

# Matrix-Valued $L^1$ -Optimal Controls in the Coefficients of Linear Elliptic Problems

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**Abstract.** We consider optimal control problems for linear degenerate elliptic equations with mixed boundary conditions. In particular, we take the matrix-valued coefficients  $A(x)$  of such systems as controls in  $L^1(\Omega; \mathbb{R}^{\frac{N(N+1)}{2}})$ . One of the important features of the admissible controls is the fact that eigenvalues of the coefficient matrices may vanish in  $\Omega$ . Equations of this type may exhibit non-uniqueness of weak solutions. Using the concept of convergence in variable spaces and following the direct method in the Calculus of variations, we establish the solvability of this optimal control problem in the class of weak admissible solutions.

**Keywords.** Degenerate elliptic equations, control in coefficients, weighted Sobolev spaces, Lavrentieff phenomenon, direct method in the Calculus of Variations

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## 1. Introduction

Material optimization is an emerging field in the engineering context of design of advanced materials. The notions of advanced materials and meta-materials have recently evolved where desired, possibly counterintuitive, material properties are realized via systematic model-based optimization of material parameters. Often, such an inverse engineering approach leads to micro-structures, where mathematical optimization indicates singular behavior for the material parameters. This is particularly true for optical meta-materials in the context of cloaking. Following the exploration of electromagnetic cloaking on the base physics by Pendry [24] and Leonhard [19], a subject that has become a major

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branch of modern physics, the mathematical theory of cloaking has been established by Uhlmann, Lassas and coworkers (see the review article [15] and, e.g., [16, 25]). The references given are by no means complete and rather exemplary in nature. The notion of transformational optics has been developed that allows, based on differential geometry, to construct Riemann metrics with special features, such that objects are “hidden”. This is a question typically posed in the context of inverse problems: given a set of data on the boundary, as inputs and measured outputs, is it possible to reconstruct “objects” represented by, say, different material properties? If we can provide a situation, represented by a Riemann metric, where this question can be answered in the negative sense, we deal with non-identifiable objects. In the language of electromagneto-dynamics, this means that objects can then be invisible. The studies of Uhlmann et al. [25] strongly indicate that the corresponding Riemann metrics which are represented as coefficient matrices in elliptic systems, exhibit singular behavior along the object to be cloaked. Indeed, eigenvalues of that matrix may vanish or tend to infinity. Several other physical phenomena related to equilibrium of continuous media modeled by elliptic problems concern media which are “perfect” insulators or “perfect” conductors (see [11]) necessitate eigenvalues of the matrix  $A$  either to vanish somewhere or to be unbounded. These circumstances appearing in modern technologies are the major motivation for the paper.

The aim of this work is to study the existence of optimal controls in the matrix-valued coefficients associated with a linear elliptic equation and mixed boundary condition. The controls are taken as the matrix of the coefficients in the main part of the elliptic operator. The most important feature of such controls is the fact that eigenvalues of the matrix  $A$  may either vanish on subsets with zero Lebesgue measure or be unbounded. In this case the precise answer for the question of existence or non-existence of optimal solutions heavily depends on the class of admissible controls chosen. The main questions are: what is the right setting of the optimal control problem with  $L^1$ -controls in coefficients, and what is the right class of admissible solutions to the above problem? Using the direct method in the Calculus of variations, we discuss the solvability of this optimal control problem in a class of weak admissible solutions. It should be emphasized that in contrast to the paper [8], we do not make use of any relaxations for the original optimal control problem.

In this paper we deal with an optimal control problem in coefficients for boundary value problems of the form

$$\begin{cases} -\operatorname{div}(A(x)\nabla y) = f & \text{in } \Omega \\ y = 0 & \text{on } \Gamma_D \\ \frac{\partial y}{\partial \nu_A} = g & \text{on } \Gamma_N, \end{cases} \quad (1)$$

where  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma_N)$  are given functions, the boundary of  $\Omega$  consists of two disjoint parts  $\partial\Omega = \Gamma_D \cup \Gamma_N$ , and  $A$  is a measurable positive-semidefinite square symmetric matrix on a bounded open domain  $\Omega$  in  $\mathbb{R}^N$ .

Even though numerous articles (see, for instance, [2, 7, 9, 12, 22, 23, 28] and references therein) are devoted to variational and non variational approaches to problems related to (1), only few deal with optimal control problems for degenerate partial differential equations (see for example [4–6, 17, 18]). This can be explained by several reasons. Firstly, boundary value problem (1) for locally integrable matrix-valued function  $A$  may exhibit non-uniqueness of weak solutions, as well as other surprising consequences. So, in general, the mapping  $A \mapsto y(A)$  can be multivalued. There are no reasons to expect that for every admissible given data  $f \in L^2(\Omega)$ ,  $g \in L^2(\Gamma_N)$ , and  $A \in L^1(\Omega; \mathbb{R}^{N \times N})$ , problem (1) admits at least one weak solution. Besides, for every admissible control function  $A$ , the weak solutions to the boundary value problem (1) belong to the corresponding weighted Sobolev space  $W^{1,2}(\Omega, A dx)$ . In addition, even if the elliptic equation is non-degenerate, i.e. admissible controls  $A(x)$  are such that

$$\beta \|\xi\|_{\mathbb{R}^N}^2 \geq \xi \cdot A(x)\xi \geq \alpha \|\xi\|_{\mathbb{R}^N}^2 \quad \xi \in \mathbb{R}^N$$

with  $\alpha > 0$ , the majority of optimal control problems in coefficients have no solution (see for instance [21]).

In spite of the fact that the original boundary value problem is ill-posed, in general, we show that the corresponding extremal problem has a practical sense and is well-posed. This problem is, thus, yet another example for the difference between well-posedness for optimal control problems for systems with distributed parameters and partial differential equations.

## 2. Notation and preliminaries

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$  ( $N \geq 2$ ) with Lipschitz boundary. We assume that the boundary of  $\Omega$  consists of two disjoint parts  $\partial\Omega = \Gamma_D \cup \Gamma_N$  with Dirichlet boundary conditions on  $\Gamma_D$ , and Neumann boundary conditions on  $\Gamma_N$ . Let the sets  $\Gamma_D$  and  $\Gamma_N$  have positive  $(N - 1)$ -dimensional measures. Let  $\chi_E$  be the characteristic function of a subset  $E \subset \Omega$ , i.e.  $\chi_E(x) = 1$  if  $x \in E$ , and  $\chi_E(x) = 0$  if  $x \notin E$ .

Let  $C_0^\infty(\mathbb{R}^N; \Gamma_D) = \{\varphi \in C_0^\infty(\mathbb{R}^N) : \varphi = 0 \text{ on } \Gamma_D\}$ . We define the Banach space  $W^{1,1}(\Omega; \Gamma_D)$  as the closure of  $C_0^\infty(\mathbb{R}^N; \Gamma_D)$  in the classical Sobolev space  $W^{1,1}(\Omega)$ . For any subset  $E \subset \Omega$  we denote by  $|E|$  its  $N$ -dimensional Lebesgue measure  $\mathcal{L}^N(E)$ .

*Symmetric matrices with degenerate eigenvalues.* By  $\mathbb{S}^N := \mathbb{R}^{\frac{N(N+1)}{2}}$  we denote the set of all symmetric matrices  $\xi = [\xi_{ij}]_{i,j=1}^N$ , ( $\xi_{ij} = \xi_{ji}$ ). We suppose that  $\mathbb{S}^N$  is endowed with the Euclidian scalar product  $\xi \cdot \eta = \text{tr}(\xi \eta) = \xi_{ij} \eta_{ij}$  and with the corresponding Euclidian norm  $\|\xi\|_{\mathbb{S}^N} = (\xi \cdot \xi)^{\frac{1}{2}}$ . Let

$$L^1(\Omega)^{\frac{N(N+1)}{2}} = L^1(\Omega; \mathbb{S}^N)$$

be the space of integrable functions whose values are symmetric matrices.

Let  $\alpha \in \mathbb{R}$  be a fixed positive value. Let  $\zeta_{ad} : \Omega \rightarrow [0, \alpha]$  be a given function satisfying the properties

$$\zeta_{ad} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \notin L^\infty(\Omega).$$

Let  $\Psi_*$  be a nonempty compact subset of  $L^1(\Omega)$  such that for any  $\zeta_* \in \Psi_*$  the following conditions hold true

$$\zeta_{ad}(x) < \zeta_*(x) \quad \text{a.e. in } \Omega, \tag{2}$$

$$\zeta_* : \Omega \rightarrow \mathbb{R}_+^1 \quad \text{is smooth function along the boundary } \partial\Omega, \tag{3}$$

$$\zeta_* = \alpha \quad \text{on } \partial\Omega. \tag{4}$$

By  $\mathfrak{M}_\alpha^\beta(\Omega)$  we denote the set of all matrices  $A(x) = [a_{ij}(x)] \in \mathbb{S}^N$  such that

$$A(x) \leq \beta(x)I \quad \text{a.e. in } \Omega, \tag{5}$$

$$\exists \zeta_* \in \Psi_* \text{ s.t. } \zeta_* I \leq A(x) \quad \text{a.e. in } \Omega. \tag{6}$$

Here  $\beta \in L^1(\Omega)$  is a given function such that  $\beta(x) > 0$  a.e. in  $\Omega$ ,  $I$  is the identity matrix in  $\mathbb{R}^{N \times N}$ , and (5), (6) should be considered in the sense of quadratic forms. Therefore, (5), (6) imply the following inequalities:

$$\text{if } A \in L^1(\Omega; \mathbb{S}^N), \quad \text{then } \|A(x)\|_{L^1(\Omega; \mathbb{S}^N)} \leq \|\beta\|_{L^1(\Omega)} < +\infty, \tag{7}$$

$$\zeta_*(x) \|\xi\|_{\mathbb{R}^N}^2 \leq (A(x)\xi, \xi)_{\mathbb{R}^N} \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^N. \tag{8}$$

**Remark 2.1.** For every measurable matrix-valued function  $A : \Omega \rightarrow \mathbb{S}^N$  we can define the corresponding collection of its eigenvalues  $\{\lambda_1^A, \dots, \lambda_N^A\}$ , where each  $\lambda_k^A = \lambda_k^A(x)$  is counted with its multiplicity. Then, in view of the properties (6) of the class  $\Psi_*$  and the Rayleigh quotient, we have:  $(\lambda_k^A)^{-1} \in L^1(\Omega)$  for all  $k = 1, \dots, N$ . It means that, in general, eigenvalues of matrices  $A \in \mathfrak{M}_\alpha^\beta(\Omega)$  cannot be strictly separated from zero on  $\Omega$  (in the sense of almost everywhere) by a positive constant. Because of this, these matrices are sometime referred to as matrices with degenerate spectrum. In the sequel, properties (2)–(8) play a central role in definition of the class of admissible controls for the control object (1).

To each matrix  $A \in \mathfrak{M}_\alpha^\beta(\Omega)$  we will associate two weighted Sobolev spaces:

$$W_A(\Omega; \Gamma_D) = W(\Omega; \Gamma_D; A dx) \quad \text{and} \quad H_A(\Omega; \Gamma_D) = H(\Omega; \Gamma_D; A dx),$$

where  $W_A(\Omega; \Gamma_D)$  is the set of functions  $y \in W^{1,1}(\Omega; \Gamma_D)$  for which the norm

$$\|y\|_A = \left( \int_\Omega (y^2 + (\nabla y, A(x)\nabla y)_{\mathbb{R}^N}) dx \right)^{\frac{1}{2}} \tag{9}$$

is finite, and  $H_A(\Omega; \Gamma_D)$  is the closure of  $C_0^\infty(\Omega; \Gamma_D)$  in  $W_A(\Omega; \Gamma_D)$ . Note that due to the inequality (8) and estimates

$$\begin{aligned} \int_\Omega |y| dx &\leq \left( \int_\Omega |y|^2 dx \right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \leq C \|y\|_A, \tag{10} \\ \int_\Omega \|\nabla y\|_{\mathbb{R}^N} dx &\leq \left( \int_\Omega \|\nabla y\|_{\mathbb{R}^N}^2 \zeta_* dx \right)^{\frac{1}{2}} \left( \int_\Omega \zeta_*^{-1} dx \right)^{\frac{1}{2}} \\ &\leq C \left( \int_\Omega (\nabla y, A(x)\nabla y)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} \\ &\leq C \|y\|_A, \tag{11} \end{aligned}$$

the space  $W_A(\Omega; \Gamma_D)$  is complete with respect to the norm  $\|\cdot\|_A$ . It is clear that  $H_A(\Omega; \Gamma_D) \subset W_A(\Omega; \Gamma_D)$ , and  $W_A(\Omega; \Gamma_D), H_A(\Omega; \Gamma_D)$  are Hilbert spaces. If the eigenvalues  $\{\lambda_1^A, \dots, \lambda_N^A\}$  of  $A : \Omega \rightarrow \mathbb{S}^N$  are bounded between two positive constants, then it is easy to verify that  $W_A(\Omega; \Gamma_D) = H_A(\Omega; \Gamma_D)$ . However, for a “typical” weight-matrix  $A \in \mathfrak{M}_\alpha^\beta(\Omega)$  the space of smooth functions  $C_0^\infty(\Omega)$  is not dense in  $W_A(\Omega; \Gamma_D)$ . Hence the identity  $W_A(\Omega; \Gamma_D) = H_A(\Omega; \Gamma_D)$  is not always valid (for the corresponding examples in the case when  $A(x) = \rho(x)I$ , we refer to [10, 26]).

*Weak compactness criterion in  $L^1(\Omega; \mathbb{S}^N)$ .* Throughout the paper we will often use the concept of weak and strong convergence in  $L^1(\Omega; \mathbb{S}^N)$ . Let  $\{A_n\}_{n \in \mathbb{N}}$  be a bounded sequence of matrices in  $L^1(\Omega; \mathbb{S}^N)$ . We recall that  $\{A_n\}_{n \in \mathbb{N}}$  is called equi-integrable on  $\Omega$ , if for any  $\delta > 0$  there is a  $\tau = \tau(\delta)$  such that  $\int_S \|A_n\|_{\mathbb{S}^N} dx < \delta$  for every measurable subset  $S \subset \Omega$  of Lebesgue measure  $|S| < \tau$ . Then the following assertions are equivalent for  $L^1(\Omega; \mathbb{S}^N)$ -bounded sequences (Dunford-Pettis, [13]):

- (i) a sequence  $\{A_k\}_{k \in \mathbb{N}}$  is weakly convergent in  $L^1(\Omega; \mathbb{S}^N)$ ;
- (ii) the sequence  $\{A_k\}_{k \in \mathbb{N}}$  is equi-integrable.

**Theorem 2.2** ([13]). *If a sequence  $\{A_k\}_{k \in \mathbb{N}} \subset L^1(\Omega; \mathbb{S}^N)$  is equi-integrable and  $A_k \rightarrow A$  almost everywhere in  $\Omega$  then  $A_k \rightarrow A$  in  $L^1(\Omega; \mathbb{S}^N)$ .*

*Functions with bounded variation.* Let  $f : \Omega \rightarrow \mathbb{R}$  be a function of  $L^1(\Omega)$ . Define

$$\int_{\Omega} |Df| = \sup \left\{ \int_{\Omega} f \operatorname{div} \varphi \, dx : \varphi = (\varphi_1, \dots, \varphi_N) \in C_0^1(\Omega; \mathbb{R}^N), |\varphi(x)| \leq 1 \text{ for } x \in \Omega \right\},$$

where  $\operatorname{div} \varphi = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial x_i}$ .

According to the Radon-Nikodym theorem, if  $\int_{\Omega} |Df| < +\infty$  then the distribution  $Df$  is a measure and there exist a vector-valued function  $\nabla f \in [L^1(\Omega)]^N$  and a measure  $D_s f$ , singular with respect to the  $N$ -dimensional Lebesgue measure  $\mathcal{L}^N|_{\Omega}$  restricted to  $\Omega$ , such that

$$Df = \nabla f \mathcal{L}^N|_{\Omega} + D_s f.$$

**Definition 2.3.** A function  $f \in L^1(\Omega)$  is said to have a bounded variation in  $\Omega$  if  $\int_{\Omega} |Df| < +\infty$ . By  $BV(\Omega)$  we denote the space of all functions in  $L^1(\Omega)$  with bounded variation.

Under the norm  $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_{\Omega} |Df|$ ,  $BV(\Omega)$  is a Banach space. The following compactness result for  $BV$ -functions is well-known:

**Proposition 2.4.** [1, p. 17] *Uniformly bounded sets in  $BV$ -norm are relatively compact in  $L^1(\Omega)$ .*

For our further analysis, we accept the following concept.

**Definition 2.5.** [1, p. 17] A sequence  $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$  weakly converges to some  $f \in BV(\Omega)$ , and we write  $f_k \rightharpoonup f$  iff the two following conditions hold:  $f_k \rightarrow f$  strongly in  $L^1(\Omega)$ , and  $Df_k \rightharpoonup Df$  weakly-\* in the space of Radon measures  $M(\Omega; \mathbb{R}^N)$ , i.e.

$$\lim_{k \rightarrow \infty} \int_{\Omega} (\varphi, Df_k)_{\mathbb{R}^N} = \int_{\Omega} (\varphi, Df)_{\mathbb{R}^N} \quad \forall \varphi \in C_0(\mathbb{R}^N)^N, \tag{12}$$

In the proposition below we give a compactness result related to this convergence, together with lower semicontinuity (see [1, 14]):

**Proposition 2.6.** [1, p. 18] *Let  $\{f_k\}_{k=1}^{\infty}$  be a sequence in  $BV(\Omega)$  strongly converging to some  $f$  in  $L^1(\Omega)$  and satisfying  $\sup_{k \in \mathbb{N}} \int_{\Omega} |Df_k| < +\infty$ . Then*

- (i)  $f \in BV(\Omega)$  and  $\int_{\Omega} |Df| \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |Df_k|$ ;
- (ii)  $f_k \rightharpoonup f$  in  $BV(\Omega)$ .

### 3. $\mathbb{S}^N$ -valued Radon measures and weak convergence in variable $L^2$ -spaces

By a nonnegative Radon measure on  $\Omega$  we mean a nonnegative Borel measure which is finite on every compact subset of  $\Omega$ . The space of all nonnegative Radon measures on  $\Omega$  will be denoted by  $M_+(\Omega)$ . According to the Riesz theory, each Radon measure  $\mu \in M_+(\Omega)$  can be interpreted as an element of the dual of the space  $C_0(\Omega)$  of all continuous functions with compact support. Let  $M(\Omega; \mathbb{S}^N)$  denote the space of all  $\mathbb{S}^N$ -valued Borel measures. Then

$$\mu = [\mu_{ij}] \in M(\Omega; \mathbb{S}^N) \Leftrightarrow \mu_{ij} \in C'_0(\Omega), \quad i, j = 1, \dots, N.$$

Let  $\mu$  and the sequence  $\{\mu_k\}_{k \in \mathbb{N}}$  be matrix-valued Radon measures. We say that  $\{\mu_k\}_{k \in \mathbb{N}}$  weakly- $*$  converges to  $\mu$  in  $M(\Omega; \mathbb{S}^N)$  if

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi \cdot d\mu_k = \int_{\Omega} \varphi \cdot d\mu \quad \forall \varphi \in C_0(\Omega; \mathbb{S}^N).$$

A typical example of such measures is

$$d\mu_k = A_k(x) dx, \quad d\mu = A(x) dx, \tag{13}$$

$$\text{where } A_k, A \in \mathfrak{M}_{\alpha}^{\beta}(\Omega) \cap L^1(\Omega; \mathbb{S}^N) \quad \text{and} \quad A_k \rightharpoonup A \text{ in } L^1(\Omega; \mathbb{S}^N), \tag{14}$$

$$\text{or } A_k, A \in \mathfrak{M}_{\alpha}^{\beta}(\Omega) \cap L^{\infty}(\Omega; \mathbb{S}^N) \quad \text{and} \quad A_k \overset{*}{\rightharpoonup} A \text{ in } L^{\infty}(\Omega; \mathbb{S}^N). \tag{15}$$

As we will see later (see Lemma 4.4), the sets  $\mathfrak{M}_{\alpha}^{\beta}(\Omega) \cap L^1(\Omega; \mathbb{S}^N)$  are sequentially closed with respect to strong convergence in  $L^1(\Omega; \mathbb{S}^N)$ .

In this section we suppose that the measures  $\mu$  and  $\{\mu_k\}_{k \in \mathbb{N}}$  are defined by (13)–(15) and  $\mu_k \overset{*}{\rightharpoonup} \mu$  in  $M(\Omega; \mathbb{S}^N)$ . Further, we will use  $L^2(\Omega, A dx)^N$  to denote the Hilbert space of measurable vector-valued functions  $f \in \mathbb{R}^N$  on  $\Omega$  such that

$$\|f\|_{L^2(\Omega, A dx)^N} = \left( \int_{\Omega} (f, A(x)f)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} < +\infty.$$

As follows from estimate (11) any vector-valued function of  $L^2(\Omega, A dx)^N$  is Lebesgue integrable on  $\Omega$ .

We say that a sequence  $\{v_k \in L^2(\Omega, A_k dx)^N\}_{k \in \mathbb{N}}$  is bounded if

$$\limsup_{k \rightarrow \infty} \int_{\Omega} (v_k, A_k(x)v_k)_{\mathbb{R}^N} dx < +\infty.$$

**Definition 3.1.** A bounded sequence  $\{v_k \in L^2(\Omega, A_k dx)^N\}_{k \in \mathbb{N}}$  is weakly convergent to a function  $v \in L^2(\Omega, A dx)$  in the variable space  $L^2(\Omega, A_k dx)^N$  if

$$\lim_{k \rightarrow \infty} \int_{\Omega} (\varphi, A_k(x)v_k)_{\mathbb{R}^N} dx = \int_{\Omega} (\varphi, A(x)v)_{\mathbb{R}^N} dx \quad \forall \varphi \in C_0^{\infty}(\Omega)^N. \tag{16}$$

The main property concerning the weak convergence in  $L^p(\Omega, d\mu_\varepsilon)$  can be expressed as follows (see for comparison [27]):

**Proposition 3.2.** *If a sequence  $\{v_k \in L^2(\Omega, A_k dx)^N\}_{k \in \mathbb{N}}$  is bounded and the condition (14) holds true, then it contains a weakly convergent subsequence in  $L^2(\Omega, A_k dx)^N$ .*

*Proof.* Having set  $L_k(\varphi) = \int_\Omega (\varphi, A_k(x)v_k)_{\mathbb{R}^N} dx \ \forall \varphi \in C_0^\infty(\Omega)^N$  and making use the Hölder inequality, we get

$$\begin{aligned} |L_k(\varphi)| &\leq \left( \int_\Omega |A_k^{\frac{1}{2}} v_k|_{\mathbb{R}^N}^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega |A_k^{\frac{1}{2}} \varphi|_{\mathbb{R}^N}^2 dx \right)^{\frac{1}{2}} \\ &= \left( \int_\Omega (v_k, A_k v_k)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} \left( \int_\Omega (\varphi, A_k \varphi)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} \\ &\leq C \left( \int_\Omega (\varphi, A_k \varphi)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} \\ &\leq C \left( \int_\Omega \beta(x) \|\varphi\|_{\mathbb{R}^N}^2 dx \right)^{\frac{1}{2}} \\ &\leq C \|\varphi\|_{C(\Omega; \mathbb{R}^N)} \|\beta\|_{L^1(\Omega)}^{\frac{1}{2}} \quad \forall k \in \mathbb{N}. \end{aligned} \tag{17}$$

Since the set  $C_0^\infty(\Omega)^N$  is separable with respect to the norm  $\|\cdot\|_{C(\Omega; \mathbb{R}^N)}$  and  $\{L_k(\varphi)\}_{k \in \mathbb{N}}$  is a uniformly bounded sequence of linear functionals, it follows that there exists a subsequence of positive numbers  $\{k_j\}_{j=1}^\infty$  for which the limit (in the sense of point-by-point convergence)

$$\lim_{j \rightarrow \infty} L_{k_j}(\varphi) = L(\varphi) \tag{18}$$

is well defined for every  $\varphi \in C_0^\infty(\Omega)^N$ . As a result, using (14), we have

$$|L(\varphi)| \leq C \lim_{j \rightarrow \infty} \left( \int_\Omega (\varphi, A_{k_j} \varphi)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} = C \left( \int_\Omega (\varphi, A \varphi)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}}.$$

Hence,  $L(\varphi)$  is a continuous functional on  $L^2(\Omega, A dx)^N$  admitting the following representation  $L(\varphi) = \int_\Omega (\varphi, A(x)v)_{\mathbb{R}^N} dx$ , where  $v$  is some element of  $L^2(\Omega, A dx)^N$ . Thus, taking into account Definition 3.1,  $v$  can be taken as the weak limit of

$$\{v_{k_j} \in L^2(\Omega, A_{k_j} dx)^N\}_{j \in \mathbb{N}}. \quad \square$$

The next property of weak convergence in  $L^2(\Omega, A_k dx)^N$  shows that the variable  $L^2$ -norm is lower semicontinuous with respect to weak convergence.



**Proposition 3.3.** *If the sequence  $\{v_k \in L^2(\Omega, A_k dx)^N\}_{k \in \mathbb{N}}$  converges weakly to  $v \in L^2(\Omega, A dx)^N$  and the condition (14) holds true, then*

$$\liminf_{k \rightarrow \infty} \int_{\Omega} (v_k, A_k(x)v_k)_{\mathbb{R}^N} dx \geq \int_{\Omega} (v, A(x)v)_{\mathbb{R}^N} dx. \quad (19)$$

*Proof.* Indeed, we have

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (v_k, A_k v_k)_{\mathbb{R}^N} dx &= \frac{1}{2} \int_{\Omega} |A_k^{\frac{1}{2}} v_k|_{\mathbb{R}^N}^2 dx \\ &\geq \int_{\Omega} (\varphi, A_k v_k)_{\mathbb{R}^N} dx - \frac{1}{2} \int_{\Omega} (\varphi, A_k \varphi)_{\mathbb{R}^N} dx \quad \forall \varphi \in C_0^\infty(\Omega)^N, \end{aligned}$$

$$\frac{1}{2} \liminf_{k \rightarrow \infty} \int_{\Omega} (v_k, A_k v_k)_{\mathbb{R}^N} dx \geq \int_{\Omega} (\varphi, Av)_{\mathbb{R}^N} dx - \frac{1}{2} \int_{\Omega} (\varphi, A\varphi)_{\mathbb{R}^N} dx.$$

Since the last inequality is valid for all  $\varphi \in C_0^\infty(\Omega)^N$  and  $C_0^\infty(\Omega)^N$  is a dense subset of  $L^2(\Omega, A dx)^N$ , it holds also true for  $\varphi \in L^2(\Omega, A dx)^N$ . So, taking  $\varphi = v$ , we arrive at (19).  $\square$

**Definition 3.4.** A sequence  $\{v_k \in L^2(\Omega, A_k dx)^N\}_{k \in \mathbb{N}}$  is said to be strongly convergent to a function  $v \in L^2(\Omega, A dx)^N$  if

$$\lim_{k \rightarrow \infty} \int_{\Omega} (b_k, A_k(x)v_k)_{\mathbb{R}^N} dx = \int_{\Omega} (b, A(x)v)_{\mathbb{R}^N} dx \quad (20)$$

whenever  $b_k \rightarrow b$  in  $L^2(\Omega, A_k dx)^N$  as  $k \rightarrow \infty$ .

**Remark 3.5.** Note that in the case  $A_k \equiv A$ , Definitions 3.1–3.4 leads to the usual notion of convergence in the weighted Hilbert space  $L^2(\Omega, A dx)^N$ .

The following property of strong convergence in the variable  $L^2(\Omega, A_k dx)^N$ -spaces will be later on.

**Proposition 3.6.** *Assume the condition (14) holds true. Then the weak convergence of a sequence  $\{v_k \in L^2(\Omega, A_k dx)^N\}_{k \in \mathbb{N}}$  to  $v \in L^2(\Omega, A dx)^N$  and*

$$\lim_{k \rightarrow \infty} \int_{\Omega} (v_k, A_k(x)v_k)_{\mathbb{R}^N} dx = \int_{\Omega} (v, A(x)v)_{\mathbb{R}^N} dx \quad (21)$$

*are equivalent to strong convergence of  $\{v_k\}_{k \in \mathbb{N}}$  in  $L^2(\Omega, A_k dx)^N$  to  $v \in L^2(\Omega, A dx)^N$ .*

*Proof.* It is easy to verify that strong convergence implies weak convergence and (21). Indeed, we use  $b_k = \varphi \in C_0^\infty(\Omega)^N$  in (20) and then substitute  $b_k = v_k$ .

In view of Proposition 3.2, we may assume that there exist two values  $\nu_1$  and  $\nu_2$  such that (up to subsequences)

$$\lim_{k \rightarrow \infty} \int_{\Omega} (b_k, A_k(x)v_k)_{\mathbb{R}^N} dx = \nu_1, \quad \lim_{k \rightarrow \infty} \int_{\Omega} (b_k, A_k(x)b_k)_{\mathbb{R}^N} dx = \nu_2.$$

Using lower semicontinuity (19) and (21), we obtain

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{\Omega} (v_k + tb_k, A_k(x)(v_k + tb_k))_{\mathbb{R}^N} dx \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} (v_k, A_k(x)v_k)_{\mathbb{R}^N} dx + 2t\nu_1 + t^2\nu_2 \\ &\geq \int_{\Omega} (v + tb, A(x)(v + tb))_{\mathbb{R}^N} dx \\ &= \int_{\Omega} (v, A(x)v)_{\mathbb{R}^N} dx + 2t \int_{\Omega} (b, A(x)v)_{\mathbb{R}^N} dx + t^2 \int_{\Omega} (b, A(x)b)_{\mathbb{R}^N} dx. \end{aligned}$$

From this we conclude that

$$2t\nu_1 + t^2\nu_2 \geq 2t \int_{\Omega} (b, A(x)v)_{\mathbb{R}^N} dx + t^2 \int_{\Omega} (b, A(x)b)_{\mathbb{R}^N} dx \quad \forall t \in \mathbb{R}^1.$$

Hence,  $\nu_1 = \int_{\Omega} (b, A(x)v)_{\mathbb{R}^N} dx$ . Thereby the strong convergence of the sequence  $\{v_k \in L^2(\Omega, A_k dx)\}_{k \in \mathbb{N}}$  is established.  $\square$

### 4. Auxiliary results

To begin with, we provide the following property of the set  $\Psi_* \subset L^1(\Omega)$  defined in (2)–(4).

**Lemma 4.1.** *Let  $\{\zeta_{*,n}\}_{n \in \mathbb{N}}$  be any sequence in  $\Psi_*$ . Then there is an element  $\zeta_* \in L^1(\Omega)$  such that, within a subsequence of  $\{\zeta_{*,n}\}_{n \in \mathbb{N}}$ , we have*

$$\zeta_{*,n} \rightarrow \zeta_* \quad \text{in } L^1(\Omega), \quad \zeta_* \in \Psi_*, \quad (22)$$

$$\zeta_{*,n}^{-1} \rightarrow \zeta_*^{-1} \quad \text{in } L^1(\Omega), \quad (23)$$

$$\text{and } \zeta_{*,n}^{-1} \rightarrow \zeta_*^{-1} \quad \text{in variable space } L^2(\Omega, \zeta_{*,n} dx). \quad (24)$$

*Proof.* Strong convergence in (22) is a direct consequence of the compactness property of  $\Psi_*$ . Hence,  $\zeta_* \in \Psi_*$  and we may assume that  $\zeta_{*,n}^{-1} \rightarrow \zeta_*^{-1}$  almost everywhere in  $\Omega$ . Since  $\zeta_{*,n} \rightarrow \zeta_*$  in  $L^1(\Omega)$  and  $\zeta_*^{-1} \leq \zeta_{ad}^{-1} \in L^1(\Omega)$ , it follows that the sequence  $\{\zeta_{*,n}^{-1}\}_{n \in \mathbb{N}}$  is equi-integrable. As a result, (23) immediately follows from Lebesgue’s Theorem (see Theorem 2.2). As for (24), we make use the following observation. For any  $\varphi \in C_0^\infty(\Omega)$ , we have

$$\zeta_n dx \xrightarrow{*} \zeta dx \text{ in } \mathcal{M}_+(\Omega), \quad \int_{\Omega} \zeta_n^{-1} \varphi \zeta_n dx = \int_{\Omega} \varphi dx = \int_{\Omega} \zeta^{-1} \varphi \zeta dx.$$

Hence,  $\zeta_n^{-1} \rightharpoonup \zeta^{-1}$  in  $L^2(\Omega_T, \zeta_n dx)$  (see [27]). Moreover, strong convergence in (23) implies the relation

$$\lim_{n \rightarrow \infty} \int_{\Omega} \zeta_n^{-2} \zeta_n dx = \lim_{n \rightarrow \infty} \int_{\Omega} \zeta_n^{-1} dx = \int_{\Omega} \zeta^{-2} \zeta dx.$$

Therefore,  $\zeta_n^{-1} \rightarrow \zeta^{-1}$  strongly in  $L^2(\Omega, \zeta_n dx)$  by the properties of strong convergence in variable spaces. The proof is complete.  $\square$

**Remark 4.2.** Note that the main assertion of Lemma 4.1 can be failed, if in definition of the set  $\Psi_*$ , instead of condition (2), we admit the following one

$$0 < \zeta_*(x) \leq \alpha \text{ a.e. in } \Omega, \quad \zeta_*^{-1} \in L^1(\Omega). \tag{25}$$

Indeed, let  $\Omega$  be the open ball in  $\mathbb{R}^N$  with the center at 0 and radius 1, let  $1 < \delta < N$ , and let  $\zeta_*(x) := \alpha \|x\|_{\mathbb{R}^N}^{\delta}$ . Then it is easy to see that  $\zeta_* \in L^1(\Omega)$  and  $0 < \zeta_*(x) \leq \alpha$  for every  $x \in \Omega \setminus 0$ . Since  $\zeta_*^{-1} = \alpha^{-1} \|x\|_{\mathbb{R}^N}^{-\delta}$  and  $\delta \in (1, N)$ , we have  $\zeta_*^{-1} \in L^1(\Omega)$  and  $\zeta_*^{-1} \notin L^\infty(\Omega)$ . Moreover,  $\zeta$  is smooth in  $\bar{\Omega} \setminus 0$  and  $\zeta_* = \alpha$  on  $\partial\Omega$ . This shows that the properties (3), (4), and (25) are satisfied.

Let us fix  $x_0 \in \Omega$  with  $\|x_0\|_{\mathbb{R}^N} = \frac{1}{2}$ . We consider the following sequence  $\{\zeta_{*,n}\}_{n \in \mathbb{N}}$  in  $L^1(\Omega)$ , where  $\zeta_{*,n} = \zeta_*$  for  $n \leq 2$  and

$$\zeta_{*,n}(x) = \begin{cases} \alpha \|x\|_{\mathbb{R}^N}^{\delta} & \text{if } \|x - x_0\|_{\mathbb{R}^N} \geq \frac{1}{n} \\ \frac{\alpha}{n^N} & \text{if } \|x - x_0\|_{\mathbb{R}^N} < \frac{1}{n}, \end{cases} \quad \text{if } n \geq 3.$$

Then each function  $\zeta_{*,n}$  satisfies the properties (3), (4), and (25). Indeed,  $\zeta_{*,n} \in L^1(\Omega)$  and  $0 < \zeta_{*,n}(x) \leq \alpha$  for every  $x \in \Omega$ . Since

$$\zeta_{*,n}^{-1}(x) = \begin{cases} \frac{1}{\alpha \|x\|_{\mathbb{R}^N}^{\delta}} & \text{if } \|x - x_0\|_{\mathbb{R}^N} \geq \frac{1}{n} \\ \frac{n^N}{\alpha} & \text{if } \|x - x_0\|_{\mathbb{R}^N} < \frac{1}{n}, \end{cases} \quad \text{for all } n \geq 3,$$

it follows that  $\zeta_{*,n}^{-1} \in L^1(\Omega)$  and  $\zeta_{*,n}^{-1} \notin L^\infty(\Omega)$ . Moreover, the functions  $\zeta_{*,n}$  are smooth near  $\partial\Omega$  and  $\zeta_{*,n} = \alpha$  on  $\partial\Omega$ . This shows that the properties (3), (4), and (25) are satisfied.

It is clear that  $\zeta_{*,n} \rightarrow \zeta_*$  strongly in  $L^1(\Omega)$  and pointwise a.e. in  $\Omega$ . The problem is that the sequence  $\{\zeta_{*,n}^{-1}\}_{n \in \mathbb{N}}$  does not converge to  $\zeta_*^{-1}$  strongly in  $L^1(\Omega)$ . Indeed, it is the case when the sequence  $\{\zeta_{*,n}^{-1}\}_{n \in \mathbb{N}}$  is not equi-integrable. As a result, we have

$$\int_{\Omega} |\zeta_{*,n}^{-1} - \zeta_*^{-1}| dx = \int_{B(x_0, \frac{1}{n})} \left| \frac{n^N}{\alpha} - \frac{1}{\alpha \|x\|_{\mathbb{R}^N}^{\delta}} \right| dx \rightarrow \alpha^{-1} \omega_N \quad \text{as } n \rightarrow \infty,$$

where  $B(x_0, \frac{1}{n})$  is an open ball with center at  $x_0$  and radius  $\frac{1}{n}$ , while  $\omega_N$  is the Lebesgue measure of the unit ball in  $\mathbb{R}^N$ .

For our further analysis, we make use of the following concept.

**Definition 4.3.** We say that a bounded sequence

$$\{(A_n, u_n) \in L^1(\Omega; \mathbb{S}^N) \times W_{A_n}(\Omega; \Gamma_D)\}_{n \in \mathbb{N}} \tag{26}$$

w-converges to  $(A, u) \in L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega)$  as  $n \rightarrow \infty$  (in symbols,  $(A_n, y_n) \xrightarrow{w} (A, y)$ ) if

$$A_n \rightarrow A \quad \text{in } L^1(\Omega; \mathbb{S}^N), \tag{27}$$

$$u_n \rightharpoonup u \quad \text{in } L^2(\Omega), \tag{28}$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in the variable space } L^2(\Omega, A_n dx)^N. \tag{29}$$

In particular, as follows from this definition, if  $(A_n, u_n) \xrightarrow{w} (A, u)$ , then

$$\lim_{n \rightarrow \infty} \int_{\Omega} A_n \eta dx = \int_{\Omega} A \eta dx \quad \forall \eta \in L^\infty(\Omega; \mathbb{S}^N), \tag{30}$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} u_n \lambda dx = \int_{\Omega} u \lambda dx \quad \forall \lambda \in L^2(\Omega), \tag{31}$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\xi, A_n \nabla u_n)_{\mathbb{R}^N} dx = \int_{\Omega} (\xi, A \nabla u)_{\mathbb{R}^N} dx \quad \forall \xi \in C_0^\infty(\Omega)^N. \tag{32}$$

In order to motivate this definition, we give the following result.

**Lemma 4.4.** Let  $\{(A_n, u_n) \in L^1(\Omega; \mathbb{S}^N) \times W_{A_n}(\Omega; \Gamma_D)\}_{n \in \mathbb{N}}$  be a sequence such that

- (i) the sequence  $\{u_n \in W_{A_n}(\Omega; \Gamma_D)\}_{n \in \mathbb{N}}$  is bounded, i.e.

$$\sup_{n \in \mathbb{N}} \int_{\Omega} (u_n^2 + (\nabla u_n, A_n \nabla u_n)) dx < +\infty; \tag{33}$$

- (ii)  $\{A_n\}_{n \in \mathbb{N}} \subset \mathfrak{M}_\alpha^\beta(\Omega)$  and there exists a matrix-valued function  $A(x) \in \mathbb{S}^N$  such that

$$A_n \rightarrow A \quad \text{and} \quad A_n^{-1} \rightarrow A^{-1} \quad \text{in } L^1(\Omega; \mathbb{S}^N) \quad \text{as } n \rightarrow \infty. \tag{34}$$

Then,  $A \in \mathfrak{M}_\alpha^\beta(\Omega) \cap L^1(\Omega; \mathbb{S}^N)$  and withing a subsequence the original sequence is w-convergent. Moreover, each w-limit pair  $(A, u)$  belongs to the space  $L^1(\Omega; \mathbb{S}^N) \times W_A(\Omega; \Gamma_D)$ .

*Proof.* We note that (33), (34) and (10), (11) immediately imply the boundedness of the original sequence in  $L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega; S)$ . Moreover, due to (34), we have (see the suppositions (13)–(15) of Section 3):

$$d\mu_n := A_n dx \xrightarrow{*} A dx =: d\mu \quad \text{in } \mathcal{M}(\Omega; \mathbb{S}^N).$$

Thus, the compactness criterium for weak convergence in variable spaces (see Proposition 3.2) and (33) imply the existence of a pair  $(u, v) \in L^2(\Omega) \times L^2(\Omega, A dx)^N$  such that, within a subsequence of  $\{u_n\}_{n \in \mathbb{N}}$ ,

$$u_n \rightharpoonup u \quad \text{in } L^2(\Omega), \tag{35}$$

$$\nabla u_n \rightharpoonup v \quad \text{in variable space } L^2(\Omega, A_n dx)^N. \tag{36}$$

Our aim is to show that  $A \in \mathfrak{M}_\alpha^\beta(\Omega)$ ,  $v = \nabla u$ , and  $u \in W_A(\Omega; \Gamma_D)$ . It is clear that  $A(x) \in \mathbb{S}^N$  and this matrix satisfies (5). Since  $A_n \in \mathfrak{M}_\alpha^\beta(\Omega) \cap L^1(\Omega; \mathbb{S}^N)$  for all  $n \in \mathbb{N}$ , it follows that there is a sequence  $\{\zeta_{*,n}\}_{n \in \mathbb{N}}$  in  $\Psi_*$  such that

$$\zeta_{*,n}(x)I \leq A_n(x)I \leq \beta(x)I \quad \text{a.e. in } \Omega, \quad \forall k \in \{1, \dots, N\}. \tag{37}$$

Then, by  $L^1$ -compactness of the set  $\Psi_*$ , there exists an element  $\zeta_* \in \Psi_*$  such that  $\zeta_{*,n} \rightarrow \zeta_*$  in  $L^1(\Omega)$  as  $n \rightarrow \infty$ . Moreover, Lemma 4.1 implies strong convergence

$$\zeta_{*,n}^{-1} \rightarrow \zeta_*^{-1} \quad \text{in } L^1(\Omega), \tag{38}$$

and (2)–(4). Hence, passing to the limit in (37) as  $n \rightarrow \infty$ , we come to (6). Thus,  $A \in \mathfrak{M}_\alpha^\beta(\Omega)$  and the limit matrix  $A(x) \in \mathbb{S}^N$  satisfies (7), (8).

For our further analysis, we fix any test function  $\varphi \in C_0^\infty(\Omega)^N$ , and make use of the following equality

$$\int_\Omega (A_n^{-1}\varphi, A_n\psi)_{\mathbb{R}^N} dx = \int_\Omega (\varphi, \psi)_{\mathbb{R}^N} dx = \int_\Omega (A^{-1}\varphi, A\psi)_{\mathbb{R}^N} dx, \tag{39}$$

which is obviously true for each  $\psi \in C_0^\infty(\Omega)^N$  and for all  $n \in \mathbb{N}$ . Since

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_\Omega (A_n^{-1}\varphi, A_n A_n^{-1}\varphi)_{\mathbb{R}^N} dx &= \limsup_{n \rightarrow \infty} \int_\Omega (\varphi, A_n^{-1}\varphi)_{\mathbb{R}^N} dx \\ &\leq \limsup_{n \rightarrow \infty} \int_\Omega \zeta_{*,n}^{-1} \|\varphi\|_{\mathbb{R}^N}^2 dx \\ &\stackrel{\text{by (38)}}{=} \int_\Omega \zeta_*^{-1} \|\varphi\|_{\mathbb{R}^N}^2 dx \\ &\leq \|\varphi\|_{C(\Omega)^N}^2 \|\zeta_*^{-1}\|_{L^1(\Omega)} < +\infty, \end{aligned}$$

it follows that the sequence  $\{A_n^{-1}\varphi \in L^2(\Omega, A_n dx)^N\}_{n \in \mathbb{N}}$  is bounded. Consequently, combining this fact with (39), we conclude  $A_n^{-1}\varphi \rightharpoonup A^{-1}\varphi$  in the variable space  $L^2(\Omega, A_n dx)^N$  (see Definition 3.1). At the same time, strong convergence in (34) implies the relation  $\lim_{n \rightarrow \infty} \int_\Omega (A_n^{-1}\varphi, A_n A_n^{-1}\varphi)_{\mathbb{R}^N} dx = \lim_{n \rightarrow \infty} \int_\Omega (\varphi, A_n^{-1}\varphi)_{\mathbb{R}^N} dx = \int_\Omega (\varphi, A^{-1}\varphi)_{\mathbb{R}^N} dx = \int_\Omega (A^{-1}\varphi, A A^{-1}\varphi)_{\mathbb{R}^N} dx$ . Hence (see Proposition 3.6),

$$A_n^{-1}\varphi \rightarrow A^{-1}\varphi \quad \text{strongly in } L^2(\Omega, A_n dx)^N \quad \forall \varphi \in C_0^\infty(\Omega)^N. \tag{40}$$

Further, we note that for every measurable subset  $K \subset \Omega$ , the estimate

$$\begin{aligned} \int_K \|\nabla u_n\|_{\mathbb{R}^N} dx &\leq \left( \int_K \|\nabla u_n\|_{\mathbb{R}^N}^2 \zeta_{*,n} dx \right)^{\frac{1}{2}} \left( \int_K \zeta_{*,n}^{-1} dx \right)^{\frac{1}{2}} \\ &\leq \left( \int_{\Omega} (\nabla u_n, A_n(x) \nabla u_n)_{\mathbb{R}^N} dx \right)^{\frac{1}{2}} \left( \int_K \zeta_{*,n}^{-1} dx \right)^{\frac{1}{2}} \\ &\leq C \left( \int_K \zeta_{*,n}^{-1} dx \right)^{\frac{1}{2}} \end{aligned}$$

implies equi-integrability of the family  $\{\|\nabla u_n\|_{\mathbb{R}^N}\}_{n \in \mathbb{N}}$ . Hence,  $\{\|\nabla u_n\|_{\mathbb{R}^N}\}_{n \in \mathbb{N}}$  is weakly compact in  $L^1(\Omega)$ , which means the weak compactness of the vector-valued sequence  $\{\nabla u_n\}_{n \in \mathbb{N}}$  in  $L^1(\Omega; \mathbb{R}^N)$ . As a result, by the properties of the strong convergence in variable spaces, we obtain

$$\begin{aligned} \int_{\Omega} (\xi, \nabla u_n)_{\mathbb{R}^N} dx &= \int_{\Omega} (A_n^{-1} \xi, A_n \nabla u_n)_{\mathbb{R}^N} dx \\ \text{by (20), (36), and (40)} \int_{\Omega} (A^{-1} \xi, Av)_{\mathbb{R}^N} dx &= \int_{\Omega} (\xi, v)_{\mathbb{R}^N} dx \quad \forall \xi \in C_0^\infty(\Omega)^N. \end{aligned}$$

Thus, in view of the weak compactness property of  $\{\nabla u_n\}_{n \in \mathbb{N}}$  in  $L^1(\Omega; \mathbb{R}^N)$ , we conclude

$$\nabla u_n \rightharpoonup v \text{ in } L^1(\Omega; \mathbb{R}^N) \quad \text{as } n \rightarrow \infty. \tag{41}$$

Since  $u_n \in W^{1,1}(\Omega; \Gamma_D)$  for all  $n \in \mathbb{N}$  and the Sobolev space  $W^{1,1}(\Omega; \Gamma_D)$  is complete, (35) and (41) imply  $\nabla u = v$ , and consequently  $u \in W^{1,1}(\Omega; \Gamma_D)$ . To end the proof, it remains to observe that (35), (36) guarantee the finiteness of the norm  $\|u\|_A$  (see (9)). Hence,  $u \in W_A(\Omega; \Gamma_D)$  and this concludes the proof.  $\square$

### 5. Setting of the optimal control problem

Let  $M \in \mathbb{S}^N$  be a given constant matrix satisfying the condition

$$(M\xi, \xi)_{\mathbb{R}^N} \geq m \|\xi\|_{\mathbb{R}^N}^2 \quad \text{for some } m > 0.$$

Let  $Q$  be a closed nonempty subdomain of  $\Omega$  for which  $\text{dist}(\partial\Omega, \partial Q) \geq \delta > 0$ , where  $\delta$  is a prescribed value. Let  $B \in L^\infty(Q; \mathbb{S}^N)$  be a given matrix-valued function such that

$$\sigma_1 \|\xi\|_{\mathbb{R}^N}^2 \leq (B(x)\xi, \xi)_{\mathbb{R}^N} \leq \sigma_2 \|\xi\|_{\mathbb{R}^N}^2 \quad \text{a.e. in } Q \quad \forall \xi \in \mathbb{R}^N$$

with some  $\sigma_2 > \sigma_1 > 0$ .

Let  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma_N)$  be given functions. We consider the following boundary value problem

$$-\operatorname{div}(A(x)\nabla y) = f \quad \text{in } \Omega, \tag{42}$$

$$y = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial y}{\partial \nu_A} = g \quad \text{on } \Gamma_N. \tag{43}$$

Here

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial y}{\partial x_j} \cos(n, x_i),$$

$\cos(n, x_i)$  is  $i$ -th directional cosine of  $n$ , and  $n$  is the outward unit normal at  $\Gamma_N$  to  $\Omega$ .

To introduce the class of admissible controls in coefficients, we adopt the following concept:

**Definition 5.1.** We say that a matrix-valued function  $A = A(x) \in \mathbb{S}^N$  is an admissible control for the boundary value problem (42), (43) (it is written as  $A \in \mathfrak{A}_{ad}$ ) if

$$A \in BV(\Omega \setminus Q; \mathbb{S}^N), \quad \int_{\Omega \setminus Q} A(x) dx = M, \tag{44}$$

$$A \in \mathfrak{M}_\alpha^\beta(\Omega \setminus Q), \quad A(x) = B(x) \text{ a.e. in } Q. \tag{45}$$

Hereafter we assume that the set  $\mathfrak{A}_{ad}$  is nonempty. Moreover, it is easy to see that for a given  $B \in L^\infty(Q; \mathbb{S}^N)$ , we can always guarantee the fulfilment of condition  $\mathfrak{A}_{ad} \neq \emptyset$  by an appropriate choice of the matrix  $M \in \mathbb{S}^N$  and functions  $\zeta_{ad} \in L^1(\Omega)$  and  $\beta \in L^1(\Omega)$ .

**Remark 5.2.** As immediately follows from definition of the set  $\mathfrak{M}_\alpha^\beta(\Omega \setminus Q)$  and the properties of the matrix  $B$ , (45) implies

$$A \in \mathfrak{M}_\alpha^{\tilde{\beta}}(\Omega) \cap L^1(\Omega; \mathbb{S}^N) \quad \text{with } \tilde{\beta} = \max\{\beta, \sigma_2\}.$$

**Remark 5.3.** In view of (45)<sub>1</sub> and (6) (see also Remark 2.1), we deal with a boundary value problem for the degenerate elliptic equation. It means that for some admissible controls  $A \in \mathfrak{A}_{ad}$  the boundary value problem (42), (43) can exhibit the Lavrentieff phenomenon and nonuniqueness of the weak solutions.

**Definition 5.4.** We say that a function  $y = y(A, f, g)$  is a weak solution to the boundary value problem (42), (43) for a fixed control  $A \in \mathfrak{A}_{ad}$  and given functions  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma_N)$  if  $y \in W_A(\Omega; \Gamma_D)$  and the integral identity

$$\int_{\Omega} (\nabla \varphi, A(x)\nabla y)_{\mathbb{R}^N} dx = \int_{\Omega} f \varphi dx + \int_{\Gamma_N} g \varphi d\mathcal{H}^{N-1} \tag{46}$$

holds for any  $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$ .

**Remark 5.5.** It is worth to notice that the original boundary value problem (42), (43) is ill-posed, in general. This means that there are no reasons to expect that for every admissible given data  $f \in L^2(\Omega)$ ,  $g \in L^2(\Gamma_N)$ , and  $A \in \mathfrak{A}_{ad}$ , this problem admits at least one weak solution  $y \in W_A(\Omega; \Gamma_D)$  in the sense of Definition 5.4. So, it is not possible to write in this case  $y = y(A, f, g)$ . Moreover, it should be emphasized that, to the best knowledge of authors, the existence of a weak solution to (42), (43) for fixed  $A \in \mathfrak{A}_{ad}$ ,  $f \in L^2(\Omega)$ , and  $g \in L^2(\Gamma_N)$  is an open question. On the other hand, even if a weak solution to the above problem exists, the question about its uniqueness leads us to the problem of density of the subspace of smooth functions  $C_0^\infty(\Omega; \Gamma_D)$  in  $W_A(\Omega; \Gamma_D)$ . However, as was indicated in [28], there exists a diagonal matrix-valued function  $A(x) = \rho(x)I$  with  $\rho \in \Psi_*$  such that the subspace  $C_0^\infty(\Omega; \Gamma_D)$  is not dense in  $W_A(\Omega; \Gamma_D)$ , and, hence, there is no uniqueness of weak solutions (for more details and other types of solutions we refer to [3, 26, 28]). Thus, the mapping  $A \mapsto y(A, f, g)$  can be multivalued, in general.

To avoid this situation in our analysis, we introduce the set of admissible solutions to the original optimal control problem as follows:

$$\Xi_w = \{(A, y) \mid A \in \mathfrak{A}_{ad}, y \in W_A(\Omega; \Gamma_D), (A, y) \text{ are related by (46)}\}. \quad (47)$$

In what follows, we make use the following result:

**Proposition 5.6.** *Let  $A \in \mathfrak{A}_{ad}$  be a given matrix-valued function. Then there exist bounded linear operators*

$$\gamma_A^0 : W_A(\Omega; \Gamma_D) \rightarrow H^{\frac{1}{2}}(\partial\Omega) \quad \text{and} \quad \gamma_A^1 : W_A(\Omega; \Gamma_D) \rightarrow H^{-\frac{1}{2}}(\partial\Omega) \quad (48)$$

such that

- (i)  $\gamma_A^0(y) = y|_{\partial\Omega}$  and  $\gamma_A^1(y) = \frac{\partial y}{\partial \nu_A} \Big|_{\partial\Omega}$  provided  $y \in W_A(\Omega; \Gamma_D) \cap C(\bar{\Omega})$ ;
- (ii) for any function  $y \in W_A(\Omega; \Gamma_D)$

$$\|\gamma_A^0(y)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|y\|_{W_A(\Omega; \Gamma_D)}, \quad \|\gamma_A^1(y)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C_1\|y\|_{W_A(\Omega; \Gamma_D)}$$

with the positive constants  $C$  and  $C_1$  independent of  $y$ .

*Proof.* To begin with, we note that the matrix  $A \in \mathfrak{A}_{ad}$  belongs to  $\mathfrak{M}_\alpha^\beta(\Omega \setminus Q)$ . Hence, it can be associated with an element  $\zeta_*$  of  $\Psi$  such that  $A(x) \geq \zeta_* I$  and  $\zeta_{ad} < \zeta_* \leq \alpha$  almost everywhere in  $\Omega \setminus Q$  (see (2)–(4)). Moreover, in this case  $\zeta_* : \Omega \setminus Q \rightarrow [0, \alpha]$  is a smooth function along the surface  $\partial\Omega$  satisfying the condition  $\zeta_* = \alpha$  on  $\partial\Omega$ . Hence, there exist an open set  $\mathcal{O}$  with Lipschitz continuous boundary and a positive constant  $\tilde{\alpha} \leq \alpha$  such that

$$\mathcal{O} \subset \Omega \setminus Q, \quad \partial\Omega \subset \partial\mathcal{O}, \quad \text{and} \quad \zeta_* \geq \tilde{\alpha} \text{ a.e. in } \mathcal{O}.$$



As a result, for any element  $y \in W_A(\Omega; \Gamma_D)$ , we have  $y \in W_A(\mathcal{O}; \Gamma_D)$ , and, therefore,

$$\begin{aligned} \|y\|_{W_A(\Omega; \Gamma_D)}^2 &\geq \|y\|_{W_A(\mathcal{O}; \Gamma_D)}^2 \\ &= \int_{\mathcal{O}} (y^2 + (\nabla y, A(x)\nabla y)_{\mathbb{R}^N}) dx \\ &\geq \min\{1, \tilde{\alpha}\} \int_{\mathcal{O}} (y^2 + \|\nabla y\|_{\mathbb{R}^N}^2) dx. \end{aligned}$$

Thus,  $y \in W^{1,2}(\mathcal{O})$ , and, therefore, the existence of the trace operators  $\gamma_A^0$  and  $\gamma_A^1$  with (i), (ii) immediately follows from the Trace Sobolev Theorem (see [20, Section 3.]) □

As an evident consequence of this result, we can give the following observation.

**Corollary 5.7.** *Let  $\{(A_n, y_n) \in L^1(\Omega; \mathbb{S}^N) \times W_{A_n}(\Omega; \Gamma_D)\}_{n \in \mathbb{N}}$  be a sequence such that  $\{A_n\}_{n \in \mathbb{N}} \subset \mathfrak{M}_\alpha^\beta(\Omega)$  and  $(A_n, y_n) \xrightarrow{w} (A, y)$  in the sense of Definition 4.3, where  $(A, y) \in L^1(\Omega; \mathbb{S}^N) \times W_A(\Omega; \Gamma_D)$ . Then, up to a subsequence, we have*

$$\frac{\partial y_n}{\partial \nu_{A_n}} \rightharpoonup \frac{\partial y}{\partial \nu_A} \quad \text{in } H^{-\frac{1}{2}}(\Gamma_D).$$

The optimal control problem we consider here is to minimize the discrepancies (tracking error) between given distributions  $y_d \in L^2(\Omega)$ ,  $y^* \in L^2(\Gamma_D)$  and the solution of boundary valued problem (42), (43) by choosing an appropriate coefficients matrix  $A \in \mathfrak{A}_{ad}$ . More precisely, we are concerned with the following optimal control problem

$$\begin{aligned} \text{Minimize } \left\{ I(A, y) = \int_{\Omega} |y(x) - y_d(x)|^2 dx + \int_{\Omega} (\nabla y(x), A(x)\nabla y(x))_{\mathbb{R}^N} dx \right. \\ \left. + \sum_{i,j=1}^N \int_{\Omega \setminus Q} |D a_{ij}(x)| + \left\| \frac{\partial y}{\partial \nu_A} - y^* \right\|_{H^{-\frac{1}{2}}(\Gamma_D)}^2 \right\} \end{aligned} \quad (49)$$

subject to the constraints (42)–(45).

**Remark 5.8.** The second term in (49) plays a special role in this problem. Its appearance in the cost function (49) is motivated by the fact that there are no appropriate a priori estimates in the  $W_A(\Omega; \Gamma_D)$ -norm for weak solutions  $y = y(A, f, g)$  of the degenerate boundary value problem (42), (43) (in the sense of Definition 5.4). Hence, the term  $\int_{\Omega} (\nabla y(x), A(x)\nabla y(x))_{\mathbb{R}^N} dx$  together with the first one in (49) ensures the coercivity of the cost function on the space of weak solutions  $W_A(\Omega; \Gamma_D)$ .

**Remark 5.9.** Note that due to (10), (11), we have the following obvious inclusion for the set of admissible solutions

$$\Xi_w \subset L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega; \Gamma_D).$$

However, the characteristic feature of this set is the fact that for different admissible controls  $A \in \mathfrak{A}_{ad}$  the corresponding admissible solutions  $y$  of optimal control problem (49) belong to different weighted spaces. It is a non-typical situation from the point of view of classical optimal control theory.

It is worth noticing that for any admissible given data  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma_N)$ , verification of  $\Xi_w \neq \emptyset$  is a non-trivial matter, in general. In the particular case, when the set of admissible controls  $\mathfrak{A}_{ad}$  possesses the property:

$$A \in L^\infty(\Omega; \mathbb{S}^N), \quad A(x) \geq \nu I \text{ a.e. in } \Omega \quad \forall A \in \mathfrak{A}_{ad},$$

$\Xi_w \neq \emptyset$  is obvious since the corresponding boundary value problem (42), (43) has a unique weak solution  $y = y(A)$ . Therefore, we adopt the following hypothesis, which is mainly motivated by Remark 5.5.

**Hypothesis A.** *The set of admissible solutions  $\Xi_w$  is nonempty.*

We say that a pair  $(A^0, y^0) \in L^1(\Omega; \mathbb{S}^N) \times W_A(\Omega; \Gamma_D)$  is optimal for problem (49) on the class  $\Xi_w$  (or shortly, weakly optimal), if

$$(A^0, y^0) \in \Xi_w \quad \text{and} \quad I(A^0, y^0) = \inf_{(A,y) \in \Xi_w} I(A, y). \quad (50)$$

## 6. Existence of weak optimal solutions

Since our prime interest is the solvability of optimal control problem (49), we begin with the study of the topological properties of the set of admissible solutions  $\Xi_w$ . To do so, we give a some auxiliary results.

**Definition 6.1.** We say that a sequence  $\{(A_n, y_n) \in \Xi_w\}_{n \in \mathbb{N}}$  is bounded if

$$\sup_{n \in \mathbb{N}} \left[ \|A_n\|_{BV(\Omega \setminus Q; \mathbb{S}^N)} + \|y_n\|_{A_n} \right] < +\infty.$$

**Lemma 6.2.** *Let  $\{(A_n, y_n) \in \Xi_w\}_{n \in \mathbb{N}}$  be a bounded sequence in the sense of Definition 6.1. Then there exists a pair  $(A, y) \in L^1(\Omega; \mathbb{S}^N) \times W^{1,1}(\Omega; \Gamma_D)$  such that, up to a subsequence,*

$$(A_n, y_n) \xrightarrow{w} (A, y) \text{ see Definition 4.3, } A \in \mathfrak{A}_{ad}, \quad \text{and } y \in W_A(\Omega; \Gamma_D). \quad (51)$$

*Proof.* By the compactness result for  $BV$ -functions (see Proposition 2.6), there exists a subsequence of  $\{A_n\}_{n \in \mathbb{N}}$ , still denoted by the same indices, and a matrix  $A \in BV(\Omega \setminus Q; \mathbb{S}^N)$  such that  $A_n \rightarrow A$  in  $L^1(\Omega \setminus Q; \mathbb{S}^N)$ . Since  $A_n(x) = B(x)$  a.e. in  $Q$  and  $B \in L^\infty(Q; \mathbb{S}^N)$ , it follows that the strong  $L^1$ -convergence  $A_n \rightarrow A$  can be extended to the entire domain  $\Omega$ . Thus,

$$A \in BV(\Omega \setminus Q; \mathbb{S}^N), \quad A(x) = B(x) \text{ a.e. in } Q, \tag{52}$$

$$\int_{\Omega \setminus Q} A(x) dx = \lim_{n \rightarrow \infty} \int_{\Omega \setminus Q} A_n(x) dx = M, \tag{53}$$

and the condition (27) of Definition 4.3 holds true. In order to check the remaining conditions (28), (29) of this definition and to show that  $A \in \mathfrak{A}_{ad}$ , we make use of the following observation.

We have  $(A_n, y_n) \in \Xi_w$  for all  $n \in \mathbb{N}$ . Hence, there is a sequence  $\{\zeta_{*,n}\}_{n \in \mathbb{N}}$  in  $\Psi_*$  such that (see Lemma 4.1 for the details)  $\zeta_{*,n} \rightarrow \zeta_*$  and  $\zeta_{*,n}^{-1} \rightarrow \zeta_*^{-1}$  in  $L^1(\Omega)$  as  $n \rightarrow \infty$ . Moreover, by properties of  $\Psi_*$ , the  $L^1$ -limit element  $\zeta_*$  satisfies (2)–(4). Then, in view of  $L^1$ -convergence  $A_n \rightarrow A$ , we may assume that  $A_n^{-1} \rightarrow A^{-1}$  almost everywhere in  $\Omega$ . Since  $A_n(x) \geq \zeta_{*,n} I$  a. e. in  $\Omega$ , it follows that

$$\int_K (\xi, A_n^{-1} \xi)_{\mathbb{R}^N} dx \leq \int_K \zeta_{*,n}^{-1} dx \|\xi\|_{\mathbb{R}^N}^2 \quad \forall n \in \mathbb{N}$$

for any subset  $K \subset \Omega$ . Hence, due to the strong  $L^1$ -convergence  $\zeta_{*,n}^{-1} \rightarrow \zeta_*^{-1}$ , the sequence  $\{A_n^{-1}\}_{n \in \mathbb{N}}$  is equi-integrable. Then, by Lebesgue’s Theorem (see Theorem 2.2) we obtain  $A_n^{-1} \rightarrow A^{-1}$  in  $L^1(\Omega; \mathbb{S}^N)$  as  $n \rightarrow \infty$ . As a result,

$$A \in \mathfrak{M}_\alpha^\beta(\Omega \setminus Q) \cap L^1(\Omega; \mathbb{S}^N)$$

by Lemma 4.4. Combining this fact with properties (52), (53), we conclude  $A \in \mathfrak{A}_{ad}$ .

To end of this proof, it remains to observe that the remaining conditions (28), (29) of Definition 4.3 and  $y \in W_A(\Omega; \Gamma_D)$  for the  $w$ -limiting component  $(A, y)$  of the sequence  $\{(A_n, y_n)\}_{n \in \mathbb{N}}$ , are ensured by Lemma 4.4. This concludes the proof. □

Our next step deals with the study of topological properties of the set of admissible solutions  $\Xi_w$  to the problem (49). The following theorem is crucial for our next analysis.

**Theorem 6.3.** *Assume that the Hypothesis A is valid. Then for any admissible given data  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma_N)$ , the set of admissible solutions  $\Xi_w$  is sequentially closed with respect to  $w$ -convergence.*

*Proof.* Let  $\{(A_n, y_n) \in \Xi_w\}_{n \in \mathbb{N}}$  be a bounded  $w$ -convergent sequence of admissible solutions to the optimal control problem (49). Let  $(\widehat{A}, \widehat{y})$  be its  $w$ -limit. Our aim is to prove that  $(\widehat{A}, \widehat{y}) \in \Xi_w$ . By Lemma 6.2, we have:  $\widehat{A} \in \mathfrak{A}_{ad}$  and  $\widehat{y} \in W_{\widehat{A}}(\Omega; \Gamma_D)$ .

It remains to show that the pair  $(\widehat{A}, \widehat{y})$  is related to (46) for all  $\varphi \in C_0^\infty(\Omega; \Gamma_D)$ . To do so, we note that for every  $n \in \mathbb{N}$ , the integral identity (46) (with  $A_n$  and  $y_n$  instead of  $A$  and  $y$ , respectively), has to be satisfied for the test functions  $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$ . Then  $\nabla\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)^N$ . However, this class is essentially wider than the space  $C_0^\infty(\Omega)^N$  in the definition of the weak convergence in the variable space  $L^2(\Omega, A_n dx)^N$  (see (16)). Therefore, in order to pass to the limit in that integral identity as  $n \rightarrow \infty$ , we make use the following argument (see Buttazzo & Kogut [6]).

For every fixed  $n \in \mathbb{N}$  we denote by

$$(\widetilde{A}_n, \widetilde{y}_n) \in BV_{loc}(\mathbb{R}^N \setminus Q) \times W_{loc}^{1,1}(\mathbb{R}^N; \Gamma_D)$$

an extension of the functions  $(A_n, y_n)$  to the whole of space  $\mathbb{R}^N$  such that the sequence  $\{(\widetilde{A}_n, \widetilde{y}_n)\}_{n \in \mathbb{N}}$  satisfies the properties:

$$\widetilde{A} \in \mathfrak{M}_\alpha^\beta(D \setminus Q), \quad \widetilde{A}(x) = B(x) \quad \text{a.e. in } Q, \tag{54}$$

$$\widetilde{A}(x) \leq \widetilde{\beta}(x)I \quad \text{a.e. in } D \setminus Q, \tag{55}$$

$$\widetilde{\zeta}_* I \leq \widetilde{A}(x) \quad \text{a.e. in } D \setminus Q, \tag{56}$$

$$\widetilde{\zeta}_*^{-1} \in L^1(D \setminus Q), \quad \widetilde{\zeta}_{ad} < \widetilde{\zeta}_* \leq \alpha \quad \text{a.e. in } D \setminus Q, \tag{57}$$

$$\sup_{n \in \mathbb{N}} \left[ \|\widetilde{A}_n\|_{BV(D \setminus Q; \mathbb{S}^N)} + \|\widetilde{y}_n\|_{L^2(D)} + \|\nabla\widetilde{y}_n\|_{L^2(D, \widetilde{A}_n dx)^N} \right] < +\infty \tag{58}$$

for any bounded open domain  $D$  in  $\mathbb{R}^N$  such that  $\Omega \subset D$ .

Here  $\widetilde{\beta} \in L_{loc}^1(\mathbb{R}^N)$ ,  $\widetilde{\zeta}_{ad} \in L_{loc}^1(\mathbb{R}^N)$ , and  $\widetilde{\zeta}_* \in L_{loc}^1(\mathbb{R}^N)$  are non-negative functions such that  $\widetilde{\beta}|_\Omega = \beta$ ,  $\widetilde{\zeta}_{ad}|_\Omega = \zeta_{ad}$ , and  $\widetilde{\zeta}_*|_\Omega \in \Psi_*$ , respectively.

Then by analogy with Lemmas 4.4 and 6.2, it can be proved that for every bounded domain  $D \subset \mathbb{R}^N$  there exists a matrix  $\widetilde{A} \in BV(D \setminus Q; \mathbb{S}^N)$  and a function  $\widetilde{y} \in W_{\widetilde{A}}(D; \Gamma_D)$  such that

$$\widetilde{A}_n \rightarrow \widetilde{A} \text{ in } L^1(D), \quad \widetilde{y}_n \rightarrow \widetilde{y} \text{ in } L^2(D), \tag{59}$$

$$\nabla\widetilde{y}_n \rightharpoonup \nabla\widetilde{y} \text{ in the variable space } L^2(D, \widetilde{A}_n dx)^N. \tag{60}$$

It is important to note that in this case we have

$$\widetilde{y} = \widehat{y} \quad \text{and} \quad \widetilde{A} = \widehat{A} \quad \text{a.e. in } \Omega. \tag{61}$$

Taking this fact into account, we can rewrite (46) in the equivalent form

$$\int_{\mathbb{R}^N} (\nabla\varphi, \tilde{A}_n(x)\nabla\tilde{y}_n)_{\mathbb{R}^N} \chi_\Omega dx = \int_{\mathbb{R}^N} f\varphi\chi_\Omega dx + \int_{\Gamma_N} g\varphi d\mathcal{H}^{N-1} \quad (62)$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D)$ . In what follows, we note that due to (59), we have  $\tilde{A}_n \rightarrow \tilde{A}$  strongly in  $L^1_{loc}(\mathbb{R}^N; \mathbb{S}^N)$ . Hence,

$$\begin{aligned} \int_{\mathbb{R}^N} (\chi_\Omega \xi, \tilde{A}_n \chi_\Omega \xi)_{\mathbb{R}^N} dx &= \int_{\mathbb{R}^N} (\xi, \tilde{A}_n \xi)_{\mathbb{R}^N} \chi_\Omega dx \\ &\longrightarrow \int_{\mathbb{R}^N} (\xi, \tilde{A} \xi)_{\mathbb{R}^N} \chi_\Omega dx = \int_{\mathbb{R}^N} (\chi_\Omega \xi, \tilde{A} \chi_\Omega \xi)_{\mathbb{R}^N} dx \end{aligned} \quad (63)$$

for any vector  $\xi \in \mathbb{R}^N$ . As follows from convergence property (21), (63) implies strong convergence  $\chi_\Omega \xi \rightarrow \chi_\Omega \xi$  in the variable space  $L^2(\mathbb{R}^N, \tilde{A}_n dx)^N$ . Taking (59), (60) into account, we can pass to the limit in (62) as  $n \rightarrow \infty$  and obtain

$$\int_{\mathbb{R}^N} (\nabla\varphi, \tilde{A}(x)\nabla\tilde{y})_{\mathbb{R}^N} \chi_\Omega dx = \int_{\mathbb{R}^N} f\varphi\chi_\Omega dx + \int_{\Gamma_N} g\varphi d\mathcal{H}^{N-1}$$

which, due to (61), is equivalent to

$$\int_{\Omega} (\nabla\varphi, \hat{A}(x)\nabla\hat{y})_{\mathbb{R}^N} dx = \int_{\Omega} f\varphi dx + \int_{\Gamma_N} g\varphi d\mathcal{H}^{N-1} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_D).$$

Hence,  $\hat{y} \in W_{\hat{A}}(\Omega; \Gamma_D)$  is a weak solution to (42), (43) under  $A = \hat{A}$  in the sense of Definition 5.4. Thus, the  $w$ -limit pair  $(\hat{A}, \hat{y})$  belongs to set  $\Xi_w$ , and this concludes the proof.  $\square$

We are now in a position to state the existence of weak optimal solution to the problem (49).

**Theorem 6.4.** *Let  $f \in L^2(\Omega)$ ,  $g \in L^2(\Gamma_N)$ ,  $y_d \in L^2(\Omega)$ , and  $y^* \in L^2(\Gamma_D)$  be given functions. Assume that the Hypothesis A is valid. Then the optimal control problem (49) admits at least one solution*

$$(A^0, y^0) \in L^1(\Omega; \mathbb{S}^N) \times W_{A^0}(\Omega; \Gamma_D).$$

*Proof.* Since the cost functional  $I = I(A, y)$  is bounded below and  $\Xi_w \neq \emptyset$ , it provides the existence of a minimizing sequence  $\{(A_n, y_n) \in \Xi_w\}_{n \in \mathbb{N}}$  to the problem (50). Then,

$$\begin{aligned} \inf_{(A,y) \in \Xi_w} I(A, y) &= \lim_{n \rightarrow \infty} I(A_n, y_n) \\ &= \lim_{n \rightarrow \infty} \left[ \int_{\Omega} |y_n(x) - y_d(x)|^2 dx + \int_{\Omega} (\nabla y_n(x), A(x)\nabla y_n(x))_{\mathbb{R}^N} dx \right. \\ &\quad \left. + \sum_{i,j=1}^N \int_{\Omega \setminus Q} |D a_{ij}^n(x)| dx + \left\| \frac{\partial y_n}{\partial \nu_{A_n}} - y^* \right\|_{H^{-\frac{1}{2}}(\Gamma_D)}^2 \right] \\ &< +\infty \end{aligned} \quad (64)$$

implies the existence of a constant  $C > 0$  such that

$$\sup_{n \in \mathbb{N}} \|\nabla y_n\|_{L^2(\Omega, A_n dx)^N} \leq C, \quad \left\| \frac{\partial y_n}{\partial \nu_{A_n}} \right\|_{H^{-\frac{1}{2}}(\Gamma_D)} \leq C, \quad (65)$$

$$\sup_{n \in \mathbb{N}} \|y_n\|_{L^2(\Omega)} \leq C, \quad \sup_{n \in \mathbb{N}} \|A_n\|_{BV(\Omega \setminus Q; \mathbb{S}^N)} \leq C. \quad (66)$$

Hence, the minimizing sequence  $\{(A_n, y_n) \in \Xi_w\}_{n \in \mathbb{N}}$  is bounded in the sense of Definition 6.1. Hence, by Lemma 6.2 there exist functions  $A^0 \in L^1(\Omega; \mathbb{S}^N)$  and  $y^0 \in W_{A^0}(\Omega; \Gamma_D)$  such that, up to a subsequence,  $(A_n, y_n) \xrightarrow{w} (A^0, y^0)$ . Since the set  $\Xi_w$  is sequentially closed with respect to the  $w$ -convergence (see Theorem 6.3), it follows that the  $w$ -limit pair  $(A^0, y^0)$  is an admissible pair of (49) (i.e.  $(A^0, y^0) \in \Xi_w$ ). Moreover, by (65)<sub>2</sub> and Corollary 5.7 of Proposition 5.6, we have

$$\frac{\partial y_n}{\partial \nu_{A_n}} \rightharpoonup \frac{\partial y^0}{\partial \nu_{A^0}} \quad \text{in } H^{-\frac{1}{2}}(\Gamma_D). \quad (67)$$

To conclude the proof it is enough to observe that by (67) and  $(A_n, u_n) \xrightarrow{w} (A^0, y^0)$ , the cost functional  $I$  is sequentially lower  $w$ -semicontinuous. Hence,

$$I(A^0, y^0) \leq \liminf_{n \rightarrow \infty} I(A_n, y_n) = \inf_{(A, y) \in \Xi_w} I(A, y),$$

i.e.  $(A^0, y^0)$  is an optimal solution. The proof is complete.  $\square$

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