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A Smooth Solution to a Linear System of Singular ODEs

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Abstract. For a linear system of singular ordinary differential equations, necessary and sufficient conditions are established for the existence of a unique C^m-smooth solution. A reduction to cordial Volterra integral equations is used.

Keywords. Differential equations with singularities, cordial Volterra integral equations, smooth solutions

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1. Introduction, main results and discussion

We consider a linear system of singular ordinary differential equations (ODEs) of the form

$$tu'(t) = A(t)u(t) + f(t), \quad 0 < t \le T, \tag{1}$$

where $A = (a_{p,q})_{p,q=1}^n \in C^m_{n\times n}[0,T]$, $m \ge 0$, $n \in \mathbb{N}$, is a given matrix function, and $f = (f_1, \dots, f_n)^{\mathrm{T}} \in C^m_n[0,T]$ is a given vector function. We are interested in conditions guaranteeing the existence of a unique solution $u \in C^m_n[0,T]$. Also the computation of such solution will be commented.

A system of type (1) for $n = 2$ appears, e.g., when a solution $u = u(|x|)$ of the PDE $\triangle u + au = f$ with $a = a(|x|)$, $f = f(|x|)$ is determined where $x = (x_1, x_2, x_3)$ or $x = (x_1, x_2)$, see [5, 10]. Another example is connected with the regular system of ODEs $v'(x) = B(x)v(x) + g(x)$, $0 \le x < \infty$, of arbitrary dimension n assuming that finite limits $\lim_{x\to\infty} B(x)$ and $\lim_{x\to\infty} g(x)$ exist, and a solution is required to have a finite limit $\lim_{x\to\infty} u(x)$. With the change of variables $x = -\log t$, $u(t) = v(-\log t)$ the problem takes the form (1) with $T = 1$, $A(t) = -B(-\log t)$, $f(t) = -g(-\log t)$ having finite limits as $t \to 0$.

Unique solvability of system (1) in $C^m_n[0,T]$ can be described completely in terms of the spectrum $\sigma(A(0))$, i.e. the set of eigenvalues of the matrix $A(0)$.

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The main results of the paper, the following Theorems 1.1 and 1.3, provide the precise statement; see also Theorem 1.11. Notations used are fairly standard, see Section 2 if needed.

Theorem 1.1. *Assume that* $A \in C_{n\times n}[0,T]$ *and*

$$\max_{\lambda_k\in\sigma(A(0))} \mathrm{Re}\lambda_k < 0. \tag{2}$$

Then system (1) *has a unique solution* $u \in C_n[0,T]$ (*with* $u' \in C_n(0,T]$) *for any* $f \in C_n[0,T]$. *If* $A \in C_{n\times n}[0,T]$ *is such that the limit*

$$\lim_{t\to 0} \frac{A(t)-A(0)}{t^\beta} \in \mathbb{C}_{n\times n} \quad \textit{exists for a} \quad \beta > 0, \tag{3}$$

then condition (2) *is also necessary for the unique solvability of* (1) *in* $C_n[0,T]$ *for all* $f \in C_n[0,T]$.

Observe that $u = t^\mu v$ is a solution to (1) iff v is a solution to system

$$tv'(t) = (A(t)-\mu I)v(t) + g(t), \quad g(t) = t^{-\mu} f(t). \tag{4}$$

Applying to system (4), Theorem 1.1 we obtain the following

Corollary 1.2. *Assume* $A \in C_{n\times n}[0,T]$ *and*

$$\max_{\lambda_k\in\sigma(A(0))} \mathrm{Re}(\lambda_k - \mu) < 0 \quad \text{for a} \quad \mu \in \mathbb{C}. \tag{5}$$

Then system (1) *has a unique solution of the form* $u = t^\mu v$, $v \in C_n[0,T]$, *for any* $f = t^\mu g$, $g \in C_n[0,T]$; v *is a unique solution in* $C_n[0,T]$ *of system* (4).

Assuming (3), condition (5) becomes necessary and sufficient for the unique solvability of system (4) in $C_n[0,T]$ for all $g \in C_n[0,T]$.

Theorem 1.3. *Assume that* $A \in C^m_{n\times n}[0,T]$ *for an* $m \geq 1$. *Then the following assertions* (i), (ii) *and* (iii) *are equivalent:*

(i) *it holds that*

$$m > \max_{\lambda_k\in\sigma(A(0))} \mathrm{Re}\lambda_k \quad \textit{and} \quad \sigma(A(0)) \cap \mathbb{N}_0 = \varnothing; \tag{6}$$

(ii) *system* (1) *has a unique solution* u *in* $C^m_n[0,T]$ *for any* $f \in C^{m-1}_n[0,T]$ *such that* $f^- \in C^{m-1}_n[0,T]$, *where* $f^-(t) := \frac{f(t)-f(0)}{t}$, $0 < t \leq T$, $f^-(0) = f'(0)$;

(iii) *system* (1) *has a unique solution* u *in* $C^m_n[0,T]$ *for any* $f \in C^m_n[0,T]$.

Corollary 1.4. *Let* $A \in C^m_{n\times n}[0,T]$ *for an* $m \geq 1$. *Then the following assertions* (i′), (ii′) *and* (iii′) *are equivalent:*

(i′) *for a* $\mu \in \mathbb{C}$, *it holds that*

$$m + \mathrm{Re}\mu > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k \quad \textit{and} \quad \big(\sigma(A(0)) - \mu\big) \cap \mathbb{N}_0 = \varnothing; \tag{7}$$

(ii′) *for any* f *of the form* $f(t) = t^\mu g(t)$, $g \in C_n^{m-1}[0,T]$, $g^- \in C_n^{m-1}[0,T]$, *system* (1) *has a unique solution of the form* $u(t) = t^\mu v(t)$, $v \in C_n^m[0,T]$;

(iii′) *for any* f *of the form* $f(t) = t^\mu g(t)$, $g \in C_n^m[0,T]$, *system* (1) *has a unique solution of the form* $u(t) = t^\mu v(t)$, $v \in C_n^m[0,T]$.

In (ii′) *and* (iii′), v *is a unique solution of system* (4) *in* $C_n^m[0,T]$.

Remark 1.5. Let $A \in C_{n\times n}^m[0,T]$, $f(t) = t^\mu g(t)$, $\mu \in \mathbb{N}$, $g \in C_n^m[0,T]$, and let (7) be fulfilled. Then a solution of system (1) of the form $u(t) = t^\mu v(t)$ with $v \in C_n^m[0,T]$, although unique among the solutions of this form, need not be a unique in $C_n^m[0,T]$.

For a unique solution $u_\star \in C_n^m[0,T]$ of (1), existence of which is ensured by condition (6), no boundary conditions are permitted. Imposing boundary conditions may lead to a solution of lesser regularity. About the solvability of boundary value problems for linear and nonlinear singular systems of ODEs and about discretization methods for solving the boundary value problems see [1, 10, 11, 18]. In linear case, the problem setting in these works concerns (1) with $f(t) = tg(t)$, smooth g; in nonlinear case, a restriction of similar type is set.

If the problem setting requires additional linear constraints, such as for example initial value, final value, multi-point or integral constraints, a solution $u = u_\star + \sum c_k u_k$ may be constructed due to linearity of the problem by combining $u_\star$ with a suitable linear combination of solutions u_k of the associated homogeneous system $tu'(t) = A(t)u(t)$. Fortunately, under generic conditions, the computation of u_k can be reduced to the solving inhomogeneous problems like (1) with smooth free terms. Below we formulate some results in this direction. The proof of the following Theorem 1.6 will be omitted since it consists in an elementary check of the claims of the theorem.

Theorem 1.6. *Let* $d^1, d^2, \dots, d^\ell$, $1 \le \ell \le n$, *be a Jordan chain of root vectors of* $A(0)$ *to an eigenvalue* $\lambda_0 \in \sigma(A(0))$, *i.e.*

$$d^1 \neq 0, \quad (A(0) - \lambda_0 I)d^1 = 0, (A(0) - \lambda_0 I)d^k = d^{k-1}, \quad k = 2, \dots, \ell,$$

and let $v_1, \dots, v_\ell$ *satisfy the following recursive systems of* ODEs*:*

$$\begin{aligned} &tv_1'(t) = (A(t) - \lambda_0 I)v_1(t) + (A(t) - A(0))d^1, \\ &tv_k'(t) = (A(t) - \lambda_0 I)v_k(t) + (A(t) - A(0))d^k - v_{k-1}, \quad k = 2, \dots, \ell. \end{aligned} \tag{8}$$

Then the vector functions

$$
\begin{aligned}
u_1(t) &= t^{\lambda_0}\big(d^1 + v_1(t)\big),\\
u_2(t) &= t^{\lambda_0}\big[\big(d^2 + v_2(t)\big) + \log t\,\big(d^1 + v_1(t)\big)\big],\\
u_3(t) &= t^{\lambda_0}\left[\big(d^3 + v_3(t)\big) + \log t\,\big(d^2 + v_2(t)\big) + \frac{(\log t)^2}{2!}\big(d^1 + v_1(t)\big)\right],\\
&\ \ \vdots\\
u_\ell(t) &= t^{\lambda_0}\left[\big(d^\ell + v_\ell(t)\big) + \log t\,\big(d^{\ell-1} + v_{\ell-1}(t)\big) + \cdots + \frac{(\log t)^{\ell-1}}{(\ell-1)!}\big(d^1 + v_1(t)\big)\right]
\end{aligned}
$$

are solutions to the homogeneous system $tu'(t) = A(t)u(t)$.

Note that the vector functions

$$
\begin{aligned}
u_1(t) &= t^{\lambda_0} d^1,\\
u_2(t) &= t^{\lambda_0}\big[d^2 + \log t\, d^1\big],\\
&\ \ \vdots\\
u_\ell(t) &= t^{\lambda_0}\left[d^\ell + \log t\, d^{\ell-1} + \cdots + \frac{(\log t)^{\ell-1}}{(\ell-1)!} d^1\right]
\end{aligned}
$$

are solutions to the system $tu'(t) = A(0)u(t)$, cf. [4,10].

Let us comment on the solvability of systems (8). The following Theorem 1.7 is a consequence of Corollary 1.2 (case $m = 0$) and of the implication (i′)⇒(iii′) in Corollary 1.4 (case $m \ge 1$) recursively applied to systems (8); a more detailed argument is presented in Section 6.

Theorem 1.7. *Let* $A \in C^m_{n\times n}[0,T]$, $m \ge 0$. *Let* $d^1, d^2, \ldots, d^\ell$ *be a Jordan chain of the root vectors of* $A(0)$ *to an eigenvalue* $\lambda_0 \in \sigma(A(0))$.

In case $m = 0$, *assuming* (3) *and*

$$\max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}(\lambda_k - \lambda_0) < \beta \tag{9}$$

(*with* $\beta > 0$ *from* (3)), *systems* (8) *have unique solutions of the form* $v_k = t^\beta w_k \in C_n[0,T]$, $k = 1, \ldots, \ell$; w_k *are unique solutions in* $C_n[0,T]$ *of the recursive systems*

$$
\begin{aligned}
tw_1'(t) &= (A(t) - (\lambda_0 + \beta)I)w_1(t) + t^{-\beta}(A(t) - A(0))d^1,\\
tw_k'(t) &= (A(t) - (\lambda_0 + \beta)I)w_k(t) + t^{-\beta}(A(t) - A(0))d^k - w_{k-1},\\
&\qquad k = 2, \ldots, \ell.
\end{aligned} \tag{10}
$$

In case $m \ge 1$, *assuming that*

$$m > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}(\lambda_k - \lambda_0) \quad \textit{and} \quad \lambda_k - \lambda_0 \notin \mathbb{N} \quad \textit{for} \quad \forall \lambda_k \in \sigma(A(0)), \tag{11}$$

systems (8) *have unique solutions of the form* $v_k = tw_k \in C_n^m[0,T]$; $w_k \in C_n^{m-1}[0,T]$, $k = 1, \dots, \ell$, *are unique solutions in* $C_n^{m-1}[0,T]$ *of recursive systems* (10) *for* $\beta = 1$.

Remark 1.8. In case $m \geq 1$, assuming $A^- \in C_{n\times n}^m[0,T]$,

$$m > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}(\lambda_k - \lambda_0) - 1 \quad \text{and} \quad \lambda_k - \lambda_0 \notin \mathbb{N} \quad \text{for} \quad \forall \lambda_k \in \sigma(A(0)), \tag{12}$$

it holds that $w_k \in C_n^m[0,T]$, $k = 1, \dots, \ell$, for the solutions of recursive systems (10) with $\beta = 1$. Here $A^-(t) = \frac{1}{t}(A(t) - A(0))$ for $0 < t \leq T$, $A^-(0) = A'(0)$.

Remark 1.9. For $\lambda_0 \in \sigma(A(0))$ of a biggest real part compared with the real parts of other $\lambda_k \in \sigma(A(0))$, assumptions (9) and $\lambda_k - \lambda_0 \notin \mathbb{N}$ in (11), (12) are fulfilled.

Remark 1.10. Besides the solutions $v_k = t^\beta w_k$ or $v_k = tw_k$ introduced in Theorem 1.7, systems (8) have solutions $v_k = -d^k$, $k = 1, \dots, \ell$, to which there corresponds the trivial solution of the system $tu'(t) = A(t)u(t)$.

Consider the case of possibly nonempty $\sigma(A(0)) \cap \mathbb{N}_0$, cf. Theorem 1.3. If $f \in C_n^m[0,T]$, $A \in C_{n\times n}^m[0,T]$, $m \geq 1$, and if (1) still has a solution $u \in C_n^m[0,T]$ then differentiating k times the equality $tu'(t) = A(t)u(t) + f(t)$ and setting $t = 0$ we see that $u^{(k)}(0)$, $k = 0, \dots, \ell - 1$, is a solution of the recursive algebraic system

$$(kI - A(0))\, u_0^{(k)} = f^{(k)}(0) + \sum_{j=0}^{k-1} \binom{k}{j} A^{(k-j)}(0) u_0^{(j)}, \quad k = 0, \dots, \ell - 1, \tag{13}$$

where $1 \leq \ell \leq m$; for $k = 0$ (13) means that $A(0)u_0 = -f(0)$. Thus the solvability of (1) implies the consistency of (13).

Theorem 1.11. *Let* $f \in C_n^m[0,T]$, $A \in C_{n\times n}^m[0,T]$ *for an* $m \geq 1$, $m > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k$, *and* $1 \leq \ell \leq m$,

$$\sigma(A(0)) \cap \{\ell, \ell + 1, \cdots\} = \varnothing. \tag{14}$$

Then (1) *is solvable in* $C_n^m[0,T]$ *iff* (13) *is consistent. To any solution* $u_0^{(j)}$, $j = 0, \dots, \ell - 1$, *of* (13) *there corresponds a unique solution* $u \in C_n^m[0,T]$ *of the problem*

$$tu'(t) = A(t)u(t) + f(t), \quad 0 < t \leq T, \quad u^{(j)}(0) = u_0^{(j)}, \quad j = 0, \dots, \ell - 1. \tag{15}$$

Remark 1.12. If (13) is consistent for an $\ell \in \mathbb{N}$ satisfying (14) then (13) remains to be consistent for bigger ℓ. So we always can use the smallest $\ell \in \mathbb{N}$ satisfying (14). Sometimes $\ell = \min\{\ell' \in \mathbb{N} : \ell' > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k\}$ is preferable in numerics.

Remark 1.13. Theorem 1.11 is in a good accordance with the results of [2] about (not necessarily linear) Fuchsian systems of PDEs. If either $\sigma(A(0))\cap\mathbb{N}_0 = \varnothing$ or $\sigma(A(0))\cap\mathbb{N}_0 = \{0\}$, then Theorem 1.11 can be applied for $\ell = 1$, and under conditions $f \in C_n^m[0,T]$, $A \in C_{n\times n}^m[0,T]$, $m \geq 1$, $m > \max_{\lambda_k\in\sigma(A(0))} \mathrm{Re}\lambda_k$, $A(0)u_0 = -f(0)$, we obtain that the problem $tu' = Au + f$, $u(0) = u_0$ has a unique solution $u \in C_n^m[0,T]$; of course, for infinite smooth f and A also the the solution is infinite smooth. The last formulation can be derived also by interpreting the results of [2]; moreover, a further consequence of [2] is that the solution is analytic if A and f are analytic. This analyticity result can be extended to the case of a more general structure of $\sigma(A(0))\cap\mathbb{N}_0$ as in Theorem 1.11.

Example 1.14. For the scalar equation

$$tu'(t) = u(t) + f(t)$$

we have $\sigma(A(0)) = \{1\}$, (14) is fulfilled for $\ell = 2$, and the consistency conditions (13) have the form $u_0^{(0)} + f(0) = 0$, $f'(0) = 0$. By Theorem 1.11 the equation has a unique solution $u \in C^m[0,T]$, $m \geq 2$, satisfying $u(0) = u_0^{(0)}$, $u'(0) = u_0^{(1)}$ with an arbitrary $u_0^{(1)}$, provided that $f \in C^m[0,T]$ satisfies the consistency conditions; this can be easily seen also directly. On the other hand, for $f(t) = \int_0^t \frac{ds}{\log s}$, $0 \leq t \leq T$, $T < 1$, which belongs to $C^1[0,T]\backslash C^2[0,T]$ and satisfies the consistency conditions for $u_0^{(0)} = 0$, all solutions $u(t) = \int_0^t \log(-\log s)ds + ct$ of the equation live outside $C^1[0,T]$. This demonstrates that the smoothness conditions of Theorem 1.11 cannot be essentially relaxed.

The proof of Theorem 1.1 and 1.3 is based on the reduction of (1) to a system of cordial Volterra integral equations and on the extension of some results [12, 13] from scalar cordial equations to systems of such equations. Namely, rewriting (1) for an $\alpha > 0$ in the form $(D_1 + \alpha I)u = (A + \alpha I)u + f$, where $(D_1u)(t) = tu'(t)$ and I is the identity operator, we obtain that system (1) is equivalent to the system of integral equations

$$u = V_{\varphi_\alpha}(A + \alpha I)u + V_{\varphi_\alpha}f, \tag{16}$$

where $V_{\varphi_\alpha} = (D_1 + \alpha I)^{-1}$, $V_{\varphi_\alpha}f = (V_{\varphi_\alpha}f_1, \ldots, V_{\varphi_\alpha}f_n)^{\mathrm{T}}$ for $f = (f_1, \ldots, f_n)^{\mathrm{T}}$,

$$(V_{\varphi_\alpha}w)(t) = t^{-\alpha}\int_0^t s^{\alpha-1}w(s)ds = \int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-1} w(s)ds \quad \text{for} \quad w \in C[0,T].$$

The operator V_{φ_α}, $\alpha \in \mathbb{C}$, $\mathrm{Re}\alpha > 0$, often occurs in literature, see, e.g., [6–9,12], it is an example of cordial Volterra integral operators [12,13]. In Section 3 we recall and extend some results about the mapping and spectral properties of (scalar) cordial Volterra integral operators. The inversion of $D_1 + \alpha I$ between appropriate spaces and the properties of $V_{\varphi_\alpha} = (D_1 + \alpha I)^{-1} \in \mathcal{L}(C^m[0,T])$ are discussed in Section 4; for our final needs we study the inversion of $D_1 + \alpha I$ not only for $\alpha > 0$ but also for complex α, in particular, for $\mathrm{Re}\alpha < 0$ when integral $\int_0^t s^{\alpha-1} w(s) ds$ may diverge. Nevertheless, for $\mathrm{Re}\alpha > -m$, $\alpha \neq -j$, $j = 0, \ldots, m-1$, the inverse $\widetilde{V}_{\varphi_\alpha} = (D_1 + \alpha I)^{-1} \in \mathcal{L}(C_n^m[0,T])$ is still well defined and corresponds to the understanding of the divergent integral $\int_0^t s^{\alpha-1} w(s) ds$ in the sense of the Hadamard finite part; it holds $\widetilde{V}_{\varphi_\alpha} = V_{\varphi_\alpha}$ for $\mathrm{Re}\alpha > 0$. In Section 5 we prove that the spectrum of $V_{\varphi_\alpha}(A + \alpha I) \in \mathcal{L}(C_n^m[0,T])$ coincides with that for of $V_{\varphi_\alpha}(A(0) + \alpha I) \in \mathcal{L}(C_n^m[0,T])$. This is a central technical result that enables to complete the proof of Theorems 1.1 and 1.3 in Section 6.

The criteria of the existence of a unique solution $u \in C_n^m[0,T]$ to system (1) or to problem (15) formulated by Theorems 1.1, 1.3 and 1.11 are needed when polynomial collocation, spline collocation and other discretization methods are constructed and justified for system (1) either directly, or through equivalent systems of cordial Volterra integral equations, cf. [13–15] in the scalar case. These are possible topics for separate works. It is sufficient to solve (1) on a small interval $[0, T_0]$, $T_0 < T$, and continue on $[T_0, T]$ using standard methods treated, e.g., in [3].

2. Notations

We use the notations $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{C} = \mathbb{R} + \mathrm{i}\mathbb{R}$, $\lambda = \mathrm{Re}\lambda + \mathrm{i}\,\mathrm{Im}\lambda$ for $\lambda \in \mathbb{C}$. Introduce the disk

$$\mathcal{K}_b = \left\{\lambda \in \mathbb{C} : \mid \lambda - \tfrac{b}{2} \mid \leq \tfrac{b}{2}\right\}, \quad b > 0. \tag{17}$$

We denote by $\mathcal{P}_m$ the set of polynomials of degree $\leq m$. We denote by D and D_1 the differential operators $(Du)(t) = u'(t)$ and $(D_1 u)(t) = t u'(t)$, respectively. A right inverse D^{-j} of D^j, $j \in \mathbb{N}$, is given by

$$(D^{-j} u)(t) := \frac{1}{(j-1)!} \int_0^t (t-s)^{j-1} u(s) ds. \tag{18}$$

In the sequel, we use the abbreviated notation $C^m = C^m[0,T]$, $m \in \mathbb{N}_0$, for the space of m times continuously differentiable (scalar) functions on $[0,T]$;

$$\|u\|_{C^m} = \max_{0 \leq k \leq m} \max_{0 \leq t \leq T} |u^{(k)}(t)|.$$

For $m \in \mathbb{N}_0$, $r \in \mathbb{R}$, the space $C^{m,r} = C^{m,r}(0,T]$ consists of the functions $u \in C^m(0,T]$ such that finite limits $\lim_{t \to 0} t^{k-r} u^{(k)}(t)$, $k = 0, \ldots, m$, exist;

$$\|u\|_{C^{m,r}} = \max_{0 \leq k \leq m} \sup_{0 < t \leq T} t^{k-r} |u^{(k)}(t)|.$$

It holds $C = C^0 = C^{0,0}$, $C^m \subset C^{m,0}$, $C^{m',r'} \subset C^{m,r}$ for $m' \geq m \geq 0$, $r' \geq r$,

$$\begin{aligned} C^{m,m} &= \{u \in C^m : u(0) = \cdots = u^{(m-1)}(0) = 0\} \subset C^m, \\ C^m &= C^{m,m} \oplus \mathcal{P}_{m-1} = (I - \Pi_m)C^m \oplus \Pi_m C^m \quad \text{for} \quad m \geq 1, \end{aligned} \tag{19}$$

where $(\Pi_m u)(t) = \sum_{j=0}^{m-1} \frac{u^{(j)}(0)}{j!} t^j$ is the Taylor projector in C^m.

As usual, $u = (u_1, \ldots, u_n)^{\mathrm{T}} \in C_n^m$ means that $u_p \in C^m$, $p = 1, \ldots, n$, and $A = (a_{p,q})_{p,q=1}^n \in C_{n\times n}^m$ means that $a_{p,q} \in C^m$, $p, q = 1, \ldots, n$. Similar sense have the inclusions $u \in C_n^{m,r}$ and $A \in C_{n\times n}^{m,r}$. We use the norms

$$\|u\|_{C_n^m} = \max_{1\leq p\leq n} \|u_p\|_{C^m}, \quad \|u\|_{C_n^{m,\nu}} = \max_{1\leq p\leq n} \|u_p\|_{C^{m,\nu}}.$$

By $L^{1,r}(0,1)$, $r \in \mathbb{R}$, we denote the space of functions $\varphi : (0,1) \to \mathbb{C}$ such that

$$\|\varphi\|_{L^{1,r}} := \int_0^1 x^r |\varphi(x)| dx < \infty.$$

For Banach spaces X and Y, $\mathcal{L}(X,Y)$ means the space of linear bounded operators from X into Y, and $\mathcal{L}(X) = \mathcal{L}(X,X)$. By $\varrho_{\mathcal{L}(X)}(V)$ we denote the resolvent set of an operator $V \in \mathcal{L}(X)$, and by $\sigma_{\mathcal{L}(X)}(V) = \mathbb{C} \setminus \varrho_{\mathcal{L}(X)}(V)$ its spectrum. We use the abbreviated notations of the type

$$\begin{aligned} \sigma_m(V) &= \sigma_{\mathcal{L}(C_n^m)}(V), & \varrho_m(V) &= \varrho_{\mathcal{L}(C_n^m)}(V) & \text{for } V \in \mathcal{L}(C_n^m), \\ \sigma_{m,r}(V) &= \sigma_{\mathcal{L}(C_n^{m,r})}(V), & \varrho_{m,r}(V) &= \varrho_{\mathcal{L}(C_n^{m,r})}(V) & \text{for } V \in \mathcal{L}(C_n^{m,r}). \end{aligned}$$

3. Scalar cordial Volterra integral operators

Let us recall and slightly extend some results [12,13,16] concerning scalar cordial Volterra integral operators. For a "core" $\varphi \in L^1(0,1)$ and a coefficient function $a \in C$, the cordial Volterra integral operators V_φ and $V_{\varphi,a}$ are defined by

$$(V_\varphi u)(t) = \int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right) u(s)ds = \int_0^1 \varphi(x)u(tx)dx, \quad 0 \leq t \leq T, \quad u \in C, \tag{20}$$

$$(V_{\varphi,a} u)(t) = \int_0^t \frac{1}{t}\varphi\left(\frac{s}{t}\right) a(s)u(s)ds, \quad 0 \leq t \leq T, \quad u \in C.$$

Denote

$$\widehat{\varphi}(\lambda) := \int_0^1 x^\lambda \varphi(x) dx$$

for $\lambda \in \mathbb{C}$ for which the integral converges. (Function $\widehat{\varphi}$ is a shifted Mellin transform of function φ extended by the zero value from interval $(0,1)$ to $(0,\infty)$.) From (20), the second representation form, it immediately follows that

$$V_\varphi w_\lambda = \widehat{\varphi}(\lambda) w_\lambda, \quad \text{where} \quad w_\lambda(t) = t^\lambda, \quad 0 < t \leq T. \tag{21}$$

Differentiating (20), the second representation form, we observe that for $u\in C^m$, $m\geq 0$,

$$(V_\varphi u)^{(m)}(t) = \int_0^1 \varphi(x)x^m u^{(m)}(tx)dx. \tag{22}$$

From this we conclude that

$$(V_\varphi u)^{(m)}(0) = \widehat{\varphi}(m)u^{(m)}(0) \quad \text{for} \quad u \in C^m, \quad m \geq 0, \quad \varphi \in L^1(0,1). \tag{23}$$

Theorem 3.1. (See [12, 13]). *For $\varphi \in L^1(0,1)$, $a \in C^m$, $m \geq 0$, it holds that $V_\varphi, V_{\varphi,a} \in \mathcal{L}(C^m)$ and*

$$\sigma_0(V_\varphi)=\big\{\widehat{\varphi}(\lambda)\colon \mathrm{Re}\lambda\geq 0\big\}\cup\big\{0\big\}, \tag{24}$$

$$\sigma_m(V_\varphi)=\big\{\widehat{\varphi}(\lambda)\colon \mathrm{Re}\lambda\geq m\big\}\cup\big\{0\big\}\cup\big\{\widehat{\varphi}(j)\colon j=0,1,\ldots,m-1\big\} \quad \text{for } m\geq 1, \tag{25}$$

$$\sigma_m(V_{\varphi,a})= \sigma_m(V_{\varphi,a(0)})=a(0)\sigma_m(V_\varphi) \quad \text{for } m\geq 0. \tag{26}$$

If $a(0) = 0$, then $V_{\varphi,a} \in \mathcal{L}(C^m)$ is compact.

Lemma 3.2. *For $\varphi \in L^1(0,1)$, $\mu \in \mathbb{C}$, the set $(\mu I - V_\varphi)C^m \subset C^m$, $m \geq 0$, is dense in C^m iff $\mu \neq \widehat{\varphi}(j)$ for $j = 0,1,\ldots,m$.*

Proof. Assume that $\mu \neq \widehat{\varphi}(j)$, $j = 0,1,\ldots,m$. For given $v \in C^m$ and $\varepsilon > 0$, take a polynomial $v_N = \sum_{j=0}^N c_j t^j$ such that $\|v_N - v\|_{C^m} \leq \frac{\varepsilon}{2}$, and define $v_{N,\delta} = \sum_{j=0}^N c_j w_{j,\delta}$, where $w_{j,\delta}(t) = t^j$ if $\widehat{\varphi}(j) \neq \mu$, and $w_{j,\delta}(t) = t^{j+\delta}$ with a parameter $\delta > 0$ if $\widehat{\varphi}(j) = \mu$ (this may happen for $j \geq m+1$); since $\widehat{\varphi}(\lambda)$ is analytic for $\mathrm{Re}\lambda > 0$, it holds $\widehat{\varphi}(j+\delta) \neq \mu$ for sufficiently small $\delta > 0$. Further, define the function $u_{N,\delta} = \sum_{j=0}^N c_j y_{j,\delta}$, where $y_{j,\delta}(t) = \frac{t^j}{\mu-\widehat{\varphi}(j)}$ if $\widehat{\varphi}(j) \neq \mu$, and $y_{j,\delta}(t) = \frac{t^{j+\delta}}{\mu-\widehat{\varphi}(j+\delta)}$ if $\widehat{\varphi}(j) = \mu$. By (21) $(\mu I - V_\varphi)y_{j,\delta} = w_{j,\delta}$, $j = 0,\ldots,N$, hence $(\mu I - V_\varphi)u_{N,\delta} = v_{N,\delta}$, thus $v_{N,\delta} \in (\mu I - V_\varphi)C^m$. We can choose $\delta > 0$ so small that $\|v_{N,\delta} - v_N\|_{C^m} \leq \frac{\varepsilon}{2}$, then $\|v_{N,\delta} - v\|_{C^m} \leq \varepsilon$, and we conclude that $(\mu I - V_\varphi)C^m$ is dense in C^m.

Conversely, let $\mu = \widehat{\varphi}(j)$ for some $j \in \{0,\ldots,m\}$. Then due to (23)

$$(\mu u - V_\varphi u)^{(j)}(0) = \mu u^{(j)}(0) - \widehat{\varphi}(j)u^{(j)}(0) = 0, \quad \forall u \in C^m,$$

thus $(\mu I - V_\varphi)C^m \subset \{v \in C^m : v^{(j)}(0) = 0\}$, and the closed subspace in the right hand side of the inclusion is not dense in C^m. □

Theorem 3.3. (See [16]). *For $\varphi \in L^{1,r}(0,1)$, $a \in C^m$, $m \geq 0$, $r \in \mathbb{R}$, it holds*

$$V_\varphi \in \mathcal{L}(C^{m,r}), \quad \sigma_{m,r}(V_\varphi) = \big\{\widehat{\varphi}(\lambda) : \mathrm{Re}\lambda \geq r\big\} \cup \big\{0\big\}, \tag{27}$$

$$V_{\varphi,a} \in \mathcal{L}(C^{m,r}), \quad \sigma_{m,r}(V_{\varphi,a}) = \sigma_{m,r}(V_{\varphi,a(0)}) = a(0)\sigma_{m,r}(V_\varphi). \tag{28}$$

If $a(0) = 0$, then $V_{\varphi,a} \in \mathcal{L}(C^{m,r})$ is compact.

4. The resolvent of D_1 as an integral operator

Clearly, $D_1 + \alpha I \in \mathcal{L}(C^{m+1,r}, C^{m,r})$. Its inversion is described in Lemma 4.1; in Lemma 4.2 the inversion in the spaces of type C^m will be treated. The operator $(D_1 + \alpha I)^{-1}$ occurs to be cordial and formulae (24)–(28) can be applied for it.

Lemma 4.1. *For $m \geq 0$, $r \in \mathbb{R}$, $\alpha \in \mathbb{C}$, $r + \mathrm{Re}\alpha > 0$, the operator $D_1 + \alpha I \in \mathcal{L}(C^{m+1,r}, C^{m,r})$ is invertible, $(D_1 + \alpha I)^{-1} = V_{\varphi_\alpha} \in \mathcal{L}(C^{m,r}, C^{m+1,r}) \subset \mathcal{L}(C^{m,r})$, where $\varphi_\alpha \in L^{1,r}(0,1)$ is defined by $\varphi_\alpha(x) = x^{\alpha-1}$, $0 < x < 1$, and V_{φ_α} is the cordial operator*

$$(V_{\varphi_\alpha} f)(t) = \int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-1} f(s)ds = t^{-\alpha}\int_0^t s^{\alpha-1} f(s)ds, \quad 0 < t \leq T. \tag{29}$$

It holds (see notation (17)*)*

$$\sigma_{m,r}(V_{\varphi_\alpha}) = \mathcal{K}_{\frac{1}{r+\mathrm{Re}\,\alpha}}. \tag{30}$$

Proof. First, for $f \in C^{m,r}$ it is easy to check that $V_{\varphi_\alpha} f \in C^{m+1,r}$ and that $u = V_{\varphi_\alpha} f$ satisfies $(D_1 + \alpha I)u = f$. Second, the nontrivial solutions of the first order linear homogeneous ODE $(D_1 + \alpha I)u = 0$ are given by $u(t) = ct^{-\alpha}$, $c = \mathrm{const} \neq 0$, and they live outside $C^{m,r}$ due to the condition $r + \mathrm{Re}\alpha > 0$. This proves the first claim of the Lemma. For $\varphi_\alpha(x) = x^{\alpha-1}$, formula (30) is a consequence of (27), see [12]. $\square$

For $m \geq 1$, $u \in C^m = C^{m,m} \oplus \mathcal{P}_{m-1}$, $\Pi_m u = \sum_{j=0}^{m-1} \frac{u^{(j)}(0)}{j!} t^j \in \mathcal{P}_{m-1}$, it holds $u - \Pi_m u \in C^{m,m}$; by Lemma 4.1 $V_{\varphi_\alpha}(u - \Pi_m u) \in C^{m+1,m} \subset C^m$ provided that $m + \mathrm{Re}\alpha > 0$. Setting onto $\alpha \in \mathbb{C}$ the conditions

$$\mathrm{Re}\alpha > -m, \quad \alpha \neq -j, \quad j = 0, \ldots, m-1, \tag{31}$$

we can define the Hadamard finite part integral operator $\widetilde{V}_{\varphi_\alpha} = \mathrm{f.p.} V_{\varphi_\alpha} \in \mathcal{L}(C^m)$ by (cf. [5,17])

$$\widetilde{V}_{\varphi_\alpha} u = V_{\varphi_\alpha}(u - \Pi_m u) + \widetilde{V}_{\varphi_\alpha}\Pi_m u = V_{\varphi_\alpha}(u - \Pi_m u) + \sum_{j=0}^{m-1} \frac{u^{(j)}(0)}{j!(j+\alpha)} t^j, \quad u \in C^m.$$

Formally $\widetilde{V}_{\varphi_\alpha}\Pi_m u$ differs from

$$(V_{\varphi_\alpha}\Pi_m u)(t) = t^{-\alpha}\int_0^t s^{\alpha-1}\sum_{j=0}^{m-1}\frac{u^{(j)}(0)}{j!} s^j ds = t^{-\alpha}\sum_{j=0}^{m-1}\frac{u^{(j)}(0)}{j!(j+\alpha)}\lim_{\varepsilon\to 0} s^{j+\alpha}\Big|_{s=\varepsilon}^{t}$$

by omitting the divergent terms that correspond to negative $j + \mathrm{Re}\alpha$. Note that $\widetilde{V}_{\varphi_\alpha} = V_{\varphi_\alpha}$ for $\mathrm{Re}\alpha > 0$. For $0 \leq m \leq m'$, operator $\widetilde{V}_{\varphi_\alpha} \in \mathcal{L}(C^{m'})$ is a restriction of $\widetilde{V}_{\varphi_\alpha} \in \mathcal{L}(C^m)$ from C^m to $C^{m'}$. Note also that $(\widetilde{V}_{\varphi_\alpha})t^j = \frac{t^j}{j+\alpha} = (D_1+\alpha I)^{-1}t^j$, $j \in \mathbb{N}_0$, independently of $m > -\mathrm{Re}\alpha$ used in the construction of $\widetilde{V}_{\varphi_\alpha}$.

As easily seen, $C^{m+1,m} \cap \mathcal{P}_{m-1} = \{0\}$, thus we can generate the direct sum $C^{m+1,m} \oplus \mathcal{P}_{m-1}$; we equip it with a norm

$$\|u + v_m\|_{C^{m+1,m}\oplus\mathcal{P}_{m-1}} = \|u\|_{C^{m+1,m}} + \|v_m\|_{\mathcal{P}_{m-1}} \quad \text{for} \quad u \in C^{m+1,m}, \quad v_m \in \mathcal{P}_{m-1}$$

(in our considerations, it is not essential which norm is used in $\mathcal{P}_{m-1}$). The operator $D_1+\alpha I \in \mathcal{L}(C^{m+1,m}, C^{m,m})$ is invertible by Lemma 3.2, and $D_1+\alpha I \in \mathcal{L}(\mathcal{P}_{m-1}, \mathcal{P}_{m-1})$ is invertible due to (31). We obtain that the inverse to

$$D_1 + \alpha I \in \mathcal{L}(C^{m+1,m} \oplus \mathcal{P}_{m-1}, C^m) = \mathcal{L}(C^{m+1,m} \oplus \mathcal{P}_{m-1}, C^{m,m} \oplus \mathcal{P}_{m-1})$$

exists and is given by $\widetilde{V}_{\varphi_\alpha} = (D_1 + \alpha I)^{-1} \in \mathcal{L}(C^m, C^{m+1,m} \oplus \mathcal{P}_{m-1}) \subset \mathcal{L}(C^m)$. Let us summarize.

Lemma 4.2. *For* $\mathrm{Re}\alpha > 0$, *the operator* $D_1 + \alpha I \in \mathcal{L}(C^{1,0}, C)$ *is invertible,* $(D_1 + \alpha I)^{-1} = V_{\varphi_\alpha} \in \mathcal{L}(C, C^{1,0}) \subset \mathcal{L}(C)$ *(see* (29)*), and*

$$\sigma_0(V_{\varphi_\alpha}) = \mathcal{K}_{\frac{1}{\mathrm{Re}\alpha}}. \tag{32}$$

For $m \geq 1$, $m+\mathrm{Re}\alpha > 0$, $\alpha \neq -j$, $j = 0, \ldots, m-1$, *the operator* $D_1+\alpha I \in \mathcal{L}(C^{m+1,m} \oplus \mathcal{P}_{m-1}, C^m)$ *is invertible, and*

$$\begin{gathered}(D_1 + \alpha I)^{-1} = \widetilde{V}_{\varphi_\alpha} \in \mathcal{L}(C^m, C^{m+1,m} \oplus \mathcal{P}_{m-1}) \subset \mathcal{L}(C^m),\\ \sigma_m(\widetilde{V}_{\varphi_\alpha}) = \mathcal{K}_{\frac{1}{m+\mathrm{Re}\alpha}} \cup \big\{\tfrac{1}{\alpha+j} : j = 0, \ldots, m-1\big\},\end{gathered} \tag{33}$$

$$\mathcal{K}_{\frac{1}{m+\mathrm{Re}\alpha}} \cap \big\{\tfrac{1}{\alpha+j} : j = 0, \ldots, m-1\big\} = \varnothing.$$

Proof. The first claim is a reformulation of Lemma 4.1 for $m = 0$, $r = 0$. The second claim follows from Lemma 4.1 and the considerations after it. □

Lemma 4.3. *For* $m \geq 1$, $\alpha > 0$, *it holds that*

$$(D_1 + \alpha I)C^m = tC^{m-1} \oplus \mathbb{C}, \quad V_{\varphi_\alpha}(tC^{m-1} \oplus \mathbb{C}) = C^m, \tag{34}$$

where $tC^{m-1} = \{f : f(t) = tg(t),\ g \in C^{m-1}\}$.

Proof. Since $(D_1 + \alpha I)^{-1} = V_{\varphi_\alpha}$ for $\alpha > 0$, it is sufficient to prove the first one of formulae (34). For $g \in C^m$, it holds $t^{-1}\big(g(t) - g(0)\big) \in C^{m-1}$ and

$$(D_1 + \alpha I)g = t\big(g' + \alpha t^{-1}(g(t) - g(0))\big) + \alpha g(0) \in tC^{m-1} \oplus \mathbb{C},$$

thus $(D_1 + \alpha I)C^m \subset tC^{m-1} \oplus \mathbb{C}$. To prove the inverse inclusion, we take an $f = tg + f(0) \in tC^{m-1} \oplus \mathbb{C}$ with $g \in C^{m-1}$, and we show that that there is a $u \in C^m$ such that $(D_1 + \alpha I)u = tg + f(0)$. Clearly $u = V_{\varphi_\alpha}(tg + f(0))$, so we have to check that $u \in C^m$. Since (see (20))

$$V_{\varphi_\alpha}(f(0)) = f(0) \int_0^1 x^{\alpha-1} dx = \frac{f(0)}{\alpha} \in \mathbb{C},$$

it remains to prove that $V_{\varphi_\alpha}(tg) \in C^m$ for $g \in C^{m-1}$. Consider the case $m = 1$. To check that $V_{\varphi_\alpha}(tg) \in C^1$ for $g \in C$, observe that, of course, $V_{\varphi_\alpha}(tg) \in C$, and also $DV_{\varphi_\alpha}(tg) \in C$. Indeed,

$$\begin{aligned} DV_{\varphi_\alpha}(tg) &= \frac{d}{dt}\left(\int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-1} sg(s)ds\right) \\ &= g(t) - \int_0^t \frac{1}{t^2}\left(\frac{s}{t}\right)^{\alpha-1} sg(s)ds - (\alpha-1)\int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-2} \frac{s}{t} g(s)ds \\ &= g(t) - \int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha} g(s)ds - (\alpha-1)\int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-1} g(s)ds \\ &= g - V_{\varphi_{\alpha+1}} g - (\alpha-1)V_{\varphi_\alpha} g \in C. \end{aligned}$$

For $m \geq 2$, we have (see (22))

$$V_{\varphi_\alpha}(tg) = \int_0^1 x^{\alpha-1} xtg(xt)dx = \int_0^1 x^\alpha tg(xt)dx \in C,$$

$$\begin{aligned} D^k V_{\varphi_\alpha}(tg) &= \int_0^1 x^\alpha \left(\frac{d}{dt}\right)^k \big(tg(xt)\big)dx \\ &= \int_0^1 x^\alpha \big(tx^k g^{(k)}(xt) + kx^{k-1} g^{(k-1)}(xt)\big)dx \\ &= V_{\varphi_{\alpha+k+1}}(tg^{(k)}) + kV_{\varphi_{\alpha+k}} g^{(k-1)} \in C, \quad k = 1, \dots, m-1, \end{aligned}$$

$$D^m V_{\varphi_\alpha}(tg) = DV_{\varphi_{\alpha+m+1}}(tg^{(m-1)}) + (m-1)V_{\varphi_{\alpha+m}} g^{(m-1)};$$

treating the term $DV_{\varphi_{\alpha+m+1}}(tg^{(m-1)})$ with $g^{(m-1)} \in C$ similarly as above the term $DV_{\varphi_\alpha}(tg)$ for $g \in C$ we obtain that also $D^m V_{\varphi_\alpha}(tg) \in C$, hence $V_{\varphi_\alpha}(tg) \in C^m$. □

5. Operator $V_{\varphi_\alpha,A} = V_{\varphi_\alpha}A$, $\alpha > 0$, $n \in \mathbb{N}$

For a given matrix function $A \in C_{n\times n}$ and a parameter value $\alpha > 0$, consider the operator $V_{\varphi_\alpha,A} = V_{\varphi_\alpha}A$ defined by

$$(V_{\varphi_\alpha,A}u)(t) = \int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-1} A(s)u(s)ds, \quad 0 \le t \le T; \quad u \in C_n.$$

Lemma 5.1. *For $\alpha > 0$, $A \in C^m_{n\times n}$, $m \ge 0$, it holds*

$$\sigma_m(V_{\varphi_\alpha,A(0)}) = \bigcup_{\lambda_k\in\sigma(A(0))} \lambda_k \sigma_m(V_{\varphi_\alpha}), \tag{35}$$

or according to (32), (33)

$$\sigma_0(V_{\varphi_\alpha,A(0)}) = \bigcup_{\lambda_k\in\sigma(A(0))} \lambda_k \mathcal{K}_{\frac{1}{\alpha}}, \tag{36}$$

$$\sigma_m(V_{\varphi_\alpha,A(0)}) = \bigcup_{\lambda_k\in\sigma(A(0))} \lambda_k \Big(\mathcal{K}_{\frac{1}{m+\alpha}} \cup \big\{\tfrac{1}{\alpha+j} : j = 0,\dots,m-1\big\}\Big), \quad m \ge 1. \tag{37}$$

Proof. We have to prove only (35) which is equivalent to

$$\varrho_m(V_{\varphi_\alpha,A(0)}) = \bigcap_{\lambda_k\in\sigma(A(0))} \lambda_k \varrho_m(V_{\varphi_\alpha}). \tag{38}$$

It is $\mu \in \varrho_m(V_{\varphi,A(0)})$ iff the system

$$\mu u = V_\varphi A(0)u + f \tag{39}$$

has a unique solution $u \in C^m_n$ for every $f \in C^m_n$. Represent $A(0)$ in the Jordan form $A(0) = E\Lambda E^{-1}$ where $E \in \mathbb{C}_{n\times n}$ is a (constant) invertible matrix and Λ is a block diagonal matrix with the Jordan blocks

$$\Lambda_k = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ 0 & \lambda_k & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix} \in \mathbb{C}_{n_k\times n_k}, \quad \lambda_k \in \sigma(A(0)), \quad \sum_k n_k = n.$$

It holds $\varrho_m(V_{\varphi_\alpha,A(0)}) = \varrho_m(V_{\varphi_\alpha,\Lambda})$. With respect to $v = Eu$ system (39) has the form

$$\mu v = V_{\varphi_\alpha,\Lambda}v + g, \quad g = Ef, \tag{40}$$

which splits into the subsystems of the type

$$\begin{aligned} \mu v_{n_k} &= \lambda_k V_{\varphi_\alpha} v_{n,k} + g_{n_k}, \\ \mu v_j &= \lambda_k V_{\varphi_\alpha} v_j + V_{\varphi_\alpha} v_{j+1} + g_j, \end{aligned} \quad j = n_k - 1, \quad n_k - 2, \ldots, 1 \tag{41}$$

(we solve the equations recursively in the presented inverse order; actually the bounds for j depend on the position of Λ_k in Λ; for simplicity we wrote (41) as if Λ_k were on the first position). Solving the scalar equations (41) recursively we conclude with the help of Theorem 3.1 that such a subsystem has for any $g \in C^m_{n_k}$ a unique solution $v \in C^m_{n_k}$ iff $\mu \in \varrho_m(\lambda_k V_{\varphi_\alpha}) = \lambda_k \varrho_m(V_{\varphi_\alpha})$. Thus the full system (40) is uniquely solvable in C^m_n for any $g \in C^m_n$, and system (39) is uniquely solvable in C^m_n for any $f \in C^m_n$ iff $\mu \in \cap_{\lambda_k \in \sigma(A(0))} \lambda_k \varrho_m(V_{\varphi_\alpha})$. This proves (38). □

Lemma 5.2. *For $\alpha > 0$, $A \in C^m_{n\times n}$, $m \geq 0$, the range set $(\mu I - V_{\varphi_\alpha, A(0)}) C^m_n$ is dense in C^m_n iff $\mu \neq \frac{\lambda_k}{i+\alpha}$ for $i = 0, 1, \ldots, m$ and all eigenvalues $\lambda_k \in \sigma(A(0))$.*

Proof. Since $\mu I - V_{\varphi_\alpha, A(0)} = E(\mu I - V_{\varphi_\alpha, \Lambda}) E^{-1}$, the claim of the Lemma is equivalent to the following one: for any $\lambda_k \in \sigma(A(0))$, $g \in C^m_{n_k}$ and $\varepsilon > 0$, there is a $v \in C^m_{n_k}$ such that $\|(\mu I - V_{\varphi_\alpha, \Lambda_k}) v - g\|_{C^m_{n_k}} \leq \varepsilon$ if $\mu \neq \frac{\lambda_k}{i+\alpha}$, $i = 0, 1, \ldots, m$. In this form the claim is a consequence of Lemma 3.2 which we apply recursively to equations (41). □

Theorem 5.3. *For $\alpha > 0$, $A \in C^m_{n\times n}$, $m \geq 0$, it holds $V_{\varphi_\alpha, A} \in \mathcal{L}(C^m_n)$ and*

$$\sigma_m(V_{\varphi_\alpha, A}) = \sigma_m(V_{\varphi_\alpha, A(0)}). \tag{42}$$

If $A(0) = 0$, then $V_{\varphi_\alpha, A} \in \mathcal{L}(C^m_n)$ is compact.

Proof. The claim that $V_{\varphi, A} \in \mathcal{L}(C^m_n)$ and the claim about the compactness of operator $V_{\varphi, A} \in \mathcal{L}(C^m_n)$ in case $A(0) = 0$ follow from corresponding claims of Theorem 3.1, so we have to prove only (42).

We shall prove the inclusion $\sigma_m(V_{\varphi_\alpha, A}) \subset \sigma_m(V_{\varphi_\alpha, A(0)})$ by a contradiction argument. So suppose that for some $\mu_0 \in \sigma_m(V_{\varphi_\alpha, A})$ it holds $\mu_0 \in \varrho_m(V_{\varphi_\alpha, A(0)})$. Since $0 \in \mathcal{K}_{\frac{1}{m+\alpha}} \subset \sigma_m(V_{\varphi_\alpha, A(0)})$ (see (37)), it holds $\mu_0 \neq 0$. The operator $\mu_0 I - V_{\varphi_\alpha, A} \in \mathcal{L}(C^m_n)$ is Fredholm of index 0, since $\mu_0 I - V_{\varphi_\alpha, A(0)} \in \mathcal{L}(C^m_n)$ is invertible and $V_{\varphi_\alpha, A} - V_{\varphi_\alpha, A(0)} = V_{\varphi_\alpha, A - A(0)} \in \mathcal{L}(C^m_n)$ is compact. Hence μ_0 is an eigenvalue of $V_{\varphi_\alpha, A}$; let u_0 be an eigenfunction:

$$\mu_0 u_0 = V_{\varphi_\alpha} A u_0, \quad 0 \neq u_0 \in C^m_n. \tag{43}$$

Consider first the case $m = 0$. Rewrite (43) in the form

$$\mu_0 u_0 - V_{\varphi_\alpha} A(0) u_0 = V_{\varphi_\alpha}(A u_0 - A(0)) u_0, \quad u_0 = (\mu_0 I - V_{\varphi_\alpha} A(0))^{-1} V_{\varphi_\alpha}(A u_0 - A(0)) u_0.$$

Clearly $|(V_{\varphi_\alpha} f)(t)| \le \frac{1}{\alpha} \max_{0\le s\le t} |f(s)|$, $0 \le t \le T$. With the help of the Jordan representation $A(0) = E\Lambda E^{-1}$, reducing system (39) to the form (40), it is easy to see that

$$|((\mu_0 I - V_{\varphi_\alpha} A(0))^{-1} f)(t)| \le c \max_{0\le s\le t} |f(s)|, \quad 0 \le t \le T,$$

where the constant c is independent of t and $f \in C_n$. Thus

$$|u_0(t)| \le \frac{c}{\alpha} \max_{0\le s\le t} |A(s) - A(0)| \max_{0\le s\le t} |u_0(s)|, \quad 0 \le t \le T.$$

Since $|A(s) - A(0)| \to 0$ as $s \to 0$, a consequence is that $u_0(t) = 0$ on some interval $0 \le t \le t_0$, $t_0 \in (0, T]$. For $t_0 \le t \le T$ (43) is a regular Volterra system, and we obtain that $u_0(t) = 0$ on the whole interval $0 \le t \le T$. This contradicts the choice of u_0 and proves the inclusion $\sigma_m(V_{\varphi_\alpha,A}) \subset \sigma_m(V_{\varphi_\alpha,A(0)})$ for $m = 0$.

Let now $m \ge 1$. Differentiating equality (43) we show below that $u_0^{(i)}(0) = 0$, $i = 0, \ldots, m-1$, and $u_0^{(m)} = 0$, implying $u_0 = 0$ and obtaining so again a desired contradiction. For $u \in C_n^m$ we have

$$(V_{\varphi_\alpha,A} u)(t) = \int_0^t \frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-1} A(s)u(s)ds = \int_0^1 x^{\alpha-1} A(tx)u(tx)dx,$$

$$(V_{\varphi_\alpha,A} u)^{(i)}(t) = \sum_{j=0}^{i} \binom{i}{j} \int_0^1 x^{i+\alpha-1} A^{(j)}(tx)u^{(i-j)}(tx)dx, \quad i = 0, 1, \ldots, m,$$

$$(V_{\varphi,A} u)^{(i)}(0) = \frac{1}{i+\alpha} \sum_{j=0}^{i} \binom{i}{j} A^{(j)}(0)u^{(i-j)}(0), \qquad i = 0, 1, \ldots, m.$$

According to (36), (37), $\mu_0 \in \varrho_m(V_{\varphi,A(0)})$ implies that $\mu_0 \ne \frac{\lambda_k}{i+\alpha}$ for $i = 0, 1, \ldots, m–1$ and any $\lambda_k \in \sigma(A(0))$. Therefore we obtain recursively that

$$\begin{array}{lll}
\mu_0 u_0(0) = \frac{1}{\alpha} A(0)u_0(0), & \text{hence} & u_0(0) = 0,\\
\mu_0 u_0'(0) = \frac{1}{1+\alpha} A(0)u_0'(0), & \text{hence} & u_0'(0) = 0,\\
\vdots & \vdots & \vdots\\
\mu u_0^{(m-1)}(0) = \frac{1}{m-1+\alpha} A(0)u_0^{(m-1)}(0), & \text{hence} & u_0^{(m-1)}(0) = 0.
\end{array}$$

For $v_0 = u_0^{(m)}$ we obtain the equality

$$\mu_0 v_0 = V_{\varphi_{m+\alpha}} A(0)v_0 + V_{\varphi_{m+\alpha}} (A - A(0))v_0 + \sum_{j=1}^{m} \binom{m}{j} V_{\varphi_{m+\alpha}} A^{(j)} D^{-j} v_0;$$

see (18) for the definition of D^{-j}. From (36), (37) we observe that $\varrho_m(V_{\varphi_\alpha,A(0)}) \subset \varrho_0(V_{\varphi_{m+\alpha},A(0)})$, thus $\mu_0 \in \varrho_0(V_{\varphi_{m+\alpha},A(0)})$, and

$$v_0 = (\mu_0 I - V_{\varphi_{m+\alpha}} A(0))^{-1} V_{\varphi_{m+\alpha}} \Big[(A - A(0))v_0 + \sum_{j=1}^{m} \binom{m}{j} A^{(j)} D^{-j} v_0\Big]. \quad (44)$$

Similarly as above we obtain from this that $v_0(t) \equiv 0$ as claimed, and the proof of the inclusion $\sigma_m(V_{\varphi_\alpha,A}) \subset \sigma_m(V_{\varphi_\alpha,A(0)})$ is completed.

According (36) and (37), to establish the inverse inclusion $\sigma_m(V_{\varphi_\alpha,A(0)}) \subset \sigma_m(V_{\varphi_\alpha,A})$, we have to prove that

$$\lambda_k \mathcal{K}_{\frac{1}{m+\alpha}} \subset \sigma_m(V_{\varphi_\alpha,A}) \quad \text{for any} \quad \lambda_k \in \sigma(A(0)), \tag{45}$$

in case $m \geq 1$ also

$$\left\{\tfrac{\lambda_k}{\alpha+j} : j = 0, 1, \dots, m-1\right\} \subset \sigma_m(V_{\varphi_\alpha,A}) \quad \text{for any} \quad \lambda_k \in \sigma(A(0)). \tag{46}$$

Note that $\frac{1}{m+\alpha} \in \mathcal{K}_{\frac{1}{m+\alpha}}$. Since the spectrum is closed, to prove (45), it suffices to establish that

$$\lambda_k \mathcal{K}_{\frac{1}{m+\alpha}} \setminus \{\tfrac{\lambda_k}{m+\alpha}\} \subset \sigma_m(V_{\varphi_\alpha,A}) \quad \text{for any} \quad \lambda_k \in \sigma(A(0)).$$

We shall prove this inclusion by a contradiction argument. So, suppose that for some $\mu_0 \in \lambda_k \mathcal{K}_{\frac{1}{m+\alpha}} \subset \sigma_m(V_{\varphi_\alpha,A(0)})$, $\lambda_k \in \sigma(A(0))$, $\mu_0 \neq \frac{\lambda_k}{m+\alpha}$, it holds that $\mu_0 \in \varrho_m(V_{\varphi_\alpha,A})$, i.e. $\mu_0 I - V_{\varphi_\alpha,A}$ has a bounded inverse in $\mathcal{L}(C_n^m)$. Due to the compactness of $V_{\varphi_\alpha,A} - V_{\varphi_\alpha,A(0)} \in \mathcal{L}(C_n^m)$, operator $\mu_0 I - V_{\varphi_\alpha,A(0)} \in \mathcal{L}(C_n^m)$ is Fredholm of index 0, hence $(\mu_0 I - V_{\varphi_\alpha,A(0)})C_n^m$ is a closed subspace of C_n^m. Further, $\frac{1}{i+\alpha} \notin \mathcal{K}_{\frac{1}{m+\alpha}}$, thus $\mu_0 \neq \frac{\lambda_k}{i+\alpha}$ for $i = 0, 1, \dots, m$, and by Lemma 5.2 the range $(\mu_0 I - V_{\varphi_\alpha,A(0)})C_n^m$ is dense in C_n^m, hence $(\mu_0 I - V_{\varphi_\alpha,A(0)})C_n^m = C_n^m$, and $\mu_0 I - V_{\varphi_\alpha,A(0)} \in \mathcal{L}(C_n^m)$ as a Fredholm operator of index 0 has the inverse $(\mu_0 I - V_{\varphi_\alpha,A(0)})^{-1} \in \mathcal{L}(C_n^m)$, i.e. $\mu_0 \in \varrho_m(V_{\varphi_\alpha,A(0)})$. This contradicts the choice of μ_0 and proves (45).

The inclusion (46) can be proved examining spectral projectors of $V_{\varphi_\alpha,A(0)}$ and $V_{\varphi_\alpha,A}$ corresponding to the isolated point $\frac{\lambda_k}{\alpha+j}$, $0 \leq j \leq m-1$, of $\sigma_m(V_{\varphi_\alpha,A(0)})$. Technically the argument is same as in case $n = 1$, see [13], therefore we omit the details. The proof of Theorem 5.3 is finished. □

6. Proof of Theorems 1.1, 1.3, 1.7 and 1.11

Lemma 6.1. *For $A \in C_{n\times n}^m$, $m \geq 0$, $\alpha > 0$, condition* (6) *is equivalent to the condition $1 \in \rho_m(V_{\varphi_\alpha}(A(0) + \alpha I))$.*

Proof. Formulae (36), (37) for $A(0) + \alpha I$ (in the role of $A(0)$) yield

$$\sigma_0(V_{\varphi_\alpha}(A(0)+\alpha I)) = \bigcup_{\lambda_k \in \sigma(A(0))} (\lambda_k+\alpha)\mathcal{K}_{\frac{1}{\alpha}}$$

$$\sigma_m(V_{\varphi_\alpha}(A(0)+\alpha I)) = \bigcup_{\lambda_k \in \sigma(A(0))} \Big((\lambda_k+\alpha)\mathcal{K}_{\frac{1}{m+\alpha}} \cup \left\{\tfrac{\lambda_k+\alpha}{\alpha+j} : j=0,\dots,m-1\right\}\Big), \ m \geq 1.$$

The inclusion $1 \in (\lambda_k + \alpha)\mathcal{K}_{\frac{1}{m+\alpha}}$ means that

$$\left|1 - \frac{\lambda_k + \alpha}{2(m+\alpha)}\right| \leq \left|\frac{\lambda_k + \alpha}{2(m+\alpha)}\right|, \quad \text{or} \quad |2m + \alpha - \lambda_k| \leq |\lambda_k + \alpha|,$$

that after elementary simplifications takes the form $m \leq \mathrm{Re}\lambda_k$. The equality $1 = \frac{\lambda_k + \alpha)}{\alpha + j}$ takes place iff $\lambda_k = j$. The claim of the Lemma follows from these observations. □

Proof of Theorem 1.1 (*sufficiency part*). Assume that $A \in C_{n\times n}[0,T]$ satisfies (2). Observe that condition (2) is equivalent to (6) for $m = 0$ and implies by Lemma 6.1 and Theorem 5.3 that $1 \in \rho_0(V_{\varphi_\alpha}(A(0) + \alpha I)) = \rho_0(V_{\varphi_\alpha}(A + \alpha I))$. Hence equation $u = V_{\varphi_\alpha}(A+\alpha I)u + f$ is uniquely solvable in C_n for any $f \in C_n$. In particular, equation (16) is uniquely solvable in C_n since V_{φ_α} maps C_n into C_n. Equations (1) and (16) are equivalent, so (1) has for any $f \in C_n$ a unique solution $u \in C$. The sufficiency part of Theorem 1.1 and together with it also Corollary 1.2 are proved. □

Proof of Theorem 1.7 (*case* $m = 0$). Assume that $A \in C_{n\times n}$ satisfies (3) and (9). Due to (3),

$$(A(t) - A(0))d^k = t^\beta g_k(t), \quad g_k(t) := \frac{A(t) - A(0)}{t^\beta} d^k \in C_n, \quad k = 1, \ldots, \ell. \tag{47}$$

Applying Corollary 1.2 with $\mu = \beta$ to the first system in (8), condition (5) reads as $\max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}(\lambda_k - \lambda_0 - \beta) < 0$ that is fulfilled due to assumption (9), and we obtain that the system has a unique solution of the form $v_1 = t^\beta w_1$ with w_1 determined as the unique solution in C_n of the first system in (10). Now we see in the same way that the second one of systems (8) has a unique solution of the form $v_2 = t^\beta w_2$ with $w_2 \in C_n[0,T]$ determined as the unique solution of the second system in (10). Continuing in this way we obtain that all systems (8) have unique solutions of the form $v_1 = t^\beta w_1, \ldots, v_\ell = t^\beta w_\ell$ where $w_1, \ldots, w_\ell$ are unique solutions in $C_n[0,T]$ of the recursive systems (10). This proves Theorem 1.7 for $m = 0$. □

Proof of Theorem 1.1 (*necessity part*). Assume (3) and that system (1) has for any $f \in C_n$ a unique solution $u \in C_n$. Then the inverse $(D_1 - A(t))^{-1} \in \mathcal{L}(C_n)$ exists; the boundedness is a consequence of the closedness, the closedness is a consequence of the closedness of $D_1 - A(t)$ in C_n, the closedness of $D_1 - A(t)$ in C_n follows from the closedness of $D_1 - \alpha I$ in C_n, and the closedness of the latter operator is a consequence of the boundedness of $(D_1 - \alpha I)^{-1} \in \mathcal{L}(C_n)$ for $\alpha > 0$, see Lemma 4.2.

We shall prove (2) by a contradiction argument. So, suppose that $\gamma := \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k \geq 0$ and choose a $\lambda_0 \in \sigma(A(0))$ with $\mathrm{Re}\lambda_0 = \gamma$. With this choice condition (9) is satisfied. By Theorems 1.6 and 1.7 (just shown for $m = 0$), homogeneous system $tu'(t) = A(t)u(t)$ has a solution

$$u_0(t) = t^{\lambda_0}(d^0 + tw(t)), \tag{48}$$

where $w \in C_n$, $A(0)d^0 = \lambda_0 d^0$, $0 \neq d^0 \in \mathbb{C}_n$. Note that $u_0 \in C_n$ if $\gamma = \mathrm{Re}\lambda_0 > 0$ or if $\lambda_0 = 0$. In these cases (1) is solvable in C_n non-uniquely (if solvable) contradicting our assumption about unique solvability. In the case $\lambda_0 = \beta\mathrm{i}$, $0 \neq \beta \in \mathbb{R}$,

$$u_0(t) = t^{\beta\mathrm{i}}(d^0 + tw(t)) = [\cos(\log(\beta t)) + \mathrm{i}\sin(\log(\beta t))](d^0 + tw(t)).$$

Observe that $u_0 \notin C_n$ but $u_\varepsilon := t^\varepsilon u_0 \in C_n$ for any $\varepsilon > 0$, and that $(D_1 - A(t))u_\varepsilon = \varepsilon u_\varepsilon$, hence $(D_1 - A(t))^{-1}u_\varepsilon = \varepsilon^{-1}u_\varepsilon$, implying that the operator $(D_1 - A(t))^{-1}$ is unbounded in C_n. But we saw that actually $(D_1 - A(t))^{-1} \in \mathcal{L}(C_n)$. Thus the hypothesis that $\gamma := \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k \geq 0$ contradicts the unique solvability in C_n of system (1) for all $f \in C_n$. The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.3. *Implications* (i)⇒(ii)⇒(iii). Let $A \in C^m_{n\times n}$, $m \geq 1$.

(i)⇒(ii). Assume (i) and take an arbitrary $f \in C^{m-1}_n$ such that $f^- - f^-(0) \in C^{m-1}_n$, where $f^-(t) = \frac{f(t)-f(0)}{t}$. Then $f = tf^- + f(0) \in tC^{m-1}_n \oplus \mathbb{C}_n$, and by Lemma 4.3, $f = (D_1 + \alpha I)g_\alpha$, where $g_\alpha = V_\alpha f \in C^m_n$. System (16) takes the form

$$u = V_{\varphi_\alpha}(A + \alpha I)u + g_\alpha, \quad g_\alpha = V_\alpha f \in C^m_n,$$

and has a unique solution in C^m_n since by Lemma 6.1 and Theorem 5.3 (i) implies that $1 \in \rho_m(V_{\varphi_\alpha}(A(0) + \alpha I)) = \rho_m(V_{\varphi_\alpha}(A + \alpha I))$. Hence also system (1) has a unique solution in C^m_n, and the implication (i)⇒(ii) holds true.

(ii)⇒(iii). This implication is clear since $f^- \in C^{m-1}_n$ for $f \in C^m_n$. □

Together with implications (i)⇒(ii)⇒(iii) we have established also the implications (i′)⇒(ii′)⇒(iii′) for the assertions of Corollary 1.4.

Proof of Theorem 1.7 (*case* $m \geq 1$). Assume that $A \in C^m_{n\times n}$, $m \geq 1$, and that condition (11) is fulfilled. For systems (8) condition (11) can be interpreted as the assertion (i′) of Corollary 1.4 with $\mu = 1$ and m is replaced by $m - 1$. Also the free terms

$$(A(t) - A(0))d^k = tg_k(t), \quad g_k(t) := \frac{A(t) - A(0)}{t}d^k \in C^{m-1}_n, \quad k = 1, \ldots, \ell,$$

are of suitable form to apply the implication (i′)⇒(iii′) which says, on the first step of recursion, that the first one of systems (8) has a unique solution of the form $v_1 = tw_1$, where $w_1 \in C^{m-1}_n$ is the unique solution of the first system in (10). Rewriting corresponding equality (8) in the form

$$v_1' = (A(t) - \lambda_0 I)v_1 + t^{-1}(A(t) - A(0))d^1$$

we observe that $v_1' \in C_n^{m-1}$, hence $v_1 \in C_n^m$. After that we recursively obtain similar relations $v_k = tw_k$, $k = 2, \dots, \ell$, for the solutions $v_k \in C_n^m$ and $w_k \in C_n^{m-1}$ of the next systems in (8) and (10). The proof of Theorem 1.7 is completed. □

Proof of Theorem 1.3. *Implication* (iii)⇒(i). Assume (iii). We first prove by a contradiction argument that then $m > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k$. So, suppose that $m \le \gamma = \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k$. Here with small changes, we repeat the argument from the proof of the necessity part of Theorem 1.1. It follows from (iii) that a bounded inverse $(D_1 - A(t))^{-1} \in \mathcal{L}(C_n^m)$ exists. Take a $\lambda_0 \in \sigma(A(0))$ with $\mathrm{Re}\lambda_0 = \gamma$. By Theorems 1.6 and 1.7, homogeneous system $tu'(t) = A(t)u(t)$ has a solution u_0 of the form (48) with $tw \in C_n^m$. If $\gamma = \mathrm{Re}\lambda_0 > m$ or $\lambda_0 = m$, then $u_0 \in C_n^m$, and (1) is solvable in C_n^m non-uniquely (if solvable) that contradicts (iii). If $\lambda_0 = m + \beta\mathrm{i}$, $0 \ne \beta \in \mathbb{R}$, then

$$u_0(t) = t^m[\cos(\log(\beta t)) + \mathrm{i}\sin(\log(\beta t))](d^0 + tw(t)).$$

Now $u_0 \notin C_n^m$ but $u_\varepsilon := t^\varepsilon u_0 \in C_n^m$ for any $\varepsilon > 0$, and $(D_1 - A(t))u_\varepsilon = \varepsilon u_\varepsilon$, hence $(D_1 - A(t))^{-1}u_\varepsilon = \varepsilon^{-1}u_\varepsilon$, and operator $(D_1 - A(t))^{-1}$ is unbounded in C_n^m that contradicts the previous observations. Thus the relation $m \le \gamma$ contradicts (iii), and (iii) really implies that $m > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k$.

If $k \in \sigma(A(0)) \cap \mathbb{N}_0$ then $k \le m-1$; denote $k_0 = \min\{k : k \in \sigma(A(0)) \cap \mathbb{N}_0\}$. The subsystem of (13) with $k = k_0$ is unsolvable for certain $f \in C_n^m$, hence (1) is unsolvable in C_n^m for this f. So (iii) implies that $\sigma(A(0)) \cap \mathbb{N}_0 = \varnothing$. The proof of Theorem 1.3 is completed. □

Proof of Theorem 1.11. Assume the conditions of Theorem 1.11. For $f \in C_n^m$, $m \ge \ell$, a vector function $u \in C_n^m$ is a solution of system (1) iff

$$D^\ell(tu' - Au - f) = 0, \quad [D^k(tu' - Au - f)]_{t=0} = 0, \quad k = 0, \dots, \ell - 1. \quad (49)$$

Since $D^k(tu') = ku^{(k)} + tu^{(k+1)}$, $D^k Au = \sum_{j=0}^k \binom{k}{j} A^{(k-j)} u^{(j)}$, $k = 0, \dots, \ell-1$, the latter conditions in (49) mean that $u^{(k)}(0)$, $k = 0, \dots, \ell - 1$, satisfy (13). From this we immediately obtain that the consistency of system (13) is necessary for the solvability of (1) in C_n^m. Further, every $u \in C_n^m$ has a unique representation $u = D^{-\ell}v + \sum_{k=0}^{\ell-1} \frac{u_0^{(k)}}{k!} t^k$ with $v = D^\ell u \in C_n^{m-\ell}$. We obtain the following reformulation of the equivalence: a vector function $u = D^{-\ell}v + \sum_{k=0}^{\ell-1} \frac{u_0^{(k)}}{k!} t^k \in C_n^m$ with $v \in C_n^{m-\ell}$ and $u_0^{(k)}$, $k = 0, \dots, \ell - 1$, satisfying (13) is a solution of problem (15) iff $v = u^{(\ell)}$ is a solution of the equation

$$tv' = (A(0) - \ell I)v + \big[D^\ell A D^{-\ell} - A(0)\big]v + g, \quad g = f^{(\ell)} + D^\ell A \sum_{k=0}^{\ell-1} \frac{u_0^{(k)}}{k!} t^k \in C_n^{m-\ell}. \quad (50)$$

To prove Theorem 1.11, it is sufficient to show that (50) is uniquely solvable in $C_n^{m-\ell}$ for any $g \in C_n^{m-\ell}$.

Assumption $m > \max_{\lambda_k \in \sigma(A(0))} \mathrm{Re}\lambda_k$ of the Theorem can be reformulated as $m - \ell > \max_{\lambda'_k \in \sigma(A(0)-\ell I)} \mathrm{Re}\lambda_{k'}$; due to (14), we have $\sigma(A(0) - \ell I) \cap \mathbb{N}_0 = \varnothing$. Thus conditions (6) are fulfilled for the operator $A(0) - \ell I$, with $m - \ell$ in the role of m, and we conclude with the help of Theorem 1.3 (with the help of Theorem 1.1 in case $\ell = m$) that the inverse $V = (D_1 - A(0) + \ell I)^{-1} \in \mathcal{L}(C_n^{m-\ell})$ exists. Hence (50) is equivalent to the equation

$$v = Bv + Vg, \quad B = V\left[D^\ell A D^{-\ell} - A(0)\right] \in \mathcal{L}(C_n^{m-\ell}), \quad Vg \in C_n^{m-\ell}. \qquad (51)$$

Operator $B \in \mathcal{L}(C_n^{m-\ell})$ is compact. Indeed,

$$D^\ell A D^{-\ell} u - A(0)u = (A - A(0))u + \sum_{j=1}^{\ell} \binom{\ell}{j} A^{(j)} D^{-j} u,$$

operator $D^{-j} \in \mathcal{L}(C_n^k)$ is compact for $j \geq 1$, $k \geq 0$, whereas the compactness of $V(A - A(0)) \in \mathcal{L}(C_n^{m-\ell})$ is a consequence of the cordial structure of V; a more detailed argument follows soon. Hence (51) and (50) have a unique solution $v \in C_n^{m-\ell}$ for any $g \in C_n^{m-\ell}$ if the homogenous equation $v = Bv$ has in $C_n^{m-\ell}$ only the trivial solution $v = 0$. This is really the case. Indeed, to a solution $v \in C_n^{m-\ell}$ of $v = Bv$, or of (50) with $g = 0$, there corresponds a solution $u = D^{-\ell} v \in C_n^m$ of problem $tu' = Au$, $u^{(k)}(0) = 0$, $k = 0, \ldots, \ell - 1$ (note that $f^{(k)}(0) = u_0^{(k)} = 0$, $k = 0, \ldots, \ell - 1$, satisfy (13)). Due to zero initial values, u can be represented $u = t^\ell w$, $w \in C_n^{m-\ell}$. Further, (7) is fulfilled for $\mu = \ell$ and m replaced by $m - \ell$. By Corollary 1.4 system $tu' = Au$ has a unique solution of the form $u = t^\ell w$, $w \in C_n^{m-\ell}$, and it is $u = 0$. Thus $D^{-\ell} v = 0$, $v = 0$.

It remains to show that $V(A - A(0)) \in \mathcal{L}(C_n^{m-\ell})$ is compact. To examine the structure of the operator $V = (D_1 - A(0) + \ell I)^{-1}$, consider the system

$$tu' = (A(0) - \ell I)u + f, \quad f \in C_n^{m-\ell}.$$

Represent $A(0) = E\Lambda E^{-1}$, where the block diagonal matrix $\Lambda = (\Lambda_k)$ is the Jordan form of $A(0)$, see Section 5 for details. Note that $ED_1E^{-1} = D_1$, so with respect to $v = Eu$ the system takes the form $tv' = (\Lambda - \ell I)v + g$, $g = Ef$. This system splits into independent subsystems: to a $n_k \times n_k$ block $\Lambda_k - \ell I$ with $\lambda_k - \ell \in \sigma(A(0) - \ell I)$ on the main diagonal, there corresponds the subsystem

$$D_1 v_{n_k} = (\lambda_k - \ell) v_{n_k} + g_{n_k}, \;\; D_1 v_j = (\lambda_k - \ell) v_j + v_{j+1} + g_j, \;\; j = n_k - 1,\, n_k - 2, \ldots, 1.$$

The inverse $(D_1 - \lambda_k I + \ell I)^{-1} = \widetilde{V}_{\varphi_{\ell-\lambda_k}} =: W_k \in \mathcal{L}(C^{m-\ell})$ exists by Lemma 4.2 since $(m - \ell) + \mathrm{Re}(\ell - \lambda_k) > 0$, $\ell - \lambda_k \neq -j$ for $j = 0, \ldots, m - \ell - 1$, due to (14). We find recursively that

$$v_{n_k} = W_k g_{n_k}, \;\; v_{n_k - 1} = W_k g_{n_k - 1} + W_k^2 g_{n_k}, \ldots, v_1 = W_k g_1 + W_k^2 g_2 + \cdots + W_k^{n_k} g_{n_k}.$$

A summary is that

$$V := (D_1 + \ell I - A(0))^{-1} = E(D_1 + \ell I - \Lambda)^{-1}E^{-1},$$

where $(D_1 + \ell I - \Lambda)^{-1} \in \mathcal{L}(C_n^{m-\ell})$ is a block diagonal operator with the blocks $(D_1 + \ell I - \lambda_k I)^{-1} \in \mathcal{L}(C_{n_k}^{m-\ell})$ corresponding to the blocks Λ_k of $\Lambda = (\Lambda_k)$, and $(D_1 + \ell I - \lambda_k I)^{-1} \in \mathcal{L}(C_{n_k}^{m-\ell})$ is given by

$$(D_1 + \ell I - \lambda_k I)^{-1} = \begin{pmatrix} W_k & W_k^2 & \dots & W_k^{n_k} \\ 0 & W_k & \dots & W_k^{n_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & W_k \end{pmatrix}, \quad W_k = \widetilde{V}_{\varphi_{\ell-\lambda_k}} \in \mathcal{L}(C^{m-\ell}).$$

So the elements of the matrix operator $V(A - A(0))$ consist of linear combinations of some scalar operators of the form $B_{kj} = W_k^j(b_{kj} - b_{kj}(0))$ with certain $b_{kj} \in C^m$ determined by matrices $A(t)$ and E; due to Theorem 3.3, $B_{kj} \in \mathcal{L}(C^{m-\ell,m-\ell})$ are compact that due to decomposition (19) implies that $B_{kj} \in \mathcal{L}(C^{m-\ell})$ and $V(A - A(0)) \in \mathcal{L}(C_n^{m-\ell})$ are compact.

The proof of Theorem 1.11 is completed. □

The compactness of $B \in \mathcal{L}(C_n^m)$ is a helpful property of system (51) when discretization methods are constructed and justified. For a comparison, note that the operator $V_{\varphi_\alpha}(A + \alpha I) \in \mathcal{L}(C_n^m)$ of system (16) is noncompact, and a special applicability condition is needed when spline collocation type methods are applied to (16), see [8, 15] for details in the scalar case.

In case $\sigma(A(0)) \cap \mathbb{N}_0 = \varnothing$ system (1) is equivalent to (51) for $\ell = 0$, i.e. to system $u = (D_1 - A(0))^{-1}(A - A(0))u + (D_1 - A(0))^{-1}f$ with respect to u itself.

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