<\infty} U(\varepsilon_\sigma)}\right)^{-\frac{n}{p}}\right)$$

and

$$C_2 \sup_{x \in B_{r_{\mathrm{inj}}(z)}(z)} \min\left\{1, r_{\mathrm{inj}}(x)^{\frac{n}{p}}\right\} \leq C_3, \tag{19}$$

C_3 independent of z. If $z \in U(\varepsilon)$, where $r_{\text{inj}}(\varepsilon) = 0$, then

$$C_2 = C_2\left(n, p, H, K, r_{\text{inj}}(z)^{-\frac{n}{p}}\right) = O\left(r_{\text{inj}}(z)^{-\frac{n}{p}}\right)$$

for $z \to \infty$ in $I\left(\varepsilon, \frac{\pi}{12\sqrt{K}}\right)$. But in this case

$$C_2 \sup_{x \in B_{r_{\text{inj}}(z)}(z)} \min\left\{1, r_{\text{inj}}(x)^{\frac{n}{p}}\right\} \le C_2 \min\left\{1, a r_{\text{inj}}(z)^{\frac{n}{p}}\right\} \le C_4, \tag{20}$$

C_4 independent of z.

Altogether, we obtain $|\varphi(z)| \le C|\varphi|_{B_{r_{\text{inj}}(z)}(z),p,r,\xi} \le C|\varphi|_{p,r,\xi}$, i.e. $|\varphi|$ is bounded by $C|\varphi|_{p,r,\xi}$.

The remaining part are standard arguments. If $\varphi \in \Omega^{p,r}_{\xi}(E)$, $\varphi_\nu \to \varphi$ w.r.t. $|\,|_{p,r,\xi}$, φ_ν smooth, then $\varphi_\nu \to \varphi$ w.r.t. ${}^b|\,| = {}^{b,0}|\,|$, φ is C^0. Applying this to $\nabla\varphi_\nu, \dots, \nabla^m\varphi_\nu$ yields the assertion. $\square$

Examples of open manifolds as above are manifolds of the type $\Gamma \backslash G/K =$ locally symmetric manifolds of finite volume, or, a little more general

$$M = M_0 \cup U(\varepsilon_1) \cup \cdots \cup U(\varepsilon_s),$$

where ε_σ are the cusps = ends (with finite volume) of some $\Gamma \backslash G/K$.

The choice of the weight $r_{\text{inj}}(x)^{-n}$ is in a certain sense the upper limit of the set of weights if one wants to have an embedding theorem as Theorem 2.11. For practical calculations this choice is relatively inconvenient, since the calculation of the injectivity radius $r_{\text{inj}}(x)$ can be very difficult. Much more convenient is to work with an estimate for the injectivity radius. An example for such an estimate is given by inequality of Lemma 2.6, Corollary 2.7.. If $\text{vol}(\varepsilon) < \infty$, then for $U(\varepsilon)$ far enough from a "central" compactum

$$\tilde{r}_{\text{inj}}(x) = r_{\text{inj}}(x) \text{ for } x \in U(\varepsilon), \quad r_{\text{inj}}(x) \ge C\mathrm{e}^{-(n-1)\sqrt{K}d(x,p)}. \tag{21}$$

If $r_{\text{inj}}(\varepsilon) > 0$, i.e. $r_{\text{inj}}(x) \ge c > 0$ for all $x \in U(\varepsilon)$, then the estimate (21) is much to crude and does not reflect any actual behaviour of the injectivity radius. On the other side, for open complete (M^n, g) with only such ends we have already well established embedding and module structure theorems. The weight is, e.g., simply $\xi(x) \equiv 1$. Finally, in the mixed case $r_{\text{inj}}(\varepsilon) = 0$ but $\text{vol}(\varepsilon) = \infty$, the actual behaviour of the injectivity radius can be very far from exponential decay and we should work with the choice of (17).

We summarize, in the case of bounded curvature there are for an end ε exactly three alternatives.

$$r_{\text{inj}}(\varepsilon) > 0 \quad \text{and hence} \quad \text{vol}(\varepsilon) = \infty, \tag{22}$$
$$\text{vol}(\varepsilon) < \infty \quad \text{and hence} \quad r_{\text{inj}}(x) \to 0 \text{ as } x \to \infty, \tag{23}$$
$$r_{\text{inj}}(\varepsilon) = 0 \quad \text{and} \quad \text{vol}(\varepsilon) = \infty. \tag{24}$$

The case (22) is already settled. We are concerned with (23) and (24) and establish in the case (23) a first embedding theorem for the weighted spaces with the weight $\xi(x) = C\mathrm{e}^{n(n-1)\sqrt{K}d(x,p)}$. For simplicity $\xi(x) = \mathrm{e}^{\delta\rho_y}$, y fixed, $\delta = n(n-1)\sqrt{K}$, $\rho_y(x) = d(x,y)$. Then there are defined

$$\begin{aligned}
|\varphi|_{p,r,\xi} = |\varphi|_{p,r,\delta,y} &= \left(\int \sum_{i=0}^{r} |\nabla^i\varphi|_x^p \mathrm{e}^{\delta\rho_y} d\text{vol}_x(g)\right)^{\frac{1}{p}}, \\
\Omega^p_{r,\delta,y}(E) &= \Big\{\varphi \in C^\infty(E) : |\varphi|_{p,r,\delta,y} < \infty\Big\}, \\
\overline{\Omega}^{p,r}_{\delta,y}(E) &= \overline{\Omega^p_{r,\delta,y}(E)}^{|\,|_{p,r,\delta,y}}, \quad \mathring{\Omega}^{p,r}_{\delta,y}(E) = \overline{C^\infty_c(E)}^{|\,|_{p,r,\delta,y}} \\
\Omega^{p,r}_{\delta,y}(E) &= \Big\{\varphi : \varphi \text{ distributional section with } |\varphi|_{p,r,\delta,y} < \infty\Big\}.
\end{aligned}$$

Theorem 2.12. *Let (M^n, g) be open, complete,* $\text{vol}(M) < \infty$, $|K_M| \le k$, $E = ((E,h,\nabla) \to (M^n, g))$ *a Riemannian vector bundle,* $r > \frac{n}{p} + m$, $\delta = n(n-1)\sqrt{k}$*. Then there exists a continuous embedding*

$$\overline{\Omega}^{p,r}_{\delta,y}(E) \hookrightarrow {}^{b,m}\Omega(E).$$

Proof. As before for $\varphi \in \overline{\Omega}^{p,r}_{\delta,y}(E) \cap C^\infty$,

$$\begin{aligned}
&|\varphi(z)| \\
&\le C \left(\sum_{i=0}^{r} \int_{B_{r_{\text{inj}}(z)}(z)} |\nabla^i\varphi(x)|_x^p d\text{vol}_x(g)\right)^{\frac{1}{p}} \\
&= C \sup_{x \in B_{r_{\text{inj}}(z)}(z)} \left(\sum_{i=0}^{r} \int_{B_{r_{\text{inj}}(z)}(z)} (\mathrm{e}^{-\frac{n(n-1)}{p}\sqrt{k}\rho_y})^p |\mathrm{e}^{\frac{n(n-1)}{p}\sqrt{k}\rho_y} \nabla^i\varphi(x)|_x^p d\text{vol}_x(g)\right)^{\frac{1}{p}} \\
&\le C \sup_{x \in B_{r_{\text{inj}}(z)}(z)} \left(\mathrm{e}^{-(n-1)\sqrt{k}\rho_y}\right)^{\frac{n}{p}} \left(\sum_{i=0}^{r} \int_{B_{r_{\text{inj}}(z)}(z)} \mathrm{e}^{\delta\rho_y} |\nabla^i\varphi(x)|_x^p d\text{vol}_x(g)\right)^{\frac{1}{p}} \\
&\le C_1 (r_{\text{inj}}(z))^{-\frac{n}{p}} \left(\sup_{x \in B_{r_{\text{inj}}(z)}(z)} \mathrm{e}^{-(n-1)\sqrt{k}\rho_y}\right)^{\frac{n}{p}} \left(\sum_{i=0}^{r} \int_{B_{r_{\text{inj}}(z)}(z)} \mathrm{e}^{\delta\rho_y} |\nabla^i\varphi(x)|_x^p d\text{vol}_x(g)\right)^{\frac{1}{p}} \\
&\le C_2 |\varphi|_{p,r,\delta,y},
\end{aligned}$$

since, according to Corollary 2.7, at each end

$$(r_{\mathrm{inj}}(z))^{-\frac{n}{p}}\Big(\sup_{x\in B_{r_{\mathrm{inj}}(z)}(z)} \mathrm{e}^{-(n-1)\sqrt{k}\rho_y}\Big)^{\frac{n}{p}} \le D. \tag{25}$$

The remaining arguments are exactly the same as in the proof of Theorem 2.11. □

The next step in preparing module structure theorems are embedding theorems similar to Proposition 2.1.

First we restrict to the case of functions. We recall the following theorem which can be found in [7].

Theorem 2.13. *Let (M^n, g) be open, complete and suppose an embedding*

$$\overline{\Omega}^{0,1,1} \hookrightarrow \Omega^{0,\frac{n}{n-1},0} \equiv L_{\frac{n}{n-1}}$$

is valid. Then, for any real numbers $1 \le q < p$ and any integer $0 \le m \le k$ satisfying $\frac{1}{p} = \frac{1}{q} - \frac{k-m}{n}$, there is an embedding

$$\Omega^{0,q,k} \hookrightarrow \Omega^{0,p,m}.$$

Here in $\Omega^{s,q,k}$, s stands for s-forms, hence $s = 0$ for functions. One says in this case, the scale of Sobolev embeddings is valid.

Theorem 2.14. *The Sobolev scale of embeddings is valid if and only if*

$$\inf_{x\in M} \mathrm{vol}(B_1(x), g) > 0.$$

Proof. We refer to [7, p. 38] for the proof. □

Corollary 2.15. *For complete manifolds with finite volume and bounded curvature the Sobolev embeddings are not valid. In particular this holds for cusp manifolds.* □

We see once again that our search for embedding theorems in the class of weighted spaces has a substantial background.

In the next considerations, we will not fix the weight $\xi(x)$, but we will have in mind three classes, $\xi(x)$ as in Theorems 2.11, 2.12 or $\xi(x) = \frac{1}{\mathrm{vol}(B_1(x))}$.The proofs of the following Theorems 2.16, 2.17 carry over straightforward from the unweighted to the weighted case. But we present them to improve the readability of this treatise.

Theorem 2.16. *Let (M^n, g) be open, complete and suppose a continuous embedding*

$$\overline{\Omega}_{\xi}^{0,1,1} \hookrightarrow L_{\frac{n}{n-1},\xi}.$$

Then, for any real numbers $1 \leq q < p$ satisfying $1 \leq q < n$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, there holds

$$\overline{\Omega}_{\xi}^{0,q,1} \hookrightarrow L_{p,\xi}. \tag{26}$$

Proof. By definition

$$|\varphi|_{p,r,\xi} = \left(\sum_{i=0}^{r} \int |\nabla^i \varphi|^p \xi d\mathrm{vol}_x(g) \right)^{\frac{1}{p}}.$$

Let $\varphi \in C_c^\infty$ and set $\phi = \varphi|\varphi|^{p\frac{n-1}{n}-1}$. Then $|\phi|^{\frac{n}{n-1}} = |\varphi|^p$,

$$\nabla(\varphi|\varphi|^{p\frac{n-1}{n}-1}) = |\varphi|^{p\frac{n-1}{n}-1}|\nabla\varphi| + \left(p\frac{n-1}{n} - 1\right)\varphi|\varphi|^{p\frac{n-1}{n}-2}\nabla|\varphi|,$$

there holds $|\nabla|\varphi|\,| \leq |\nabla\varphi|$ and

$$\begin{aligned}
&\left(\int |\varphi|^p \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{n-1}{n}} \\
&= \left(\int |\phi|^{\frac{n}{n-1}} \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{n-1}{n}} \\
&\leq C_1 \int (|\nabla\phi| + |\phi|)\xi(x) d\mathrm{vol}_x(g) \\
&\leq C_1 \int \left(|\varphi|^{p\frac{n-1}{n}-1}|\nabla\varphi| + \left(p\frac{n-1}{n} - 1\right)|\varphi|^{p\frac{n-1}{n}-1}\nabla|\varphi| + |\varphi|^{p\frac{n-1}{n}}| \right) \xi(x) d\mathrm{vol}_x(g) \\
&\leq C_1 \int |\varphi|^{p\frac{n-1}{n}-1}|\nabla\varphi| \left(p\frac{n-1}{n} + |\varphi|^{p\frac{n-1}{n}}\right)\xi(x) d\mathrm{vol}_x(g) \\
&= C_1 p\frac{n-1}{n} \int |\varphi|^{p'}|\nabla\varphi|\xi(x) d\mathrm{vol}_x(g) + C_1 \int |\varphi|^{p\frac{n-1}{n}}\xi(x) d\mathrm{vol}_x(g) \\
&\leq C_1 p\frac{n-1}{n} \left(\int |\varphi|^{p'q'}\xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q'}} \left(\int |\nabla\varphi|^q \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q}} \\
&\quad + C_1 \left(\int |\varphi|^{p'q'}\xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q'}} \left(\int |\varphi|^q \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q}}
\end{aligned}$$

with $\frac{1}{q} + \frac{1}{q'} = 1$, $p' = p\frac{n-1}{n} - 1 = p - \frac{p}{n} - 1$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, $p'q' = p$, $|\varphi|^{p\frac{n-1}{n}} = |\varphi|^{p'}|\varphi|$ and where we applied Hölder inequality with respect to $d\mu_x = \xi(x) d\mathrm{vol}_x(g)$.

We obtain

$$\left(\int |\varphi|^p \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{n-1}{n}}$$
$$\leq C_1 p \frac{n-1}{n} \left(\int |\varphi|^p \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q'}} \left(\int |\nabla\varphi|^q \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q}}$$
$$+ C_1 \left(\int |\varphi|^p \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q'}} \left(\int |\varphi|^q \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q}}.$$

Then we multiply this inequality with $\left(\left(\int |\varphi|^p \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q'}}\right)^{-1}$ and get by means of $\frac{n-1}{n} - \frac{1}{q'} = 1 - \frac{1}{n} - q' = 1 - \frac{1}{n} - \left(1 - \frac{1}{q}\right) = \frac{1}{q} - \frac{1}{n} = \frac{1}{p}$

$$\left(\int |\varphi|^p \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{p}}$$
$$\leq C_1 p \frac{n-1}{n} \left(\left(\int |\nabla\varphi|^q \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q}} + \left(\int |\varphi|^q \xi(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{q}} \right).$$

Note that if (M^n, g) is complete, then $\overset{\circ}{\Omega}{}_\xi^{0,q,1} = \overline{\Omega}_\xi^{0,q,1}$. Since (26) holds for $\varphi \in \overset{\circ}{\Omega}{}_\xi^{0,q,1}$, it also holds for $\varphi \in \overline{\Omega}_\xi^{0,q,1}$. □

Now, the corresponding extension of Theorem 2.13 is given by

Theorem 2.17. *Let (M^n, g) be open, complete and suppose a continuous embedding*

$$\overline{\Omega}_\xi^{0,q_0,1} \hookrightarrow L_{p_0,\xi}, \quad \frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{n}, \quad 1 \leq q_0 < n. \tag{27}$$

Then there exist continuous embeddings

$$\overline{\Omega}_\xi^{0,q,k} \hookrightarrow \overline{\Omega}_\xi^{0,p_l,l}, \quad \frac{1}{p_l} = \frac{1}{q} - \frac{k-l}{n}, \quad 1 \leq q_0 < n. \tag{28}$$

Proof. By assumption, there exists $C > 0$ such that for all $u \in \overline{\Omega}_\xi^{0,q,k}$

$$|u|_{p_0,\xi} \leq C\big(|\nabla u|_{q_0,\xi} + |u|_{q_0,\xi}\big)$$

If $u = |\nabla^r w| \in \overline{\Omega}_\xi^{0,q,1}$, then we obtain

$$|\nabla^r w|_{p_0,\xi} \leq C\big(|\nabla|\nabla^r w|\,|_{q_0,\xi} + |\nabla^r w|_{q_0,\xi}\big) \leq C\big(|\nabla^{r+1} w|_{q_0,\xi} + |\nabla^r w|_{q_0,\xi}\big). \tag{29}$$

Let $w \in \overline{\Omega}_\xi^{0,q,k} \cap C^\infty$ and $q = q_0$, $r = k-1, k-2, \ldots$. Then (29) yields

$$|w|_{p_{k-1},k-1,\xi} \le 2C|w|_{q,k,\xi} \tag{30}$$

with $\frac{1}{p_{k-1}} = \frac{1}{q} - \frac{k-(k-1)}{n} = \frac{1}{q} - \frac{1}{n} = \frac{1}{q_0} - \frac{1}{n}$.

We see (28) as follows. According to (27),

$$\begin{aligned} |\nabla^{k-1} w|_{p_{k-1},\xi} &\le C\big(|\nabla^k w|_{q,\xi} + |\nabla^{k-1} w|_{q,\xi}\big), \\ |\nabla^{k-2} w|_{p_{k-1},\xi} &\le C\big(|\nabla^{k-1} w|_{q,\xi} + |\nabla^{k-2} w|_{q,\xi}\big), \\ &\vdots \\ |w|_{p_{k-1},\xi} &\le C\big(|\nabla w|_{q,\xi} + |w|_{q,\xi}\big). \end{aligned}$$

Adding up these inequalities, we obtain $|w|_{p_{k-1},k-1,\xi} \le 2C|w|_{q,k,\xi}$, which extends to non-smooth elements $\overline{\Omega}_\xi^{0,q,k} \hookrightarrow \overline{\Omega}_\xi^{0,p_{k-1},k-1}$. Repeating this procedure, we finally get the claim of embeddings

$$\overline{\Omega}_\xi^{0,q,k} \hookrightarrow \overline{\Omega}_\xi^{0,p_{k-1},k-1} \hookrightarrow \overline{\Omega}_\xi^{0,p_{k-2},k-2} \hookrightarrow \cdots \hookrightarrow \overline{\Omega}_\xi^{0,p_l,l},$$

which yields the assertion (28). □

It is immediately evident that Theorems 2.16 and 2.17 admit an extension to vector bundles.

Theorem 2.18. *Let (M^n, g) be open, complete, $E = ((E, h, \nabla) \to (M^n, g))$ be a Riemannian embedding. Suppose a continuous embedding*

$$\overline{\Omega}_\xi^{0,q_0,1}(E) \equiv \overline{\Omega}_\xi^{q_0,1}(E) \hookrightarrow L_{p_0,\xi}(E), \quad \frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{n}, \quad 1 \le q_0 < n.$$

Then there exist continuous embeddings

$$\overline{\Omega}_\xi^{q_0,1}(E) \hookrightarrow L_{p_0,\xi}(E)$$

and

$$\overline{\Omega}_\xi^{q,k}(E) \hookrightarrow \overline{\Omega}_\xi^{p_l,l}(E), \quad \frac{1}{p_l} = \frac{1}{q} - \frac{k-l}{n} > 0.$$

Proof. The proofs of Theorem 2.16 for functions and sections are identical. For this reason we already dealt functions as φ. For the proof of Theorem 2.18 let $\varphi \in \Omega_\xi^{0,q_0,1}(E) \equiv \Omega_\xi^{q_0,1}(E)$. Let $u(x) = |\varphi(x)| \in \Omega_\xi^{0,q_0,1}(E) \equiv \Omega_\xi^{q_0,1}(E)$. For $u = |\nabla^r \varphi| \in \Omega_\xi^{q_0,1}(E)$, we have

$$|\nabla^r \varphi|_{p_0,\xi} \le C\big(|\nabla|\nabla^r \varphi|\,|_{q_0,\xi} + |\nabla^r \varphi|_{q_0,\xi}\big) \le C\big(|\nabla^{r+1} \varphi|_{q_0,\xi} + |\nabla^r \varphi|_{q_0,\xi}\big).$$

Suppose now $q = q_0$, $\varphi \in \Omega_\xi^{0,q,k}(E) \equiv \Omega_\xi^{q,k}(E)$. Then with $r = k-1, k-2, \ldots$, $\frac{1}{p_{k-1}} = \frac{1}{q} - \frac{k-(k-l)}{n} = \frac{1}{q} - \frac{1}{n} = \frac{1}{q_0} - \frac{l}{n}$ and repeating the proof for functions, we obtain

$$|\varphi|_{p_{k-1},k-1,\xi} \le 2C|\varphi|_{q,k,\xi}\,, \quad \overline{\Omega}_\xi^{q,k}(E) \hookrightarrow \overline{\Omega}_\xi^{0,p_{k-1},k-1}(E),$$

and, by an analogous procedure

$$\overline{\Omega}_\xi^{q,k}(E) \hookrightarrow \overline{\Omega}_\xi^{p_{k-1},k-1}(E) \hookrightarrow \overline{\Omega}_\xi^{p_{k-2},k-2}(E) \hookrightarrow \cdots \hookrightarrow \overline{\Omega}_\xi^{p_l,l}(E). \qquad \square$$

Remark 2.19. ¿From Theorem 2.16 up to now we did not make any assumptions on the injectivity radius, curvature, volume and the weight. The weight $\xi(x)$ before Theorem 2.16 could be replaced by any other admitted weight.

The heart of the proof that the scale of Sobolev embeddings is valid is to establish the initial main assumption, the embedding

$$\overline{\Omega}^{0,1,1}_{\text{weight}} \equiv \overline{\Omega}^{1,1}_{\text{weight}} \hookrightarrow \overline{\Omega}^{\frac{n}{n-1},0}_{\text{weight}} \equiv L_{\frac{n}{n-1},\text{weight}}. \tag{31}$$

Here we essentially use geometric assumptions.

3. The initial step in the scale of Sobolev embeddings and the module structure theorem

As we already mentioned in Theorem 2.14, the scale of embeddings in the unweighted case is valid if and only if $\inf_{x\in U(\varepsilon)} \operatorname{vol}(B_1(x)) > 0$ for all ends ε. On the other hand, the case $\inf_{x\in M} \operatorname{vol}(B_1(x)) = 0$ which is satisfied in (23) and (24) is just the case we are concerned with. Fortunately, for the weight $\xi(x) = \frac{1}{\operatorname{vol}(B_1(x))}$, (31) is already positively settled.

Theorem 3.1. *Let (M^n, g) be open, complete and suppose $\operatorname{Ric}(g) \ge k\cdot g$. Then there exists a positive constant $A = A(n,k)$ such that for any $u \in C_c^\infty(M)$*

$$\left(\int_M |u|^{\frac{n}{n-1}} v(x) d\operatorname{vol}_x(g)\right)^{\frac{n-1}{n}} \le A \int_M (|\nabla u| + |u|) v(x) d\operatorname{vol}_x(g),$$

where $v(x) = \frac{1}{\operatorname{vol}(B_1(x))}$.

Proof. We refer to [8, 10] for the proof. $\square$

We recall that $|K_M| \le K$ implies $\operatorname{Ric}(g) \ge (n-1)(-K)$.

Corollary 3.2. *Under the hypotheses of Theorem 3.1, there exists a continuous embedding*

$$\overline{\Omega}_v^{1,1} \equiv \overline{\Omega}_v^{0,1,1} \hookrightarrow L_{\frac{n}{n-1},v}. \tag{32}$$

Proof. This follows from Theorem 3.1 and

$$\overline{C_c^\infty(M)}^{|\ |_{1,1,v}} = \overline{\Omega}_v^{1,1}. \qquad \square$$

Corollary 3.3. *Under the hypotheses of Theorem* 3.1*, there holds the scale of embeddings*

$$\overset{\circ}{\Omega}{}_v^{0,q_0,1} \hookrightarrow L_{p_0,v}, \quad \frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{n}, \quad 1 \leq q_0 < n \tag{33}$$

and

$$\overset{\circ}{\Omega}{}_v^{0,q,k} \hookrightarrow \overset{\circ}{\Omega}{}_v^{p_l,l}, \quad \frac{1}{p_l} = \frac{1}{q} - \frac{k-l}{n}. \tag{34}$$

Proof. We must work with $\overset{\circ}{\Omega}{}_v^{0,q,k} = \overline{C_c^\infty(M)}^{|\ |_{q,k,v}}$ for $k, l > 1$. (33) and (34) follow from (32) by means of Theorems 2.16 and 2.17. $\square$

Theorem 3.4. *Let $E = ((E, h, \nabla) \to (M^n, g))$ be a Riemannian vector bundle, (M^n, g) as in Theorem* 3.1*. Then there exists a positive constant $A = A(n, k)$ such that for any $\varphi \in C_c^\infty(E)$*

$$\left(\int_M |\varphi|^{\frac{n}{n-1}} v(x) d\mathrm{vol}_x(g)\right)^{\frac{n-1}{n}} \leq A \int_M (|\nabla\varphi| + |\varphi|) v(x) d\mathrm{vol}_x(g), \tag{35}$$

where $v(x) = \frac{1}{\mathrm{vol}(B_1(x))}$.

Proof. Let $u = |\varphi|$, use $|\nabla u| = |\nabla|\varphi|\,| \leq |\nabla\varphi|$ and apply Theorem 3.1. $\square$

Corollary 3.5. *Assume the hypotheses of Theorem* 3.1*. Then there holds the scale of embeddings*

$$\overset{\circ}{\Omega}{}_v^{q_0,1}(E) \hookrightarrow L_{p_0,v}(E), \quad \frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{n}, \quad 1 \leq q_0 < n \tag{36}$$

and

$$\overset{\circ}{\Omega}{}_v^{q,k}(E) \hookrightarrow \overset{\circ}{\Omega}{}_v^{p_l,l}(E), \quad \frac{1}{p_l} = \frac{1}{q} - \frac{k-l}{n}. \tag{37}$$

Now we are able to establish the generalized module structure theorem for weighted Sobolev spaces. First we start with the weight $v = v(x) = \frac{1}{\mathrm{vol}(B_1(x))}$.

Theorem 3.6. *Let $E_i = ((E_i, h_i, \nabla_i) \to (M^n, g))$ be Riemannian vector bundles, (M^n, g) open, complete and* $\mathrm{Ric}(g) \geq k \cdot g$. *Let $v(x) = \frac{1}{\mathrm{vol}(B_1(x))}$ and let*

$$\left.\begin{aligned} &\frac{n}{p_1} - \frac{n}{p} < m_1 < \frac{n}{p_1}, \quad m_1 \geq 0 \textit{ integer},\\ &\frac{n}{p_2} - \frac{n}{p} < m_2 < \frac{n}{p_2}, \quad m_2 \geq 0 \textit{ integer},\\ &\frac{n}{p} = \frac{n}{p_1} + \frac{n}{p_2} - (m_1 + m_2),\\ &r_1 \geq r + m_1,\\ &r_2 \geq r + m_2. \end{aligned}\right\} \tag{38}$$

Then the tensor product defines a bilinear map

$$\overset{\circ}{\Omega}{}^{p_1,r_1}_v(E_1) \times \overset{\circ}{\Omega}{}^{p_2,r_2}_v(E_2) \to \overset{\circ}{\Omega}{}^{p,r}_v(E_1 \bigotimes E_2).$$

Proof. We present the proof for $E_1 = E_2 = M \times \mathbb{R}$. The general case immediately follows by replacing functions by sections and ordinary multiplication by the tensor product.

Let $u \in \overset{\circ}{\Omega}{}^{p_1,r_1}_v(E_1)$, $w \in \overset{\circ}{\Omega}{}^{p_2,r_2}_v(E_2)$. The result would follow if

$$|\nabla^i(uw)|_{p,0,v} \leq K|u|_{p_1,r_1,v}|w|_{p_2,r_2,v}, \quad 0 \leq i \leq r, \tag{39}$$

which is equivalent to

$$\left(\int |\nabla^j u|^p |\nabla^{i-j} w|^p v(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{p}} \leq K|u|_{p_1,r_1,v}|w|_{p_2,r_2,v}, \quad 0 \leq j \leq i \leq r. \tag{40}$$

The proof consists of two steps. First we apply Hölder inequality and secondly we apply our embedding theorem Corollary 3.3, i.e. we seek s and t such that $\frac{1}{s} + \frac{1}{t} = \frac{1}{p}$ and apply it with respect to the measure $\frac{1}{\mathrm{vol}(B_1(x))} d\mathrm{vol}_x(g)$,

$$\begin{aligned} \left(\int |\nabla^j u|^p |\nabla^{i-j} w|^p v(x) d\mathrm{vol}_x(g)\right)^{\frac{1}{p}} &\equiv |\, |\nabla^j u|\, |\nabla^{i-j} w|\, |_{p,0,v} \\ &\leq |\nabla^j u|_{s,0,v} |\nabla^{i-j} w|_{t,0,v}, \quad 0 \leq j \leq i \leq r. \end{aligned}$$

The second step consists to establish

$$|\nabla^j u|_{s,0,v} \leq K_1 |u|_{p_1,\tilde{r}_1,v}, \tag{41}$$

$$|\nabla^{i-j} w|_{t,0,v} \leq K_2 |w|_{p_2,\tilde{r}_2,v}, \tag{42}$$

where we applied Corollary 3.3 and (35).

Inequality (41) concerns the embedding

$$\Omega_v^{p_1,\tilde{r}_1} \hookrightarrow \Omega_v^{s,j}, \quad \frac{1}{s} = \frac{1}{p_1} - \frac{\tilde{r}_1 - j}{n}$$

and (42) the embedding

$$\Omega_v^{p_2,\tilde{r}_2} \hookrightarrow \Omega_v^{t,i-j}, \quad \frac{1}{t} = \frac{1}{p_2} - \frac{\tilde{r}_2 - (i-j)}{n}.$$

Here we have to solve the system

$$\frac{1}{s} + \frac{1}{t} = \frac{1}{p}, \tag{43}$$
$$r_1 \geq \tilde{r}_1 \geq j, \tag{44}$$
$$s \geq p_1, \tag{45}$$
$$\tilde{r}_1 - \frac{n}{p_1} = j - \frac{n}{s}, \tag{46}$$
$$r_2 \geq \tilde{r}_2 \geq i - j, \tag{47}$$
$$t \geq p_2, \tag{48}$$
$$\tilde{r}_2 - \frac{n}{p_2} = (i-j) - \frac{n}{t} \tag{49}$$

for $0 \leq j \leq i \leq r$, where s, t, $\tilde{r}_1$, $\tilde{r}_2$ may depend on i and j. Note that (44), (46) imply (45), while (47), (49) imply (48). (43) is equivalent to the fact that there exists $0 < \alpha < 1$ such that

$$\frac{1}{t} = \frac{\alpha}{p}, \quad \frac{1}{s} = \frac{1-\alpha}{p}, \quad \text{i.e.} \quad t = \frac{p}{\alpha}, \quad s = \frac{p}{1-\alpha}.$$

Then, (48) and (49) can be written as

$$1 - \alpha = \frac{p}{n}\Big(j - \tilde{r}_1 + \frac{n}{p_1}\Big), \quad \alpha = \frac{p}{n}\Big(i - j - \tilde{r}_2 + \frac{n}{p_2}\Big),$$

which, together with $0 < \alpha < 1$, lead to

$$\frac{n}{p} = \frac{n}{p_1} + \frac{n}{p_2} - (\tilde{r}_1 + \tilde{r}_2) + i, \quad \frac{n}{p_1} - \frac{n}{p} < \tilde{r}_1 - j < \frac{n}{p_1}, \quad \frac{n}{p_2} - \frac{n}{p} < \tilde{r}_2 - (i-j) < \frac{n}{p_2}.$$

Therefore, the system (45)–(49) can be rewritten as

$$\begin{aligned} \frac{n}{p} &= \frac{n}{p_1} + \frac{n}{p_2} - (\tilde{r}_1 + \tilde{r}_2) + i, \\ \frac{n}{p_1} - \frac{n}{p} &< \tilde{r}_1 - j < \frac{n}{p_1}, \end{aligned}$$

$$\frac{n}{p_2} - \frac{n}{p} < \tilde{r}_2 - (i-j) < \frac{n}{p_2},$$
$$r_1 \geq \tilde{r}_1 \geq j,$$
$$r_2 \geq \tilde{r}_2 \geq i-j$$

for $0 \leq j \leq i \leq r$. Equivalently, there exist nonnegative integers m_1 and m_2 such that

$$\frac{n}{p_1} - \frac{n}{p} < m_1 < \frac{n}{p_1}, \quad \frac{n}{p_2} - \frac{n}{p} < m_2 < \frac{n}{p_2}$$

and

$$\frac{n}{p} = \frac{n}{p_1} + \frac{n}{p_2} - (m_1 + m_2),$$
$$\tilde{r}_1 = m_1 + j, \quad \tilde{r}_2 = m_2 + (i-j),$$
$$r_1 \geq m_1 + r, \quad r_2 \geq m_2 + r.$$

This is just the system (38). □

Remark 3.7. 1. It is very easy to see that the system (38) has many solutions, e.g., if $n \equiv 0 \mod 4$, then $p_1 = p_2 = p = 2$, $m_1 = m_2 = \frac{n}{4}$, $r_1 = r_2 = r + \frac{n}{4}$ satisfies (38).

2. Our module structure theorem is not a real module structure theorem in the strong sense because multiplication decreases the Sobolev indices r_1, r_2 to r. In the case of a module structure in the usual strong sense, the multiplication of functions with sections would preserve the Sobolev index of the sections (for r_1, r_2, r big enough).

In Theorem 2.11 we used the weight $r_{\text{inj}}(x)^{-n}$, but in Theorem 3.6 the weight $\frac{1}{\text{vol}(B_1(x))}$. It would be desirable to establish a connection between these weights, even better, to establish an equivalence of the corresponding weighted spaces. Unfortunately, the latter seems to be impossible. We will briefly discuss these questions.

We have $\text{vol}(B_{r_{\text{inj}}(x)}(x)) \leq \text{vol}(B_1(x))$ if $r_{\text{inj}}(x) \leq 1$, i.e. for $x \in U(\varepsilon)$ if $\text{vol}(\varepsilon) < \infty$. Hence

$$\frac{1}{\text{vol}(B_1(x))} \leq \frac{1}{\text{vol}(B_{r_{\text{inj}}(x)}(x))}.$$

Moreover, for $|K_M| \leq K$ and $c = 2\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})^{-1}$,

$$c \int_0^{r_{\text{inj}}(x)} \Big(\frac{\sin t\sqrt{K}}{\sqrt{K}}\Big)^{n-1} dt \leq \text{vol}(B_{r_{\text{inj}}(x)}(x)) \leq c \int_0^{r_{\text{inj}}(x)} \Big(\frac{\sinh t\sqrt{K}}{\sqrt{K}}\Big)^{n-1} dt.$$

For given a and ε there exists $\delta = \delta(a, \varepsilon)$ such that $1 - \varepsilon \leq \left(\frac{\sin at}{at}\right)^{n-1}$ for $0 < t \leq \delta$, i.e. $(1-\varepsilon)t^{n-1} \leq \left(\frac{\sin at}{a}\right)^{n-1}$, $(1-\varepsilon)\left(\frac{\sin at}{a}\right)^{1-n} \leq \frac{1}{t^{n-1}}$, which implies finally

$$\frac{1}{\mathrm{vol}(B_1(x))} \leq \frac{1}{\mathrm{vol}(B_{r_{\mathrm{inj}}(x)}(x))} \leq \frac{d}{r_{\mathrm{inj}}(x)^n}$$

for x sufficiently near to ∞ in the case $\mathrm{vol}(\varepsilon) < \infty$. Hence, in that case with $\xi(x) = r_{\mathrm{inj}}(x)^{-n}$, we have

$$\Omega_\xi^{p_1,r_1} \times \Omega_\xi^{p_2,r_2} \hookrightarrow \Omega_v^{p_1,r_1} \times \Omega_v^{p_2,r_2} \hookrightarrow \Omega_v^{p,r},$$

but we cannot assert that the image of $\Omega_\xi^{p_1,r_1} \times \Omega_\xi^{p_2,r_2}$ under multiplication is contained in $\Omega_\xi^{p,r} \subset \Omega_v^{p,r}$. The desired inequality

$$\frac{e}{r_{\mathrm{inj}}(x)^n} \leq \frac{1}{\mathrm{vol}(B_1(x))}$$

cannot hold. Take a hyperbolic end $(]0,\infty[\times S^1, dr^2 + \mathrm{e}^{-r}dv_{S^1}^2)$. Then the injectivity radius at (v,s) is $\pi\mathrm{e}^{-r}$ and

$$\mathrm{vol}(B_1(r,s)) = 2\pi \int_{r-1}^{r+1} \mathrm{e}^{-\rho} d\rho = 2\pi\big(\mathrm{e}^{-(r-1)} - \mathrm{e}^{-(r+1)}\big).$$

For the desired inequality we would need a constant e such that $e \cdot \mathrm{vol}(B_1(r,s)) \leq r_{\mathrm{inj}}(r,s)^2$, i.e.

$$e \cdot 2\pi\big(\mathrm{e}^{-(r-1)} - \mathrm{e}^{-(r+1)}\big) \leq \pi^2 \mathrm{e}^{-2r}.$$

Such a constant does not exist.

As a conclusion, to obtain substantial results, we have to work in distinct theorems with distinct weights, in Theorem 2.11 with ξ, in Theorem 3.6 with $\frac{1}{\mathrm{vol}(B_1(x))}$. Clearly, we would have Theorem 3.6 with the weight $r_{\mathrm{inj}}(x)^{-n}$ in all cases $cr_{\mathrm{inj}}(x)^{-n} \leq \frac{1}{\mathrm{vol}(B_1(x))} \leq dr_{\mathrm{inj}}(x)^{-n}$. But there is not general rule visible where this holds.This remark is supported by our example above which is far from being “exotic”. One has to check the validity of the desired inequality in any given concrete case. The same concrete task remains in the case for an abitrary other weight, where one has to establish the equivalence with $\frac{1}{\mathrm{vol}(B_1(x))}$. A substantial general rule is for the authors not visible. We will apply our results in a forthcoming paper to uniform structures of nonlinear objects in geometry.

Acknowledgement. This paper is supported by the German-Chinese research project on PDEs 446CHV113126710-1. The second author is also supported by National Natural Science Foundation of China (Grants No. 11071100 and 11222106).

References

[1] Cheeger, J., Gromov, M. and Taylor, M., Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.* 17 (1982), 15 – 53.

[2] Eichhorn, J., The manifold structure of maps between open manifolds. *Ann. Global Anal. Geom.* 11 (1993), 253 – 300.

[3] Eichhorn, J., Spaces of Riemannian metrics on open manifolds. *Results Math.* 27 (1995), 256 – 283.

[4] Eichhorn, J., Uniform structures of metric spaces and open manifolds. *Results Math.* 40 (2001), 144 – 191.

[5] Eichhorn, J., *Global Analysis on Open Manifolds.* New York: Nova Science Publishers 2007.

[6] Eichhorn, J. and Fricke, J., The module structure theorem for Sobolev spaces on open manifolds. *Math. Nachr.* 194 (1998), 35 – 47.

[7] Hebey, E., *Sobolev Spaces on Riemannian Manifolds.* Lect. Notes Math. 1635. Berlin: Springer 1996.

[8] Hebey, E., *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities.* Courant Lect. Notes Math. 5. New York: New York Univ. 1999.

[9] Müller, W. and Salomonsen, G., Scattering theory for the Laplacian on manifolds with bounded curvature. *J. Funct. Anal.* 253 (2007), 158 – 206.

[10] Varopoulos, N., Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique. *Bull. Sci. Math.* 113 (1989), 253 – 277.

Received September 26, 2012; revised April 3, 2013