

Approximation of Finitely Additive Functions with Values in Topological Groups

Paola Cavaliere, Paolo de Lucia and Hans Weber

Abstract. We discuss the problem of the pointwise approximation of finitely additive functions, which are defined on a Boolean algebra and take values in a Hausdorff topological commutative group, via strongly continuous and exhaustive finitely additive functions. Related properties of topological groups are also investigated.

Keywords. Additive set functions, approximation, topology of pointwise convergence, connected groups

Mathematics Subject Classification (2010). Primary 28B10, secondary 28A33, 22A99

1. Introduction

Let \mathcal{A} be a Boolean algebra. It is well known [2] that any bounded finitely additive function $\mu : \mathcal{A} \rightarrow \mathbb{R}$ can be pointwise approximated by strongly continuous finitely additive functions if, and only if, \mathcal{A} is atomless. This conclusion pertains to Banach space-valued exhaustive finitely additive functions, namely, any exhaustive finitely additive function $\mu : \mathcal{A} \rightarrow X$, where X is a Banach space, is the pointwise limit of exhaustive strongly continuous finitely additive functions if, and only if, \mathcal{A} is atomless. This has been recently shown in [9], employing the same arguments of [2].

The present note is aimed at dealing with the relevant approximation problem in the general setting of group-valued finitely additive functions.

P. Cavaliere: Department of Mathematics, University of Salerno, 84084 Fisciano (Sa), Italy; pcavaliere@unisa.it

P. de Lucia: Department of Mathematics and Applications “R. Caccioppoli”, University “Federico II” of Naples, 80126 Naples, Italy; padeluci@unina.it

H. Weber: Department of Mathematics and Computer Science, University of Udine, 33100 Udine, Italy; hans.weber@uniud.it

To be more specific, let \mathcal{A} be a Boolean algebra and G a Hausdorff topological commutative group. Assume that both of them are non-trivial. We denote by $a(\mathcal{A}, G)$ the group of all G -valued finitely additive functions defined on \mathcal{A} , and by $csa(\mathcal{A}, G)$ its subgroup consisting in those functions $\mu \in a(\mathcal{A}, G)$ such that

- (i) μ is exhaustive (or strongly bounded), i.e. $\lim_k \mu(a_k) = 0$ for each disjoint sequence $(a_k)_{k \in \mathbb{N}}$ in \mathcal{A} ;
- (ii) μ is strongly continuous, i.e. for any 0-neighborhood U in G there exists some finite partition $\{d_1, \dots, d_n\} \subseteq \mathcal{A}$ of the maximal element of \mathcal{A} such that $\mu(d_i \wedge a) \in U$ for all i and $a \in \mathcal{A}$.

The approximation problem is actually equivalent to the question of whether the subgroup $csa(\mathcal{A}, G)$ is dense in $(a(\mathcal{A}, G), \tau_p)$, where τ_p is the topology of pointwise convergence, namely, of whether

$$\overline{csa(\mathcal{A}, G)}^{\tau_p} = a(\mathcal{A}, G). \quad (1)$$

The validity of (1) forces the algebra \mathcal{A} to be atomless (Lemma 3.2). In fact, this is the sole assumption that will be imposed on \mathcal{A} throughout.

Our first result is a sufficient criterion for (1) to hold, in terms of G . This is the content of Theorem 3.5, Sect. 3. Let us emphasize that, unlike [2, 9], such a condition does not require that G be complete and the approximation data be exhaustive. The proof is based on a characterization of the denseness in $a(\mathcal{A}, G)$ of those sets which, loosely speaking, are closed under operations of sum and suitable restrictions (Lemma 2.1).

The criterion Theorem 3.5 immediately implies that, for any Hausdorff topological vector space G , property (1) holds if, and only if, \mathcal{A} is atomless. This is shown in Corollary 3.7, which can be regarded as a strengthening of [9, Theorem 1].

Next, we look for a necessary condition for (1) involving G only, since - as observed above - \mathcal{A} is just assumed to be atomless.

We focus on complete groups G . The motivation relies upon the fact that to include conclusions of [2, 9] as special cases requires (1) to hold independently of atomless \mathcal{A} . This is false when G is not complete, as Examples 4.4–4.5 easily exhibit. We prove that, in the framework of complete groups, the validity of (1) entails the connectedness of G , since its subset $c_o(G)$, consisting of all elements joined to 0 by some path, must be dense in G (Theorem 4.1).

As a consequence of these results, and of the duality theorem by Pontryagin and van Kampen, a fully characterization of (1) is provided for locally compact groups. In this class, see Corollary 4.3, Section 4, condition (1) turns out to be equivalent to the connectedness of G (and \mathcal{A} atomless). Note that, instead, such an equivalence fails for merely complete groups. In fact, (1) may fail even if the assumption that G be connected is reinforced by that of being arcwise connectedness - see Example 5.1, Section 5.

Merely complete groups, however, share a property with locally compact groups in connection with our approximation problem, in that if (1) holds for some atomless Boolean algebra \mathcal{A} , then it does hold, in fact, for any other such algebra (Corollary 4.7).

This allows us to focus on the (atomless) algebra \mathcal{A}_o generated by the intervals $[\alpha, \beta[$ of $[0, 1[$, and restrict our attention to Stieltjes finitely additive functions. In such a class, Lemma 4.12 below tells us that each exhaustive strongly continuous function determines a path γ satisfying the additional property that the series $\sum_{n=1}^{\infty}(\gamma(\beta_n) - \gamma(\alpha_n))$ converges, for all disjoint sequences $([\alpha_n, \beta_n])_{n \in \mathbb{N}}$ in $[0, 1[$. Thus we introduce the set $c_1(G)$ consisting of all elements in G joined to 0 by some path γ fulfilling the previous additional property. Clearly, $c_1(G) \subseteq c_o(G)$.

The answer to the approximation problem in the framework of merely complete groups reads as follows. When G is complete, property (1) holds if, and only if, the set $c_1(G)$ is dense in G -and \mathcal{A} is atomless (Corollary 4.16, Section 4). In other words, the lack of local compactness of the group G forces to replace the denseness of $c_o(G)$ in G by that of $c_1(G)$ for (1) to hold.

Let us conclude this section with some comments. In dealing with the general case of finitely additive functions taking values in a Hausdorff topological commutative group, the study of the relevant approximation problem is intimately related to different connectedness properties of topological groups. We discuss them in Section 5. In particular, at the end of Section 5, we exhibit a broad class of complete topological groups G for which (1) fails for any atomless Boolean algebra. These groups are arcwise connected (so $c_o(G)$ is dense in G), but $c_1(G)$ consists of the single point 0 (Example 5.13).

2. Preliminaries

Throughout, \mathcal{A} is a Boolean algebra and G is a Hausdorff topological commutative group written additively. We assume that both of them are non-trivial.

Then $a(\mathcal{A}, G)$ denotes the family of all finitely additive functions $\mu : \mathcal{A} \rightarrow G$, and $sa(\mathcal{A}, G)$ is its subset consisting of those functions μ which are exhaustive. Both $a(\mathcal{A}, G)$ and $sa(\mathcal{A}, G)$ are subgroups of $G^{\mathcal{A}}$, which is always assumed to be endowed with the product topology τ_p . Obviously, $a(\mathcal{A}, G)$ is closed in $(G^{\mathcal{A}}, \tau_p)$.

For $\mu \in a(\mathcal{A}, G)$ and $a \in \mathcal{A}$, we write μ_a for the finitely additive function defined by $\mu_a(b) := \mu(a \wedge b)$, $b \in \mathcal{A}$. Then $N(\mu)$ stands for the ideal $\{a \in \mathcal{A} : \mu_a(b) = 0 \text{ for all } b \in \mathcal{A}\}$ and $\hat{\mu}$ is the finitely additive function defined on $\hat{\mathcal{A}} := \mathcal{A}/N(\mu)$ by $\hat{\mu}(\hat{a}) := \mu(a)$, where $\hat{a} := a \triangle N(\mu)$ is the element of $\hat{\mathcal{A}}$ corresponding to a . Besides, we denote by τ_μ the weakest Fréchet-Nikodým topology (for short, *FN*-topology) on \mathcal{A} which makes μ continuous (see, e.g., [4, 16]). Recall that τ_μ is a group topology on (\mathcal{A}, Δ) and the sets

$\{a \in \mathcal{A} : \mu_a(b) \in U \text{ for all } b \in \mathcal{A}\}$, where U is any 0-neighborhood in G , form a 0-neighborhood base for τ_μ .

For $M \subseteq a(\mathcal{A}, G)$ and $a \in \mathcal{A}$, we set $M(a) := \{\nu(a) : \nu \in M\}$. Note that, when M is a subgroup of $a(\mathcal{A}, G)$, then $M(a)$ and $\overline{M(a)}$ are subgroups of G .

Finally, let us recall that an *ultrafilter measure* is a function $\delta : \mathcal{A} \rightarrow G$ such that $\delta(a) = x$ for $a \in \mathcal{U}$, and $\delta(a) = 0$ otherwise, where \mathcal{U} is any ultrafilter of \mathcal{A} and $x \in G \setminus \{0\}$. Plainly, an ultrafilter measure δ is a two-valued element of $sa(\mathcal{A}, G)$ and for any $a \in \mathcal{A}$ the function δ_a agrees with δ when $a \in \mathcal{U}$ and is identically 0 otherwise.

Given a non-empty $M \subseteq a(\mathcal{A}, G)$, if $\mu \in \overline{M}^{\tau_p}$ then $\mu(a) \in \overline{M(a)}$ for all $a \in \mathcal{A}$, because of the continuity of the projection maps. Here we exhibit conditions on M ensuring the converse.

Lemma 2.1. *Let M be a subset of $a(\mathcal{A}, G)$ satisfying*

$$M + M \subseteq M, \quad \nu_a \in M \quad \text{for all } \nu \in M, a \in \mathcal{A}. \tag{2}$$

Then

(α) *for any $\mu \in a(\mathcal{A}, G)$:*

$$\mu \in \overline{M}^{\tau_p} \iff \mu(a) \in \overline{M(a)} \quad \text{for each } a \in \mathcal{A};$$

(β) *M is dense in $a(\mathcal{A}, G)$ if, and only if, $\overline{M(a)} = G$ for each $a \in \mathcal{A} \setminus \{0\}$.*

Proof. (α) (\Rightarrow) This follows at once from the continuity of projection maps.

(\Leftarrow) Let U be a 0-neighborhood in G and $\{a_1, \dots, a_n\}$ a finite subset of \mathcal{A} . We claim that there exists a function $\nu \in M$ such that

$$\mu(a_i) - \nu(a_i) \in U \quad \text{for each } i \in \{1, \dots, n\}. \tag{3}$$

To see this, consider a finite disjoint set $\{d_1, \dots, d_m\}$ in \mathcal{A} such that each a_i is the union of some d_j 's, and then a 0-neighborhood V such that $V^{(m)} := V + \dots + V$ (m -times) $\subseteq U$. Then assumption provides that

$$\forall j \in \{1, \dots, m\} \exists \nu_j \in M \quad \text{such that} \quad \mu(d_j) - \nu_j(d_j) \in V. \tag{4}$$

Now let $\nu := \sum_{j=1}^m (\nu_j)_{d_j}$. By (2) and (4), the function ν belongs to M and fulfils (3).

(β) (\Leftarrow) This trivially follows from (\Leftarrow) in (α).

(\Rightarrow) Take any $a \in \mathcal{A} \setminus \{0\}$ and $x \in G$. Then, when $x \neq 0$ pick an ultrafilter measure ν on \mathcal{A} such that $\nu(a) = x$; set $\nu = 0$ otherwise. Since M is dense in $a(\mathcal{A}, G)$, then (\Rightarrow) in (α) assures that $x = \nu(a) \in \overline{M(a)}$. □

Example 2.2. For any *FN*-topology τ on \mathcal{A} the set

$$M := \{\mu \in a(\mathcal{A}, G) : \mu \text{ is } \tau\text{-continuous}\}$$

is a subgroup of $a(\mathcal{A}, G)$ satisfying (2).

As an application of Lemma 2.1, the denseness of certain typical subsets of $a(\mathcal{A}, G)$ can be established.

Lemma 2.3. *The set $ua(\mathcal{A}, G)$ consisting of all finite sums of ultrafilter measures is dense in $a(\mathcal{A}, G)$. Thus*

$$\overline{ua(\mathcal{A}, G)}^{\tau_p} = \overline{sa(\mathcal{A}, G)}^{\tau_p} = a(\mathcal{A}, G).$$

Proof. Note that $ua(\mathcal{A}, G) =: M$ fulfils (2), and

$$ua(\mathcal{A}, G) \subseteq sa(\mathcal{A}, G) \subseteq a(\mathcal{A}, G).$$

Besides, $M(a) = G$ for all $a \in \mathcal{A} \setminus \{0\}$. In fact, for any $a \in \mathcal{A} \setminus \{0\}$ and $x \in G$ it suffices to take an ultrafilter \mathcal{U} containing a and a finitely additive function taking value x on \mathcal{U} . Therefore, Lemma 2.1(β) yields the denseness of $ua(\mathcal{A}, G)$ in $a(\mathcal{A}, G)$. \square

Remark 2.4. It is worth emphasizing that

$$ua(\mathcal{A}, G) = fsa(\mathcal{A}, G) := \{\mu \in sa(\mathcal{A}, G) : \mu(\mathcal{A}) \text{ is finite}\}.$$

To see this, let μ belong to $fsa(\mathcal{A}, G)$. Then $\mu(\mathcal{A}) \cap U = \{0\}$ for some 0-neighborhood U in G . So the $\hat{\theta}$ -neighborhood

$$\hat{U} := \{\hat{a} \in \hat{\mathcal{A}} : \hat{\mu}_{\hat{a}}(\hat{b}) \in U \text{ for all } \hat{b} \in \hat{\mathcal{A}}\}$$

of the FN -topology $\tau_{\hat{\mu}}$ determined on $\hat{\mathcal{A}} := \mathcal{A}/N(\mu)$ by $\hat{\mu}$ consists of the single point $\hat{\theta}$. According to the exhaustivity of μ (henceforth of $\hat{\mu}$), then any disjoint sequence in $\hat{\mathcal{A}}$ must be eventually $\hat{\theta}$. This fact forces the algebra $\hat{\mathcal{A}}$ to be finite. Consequently, $\hat{\mu}$ may be written as finite sum of ultrafilter measures on $\hat{\mathcal{A}}$. Hence, μ is a finite sum of ultrafilter measures on \mathcal{A} , namely, $\mu \in ua(\mathcal{A}, G)$. The converse inclusion is obvious.

3. Sufficient conditions for property (1)

Let us recall that a function $\mu \in a(\mathcal{A}, G)$ is said to be strongly continuous if for any 0-neighborhood U in G there exists some finite partition $\{d_1, \dots, d_n\} \subseteq \mathcal{A}$ of the maximal element of \mathcal{A} such that $\mu_{d_i}(\mathcal{A}) \subseteq U$ for each i .

Throughout, $csa(\mathcal{A}, G)$ stands for the collection of all strongly continuous elements of $sa(\mathcal{A}, G)$.

Remark 3.1. The set $csa(\mathcal{A}, G)$ is a subgroup of $a(\mathcal{A}, G)$ satisfying (2). This follows from Example 2.2, when we observe that $csa(\mathcal{A}, G)$ may be characterized in terms of FN -topologies as follows

$$csa(\mathcal{A}, G) = \{\mu \in a(\mathcal{A}, G) : \mu \text{ is } \tau_o\text{-continuous}\}$$

where τ_o is the supremum of all strongly continuous and exhaustive FN -topologies on \mathcal{A} . Recall that an FN -topology τ on \mathcal{A} is exhaustive if $a_k \rightarrow 0$ in (\mathcal{A}, τ) for all disjoint sequence $(a_k)_{k \in \mathbb{N}}$, and strongly continuous (or chained) if for any θ -neighborhood U in (\mathcal{A}, τ) there exists some finite partition $\{d_1, \dots, d_n\} \subseteq \mathcal{A}$ of the maximal element of \mathcal{A} such that $d_i \in U$ for all i (see, e.g., [16, pp. 709, 729]).

Firstly, we observe

Lemma 3.2. *If the set $csa(\mathcal{A}, G)$ is dense in $a(\mathcal{A}, G)$ - i.e. (1) holds -, then \mathcal{A} must be atomless.*

Proof. It follows by Lemma 2.1(β) and G non-trivial, since each function in $csa(\mathcal{A}, G)$ must be zero on atoms of \mathcal{A} . □

Next, we recall the following well-known result (see, e.g., [3, Section 5.3]), that is one of the tools for proving Theorem 3.5 below.

Lemma 3.3. *Let \mathcal{A} be atomless. For any $a \in \mathcal{A} \setminus \{0\}$ and $t \in \mathbb{R}$ there exists a function $\mu \in csa(\mathcal{A}, \mathbb{R})$ such that $\mu(a) = t$.*

Remark 3.4. Combining the previous lemma and Lemma 2.1(β) directly yields the following version of [3, Theorem 5.4.2]: the set $csa(\mathcal{A}, \mathbb{R})$ is dense in $a(\mathcal{A}, \mathbb{R})$ if, and only if, \mathcal{A} is atomless.

Our first theorem gives a sufficient condition for (1), in terms of G . According to Lemma 3.2, the sole assumption of non-atomicity will in fact be imposed on \mathcal{A} throughout. A precise statement of the theorem needs to recall that a subgroup $H \subseteq G$ is said to be a *one-parameter subgroup* of G if there exists a continuous homomorphism $\phi : \mathbb{R} \rightarrow G$ such that $\phi(\mathbb{R}) = H$.

Theorem 3.5. *Let \mathcal{A} be an atomless Boolean algebra. If the smallest closed subgroup of G which contains all one-parameter subgroups is G itself, then the set $csa(\mathcal{A}, G)$ is dense in $a(\mathcal{A}, G)$ - i.e. (1) holds.*

Proof. Let $M := csa(\mathcal{A}, G)$ and $a \in \mathcal{A} \setminus \{0\}$. On account of Lemma 2.1(β), it suffices to prove that $\overline{M(a)} = G$.

To see this, consider a continuous homomorphism $\phi : \mathbb{R} \rightarrow G$. By Lemma 3.3, for any $t \in \mathbb{R}$ there exists some function $\mu \in csa(\mathcal{A}, \mathbb{R})$ such that $\mu(a) = t$. Now it is easy to verify that the function $\phi \circ \mu$ belongs to M , and so $\phi(t) = \phi \circ \mu(a) \in M(a)$. Thus $\phi(\mathbb{R}) \subseteq M(a) \subseteq \overline{M(a)}$. This means that $\overline{M(a)}$ is a closed subgroup of G containing all one-parameter subgroups of G . Henceforth $\overline{M(a)} = G$. □

Let us emphasize some of its consequences. Firstly, we consider the case of locally compact groups. For such groups, [8, Theorem 25.20] shows that the assumption on G in Theorem 3.5 is equivalent to that of its connectedness. Hence

Corollary 3.6. *Let \mathcal{A} be atomless. If G is locally compact and connected, then (1) holds.*

Next, we fully characterize property (1) in the framework of Hausdorff topological vector spaces, providing the following strengthening of [9, Theorem 1].

Corollary 3.7. *Let G be a Hausdorff topological real vector space. Then (1) holds if, and only if, \mathcal{A} is atomless.*

Proof. It suffices to couple Lemma 3.2 and Theorem 3.5. □

Remark 3.8. The latter characterization of (1) does hold under a weaker condition on G ; that is, G is a real vector space endowed with a Hausdorff group topology such that the function $\mathbb{R} \ni t \mapsto tx$ is continuous for each $x \in G$. This follows by the argument in the proof of Theorem 3.5.

We conclude this section exhibiting how Theorem 3.5 allows us to treat also those topological groups employed in [5] to extend Lyapunoff’s convexity theorem to group-valued measures. To this end, let us recall that (see, e.g., [7]) if p is a prime and H is a p -divisible commutative group without elements of order p , then for any $x \in H$ and $n \in \mathbb{N}$ there is a uniquely determined element $y \in H$ such that $p^n y = x$. We denote it as $\frac{1}{p^n}x$. For each $m \in \mathbb{Z}$ the element $\frac{m}{p^n}x$ is actually well-defined and H naturally becomes a module over the valuation ring $\mathcal{R}_p := \left\{ \frac{m}{p^n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$.

Corollary 3.9. *Let \mathcal{A} be atomless. Assume that G is complete and satisfy the following properties:*

- (i) G is p -divisible without elements of order p (p prime);
- (ii) for any $x \in G$ and 0 -neighborhood U in G there exists some $\delta > 0$ such that $\frac{m}{p^n}x \in U$ for all $n, m \in \mathbb{N}$ such that $\frac{m}{p^n} < \delta$.

Then (1) holds.

Proof. According to Remark 3.8, it suffices to prove the existence of a unique scalar multiplication, from $\mathbb{R} \times G$ to G , making G a real vector space and the function $\mathbb{R} \ni t \mapsto tx \in G$ continuous for each $x \in G$.

For this, observe that G is an \mathcal{R}_p -module by (i). Then, using (ii), we have that for each $x \in G$ the group homomorphism $\mathcal{R}_p \ni t \mapsto tx \in G$ is continuous at zero. Thus it is uniformly continuous and admits a (uniquely determined) continuous extension on \mathbb{R} , being G complete. Hence, G becomes a real vector space. □

4. The case where G is complete

In the previous section, Lemma 3.2 gives a necessary condition for the denseness of $csa(\mathcal{A}, G)$ in $(a(\mathcal{A}, G), \tau_p)$ - i.e. for (1) to hold - in terms of the Boolean algebra \mathcal{A} . Our next result presents a necessary condition for (1) in terms only of G , in the case where G is complete.

Theorem 4.1. *Let G be complete. If the set $csa(\mathcal{A}, G)$ is dense in $a(\mathcal{A}, G)$ - i.e. (1) holds -, then the subgroup*

$$c_o(G) := \{y \in G : y \text{ can be joined to } 0 \text{ by a path}\} \tag{5}$$

is dense in G ; henceforth G is connected.

Proof. Let $\mu \in csa(\mathcal{A}, G)$. We may and will assume that the FN -topology τ_μ determined by μ on \mathcal{A} is Hausdorff. Indeed, if not, we can replace \mathcal{A} by the quotient $\mathcal{A}/N(\mu)$ and μ by the finitely additive function $\hat{\mu}$ induced by μ on it. Then, as proved in [16, Section 9], μ admits a unique continuous extension $\tilde{\mu}$ on the completion $\tilde{\mathcal{A}}$ of (\mathcal{A}, τ_μ) and $\tilde{\mu}(\mathcal{A})$ is arcwise connected. Thus

$$\mu(\mathcal{A}) \subseteq \tilde{\mu}(\mathcal{A}) \subseteq c_o(G). \tag{6}$$

Application of Lemma 2.1(β) with $M := csa(\mathcal{A}, G)$ -see Remark 3.1- concludes the proof. □

Remark 4.2. Under the assumption of Theorem 4.1, the group G admits a dense arcwise connected subset, but G may fail to be arcwise connected, as shown in Example 5.1, Section 5.

Now, in the setting of locally compact groups, a complete answer to our approximation problem can be formulate as follows.

Corollary 4.3. *Let G be locally compact. Then (1) holds if, and only if, \mathcal{A} is atomless and G is connected.*

Proof. Combine Theorem 4.1 with Lemma 3.2 and Corollary 3.6. □

Let us emphasize that the previous corollary exhibits a precise behaviour of locally compact groups in connection with our approximation problem: either (1) holds for *all* atomless Boolean algebras (when G is connected) or for *none* of them (when G is not connected).

We will show - in Corollary 4.7 below - that such a behaviour pertains to merely complete groups G , although it does not still rely upon the connectedness of G (see Corollary 4.16).

On the contrary, for incomplete groups the validity of (1) does depend on the atomless Boolean algebra \mathcal{A} taken into account, as the following examples show.

Example 4.4. For any σ -algebra \mathcal{A} the set $csa(\mathcal{A}, \mathbb{Q})$ is not dense in $a(\mathcal{A}, \mathbb{Q})$, i.e. (1) does not hold.

Indeed, according to [3, Theorem 11.4.10], the range of any function in $csa(\mathcal{A}, \mathbb{Q})$ must be connected. Thus, $csa(\mathcal{A}, \mathbb{Q})$ consists only of the zero function, and (1) does fail.

Example 4.5. Let \mathcal{A} be the algebra generated by the collection

$$\mathcal{F} := \{[(i - 1)2^{-n}, i2^{-n}[: i, n \in \mathbb{N}, i \leq 2^n\}.$$

Then $csa(\mathcal{A}, \mathbb{Q})$ is dense in $a(\mathcal{A}, \mathbb{Q})$, i.e. (1) does hold.

To see this, let $M := csa(\mathcal{A}, \mathbb{Q})$. By Lemma 2.1(β) it suffices to check that $M(a) = \mathbb{Q}$ for each $a \in \mathcal{A} \setminus \{0\}$. If $a \in \mathcal{A} \setminus \{0\}$ and $q \in \mathbb{Q}$, define $\nu := \frac{q}{\lambda(a)}\lambda$, where λ stands for the Lebesgue measure on \mathcal{A} . Clearly, $\nu \in M$ and $\nu(a) = q$.

Theorem 4.6. *Let G be complete. If \mathcal{A} and \mathcal{B} are atomless Boolean algebras, then*

$$\overline{\{\mu(a) : \mu \in csa(\mathcal{A}, G)\}} = \overline{\{\nu(b) : \nu \in csa(\mathcal{B}, G)\}}$$

for each $a \in \mathcal{A} \setminus \{0\}$ and $b \in \mathcal{B} \setminus \{0\}$.

Proof. Given $\mu \in csa(\mathcal{A}, G)$, $a \in \mathcal{A} \setminus \{0\}$ and $b \in \mathcal{B} \setminus \{0\}$, it is enough to prove that

$$\mu(a) \in \overline{\{\nu(b) : \nu \in csa(\mathcal{B}, G)\}}. \tag{7}$$

We may and will assume that the FN -topology τ_μ determined by μ on \mathcal{A} is Hausdorff. Indeed, if not, we can replace \mathcal{A} by the quotient $\mathcal{A}/N(\mu)$ and μ by the finitely additive function $\hat{\mu}$ induced by μ on it.

Let $(\tilde{\mathcal{A}}, \tilde{\tau})$ be the uniform completion of (\mathcal{A}, τ_μ) . Since μ is exhaustive, [16, Theorem 6.1] provides that $\tilde{\mathcal{A}}$ is a complete Boolean algebra and $\tilde{\tau}$ is an order continuous FN -topology. Hence, by [16, Corollary 4.11], $(\tilde{\mathcal{A}}, \tilde{\tau})$ is algebraically and topologically isomorphic to the product $\prod_{\alpha \in \Lambda} (\mathcal{A}_\alpha, \tau_\alpha)$, where each \mathcal{A}_α is a complete Boolean algebra and τ_α is a metrizable order continuous FN -topology on \mathcal{A}_α . So we may assume that

$$(\tilde{\mathcal{A}}, \tilde{\tau}) = \prod_{\alpha \in \Lambda} (\mathcal{A}_\alpha, \tau_\alpha). \tag{8}$$

From (8), it easily follows that the set $S := \{s \in [0, a] \subseteq \tilde{\mathcal{A}} : \tilde{\tau}_{|[0,s]}$ is metrizable} is dense in $(\tilde{\mathcal{A}}, \tilde{\tau})$. In fact, let $(a_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} \mathcal{A}_\alpha$. For any finite F subset of Λ , define $a_F := (b_\alpha)_{\alpha \in \Lambda}$, where b_α is a_α if $\alpha \in F$ and 0 otherwise. Then $(a_F)_{F \in \mathcal{F}}$ is a net in S converging to $(a_\alpha)_{\alpha \in \Lambda}$.

Now let $\tilde{\mu}$ be the continuous extension of μ on $(\tilde{\mathcal{A}}, \tilde{\tau})$ and U a θ -neighborhood in G . The denseness of S in $(\tilde{\mathcal{A}}, \tilde{\tau})$ assures the existence of an $s_o \in S$ such that

$$\tilde{\mu}(s_o) \in \mu(a) + U.$$

Since $([0, s_o], \tilde{\tau}_{|[0, s_o]})$ is metrizable, there is a θ -neighborhood base $(W_n)_{n \in \mathbb{N}}$ in it. Set $D_0 := \{s_o\}$. The strong continuity of $\tilde{\tau}$ allows to select for each n a finite partition D_n of s_o such that $D_n \subseteq W_n$ and any $d \in D_{n-1}$ is a union of members of D_n as well.

Let \mathcal{A}_o be the Boolean algebra generated by $\cup D_n$. Plainly, \mathcal{A}_o is a countable atomless algebra whose maximal element is s_o . Moreover, the function $\mu_o := \tilde{\mu}|_{\mathcal{A}_o}$ belongs to $csa(\mathcal{A}_o, G)$, because $\tilde{\tau}$ is exhaustive and strongly continuous. Now take a countable atomless Boolean algebra \mathcal{B}_o , with $\mathcal{B}_o \subseteq \mathcal{B}$, having b as maximal element. Then, consider a Boolean algebra isomorphism φ from \mathcal{B}_o onto \mathcal{A}_o (it does exist because any two countable atomless Boolean algebras are isomorphic -see, e.g., [10, Corollary 5.16]). Clearly, $\nu_o := \mu_o \circ \varphi \in csa(\mathcal{B}_o, G)$. Next, Lipecki's extension theorem (see [4, 11]) guarantees that ν_o admits an exhaustive strongly continuous extension $\tilde{\nu}_o$ on $\{c \in \mathcal{B} : c \leq b\}$. Thus the function ν defined by $\nu(d) := \tilde{\nu}_o(b \wedge d)$, $d \in \mathcal{B}$, actually belongs to $csa(\mathcal{B}, G)$ and $\nu(b) = \tilde{\mu}(s_o) \in \mu(a) + U$. Henceforth $\mu(a)$ belongs to $\overline{\{\nu(b) : \nu \in csa(\mathcal{B}, G)\}}$. This completes the proof of (7). \square

Corollary 4.7. *Let G be complete. If (1) holds for some atomless algebra \mathcal{A} , then it does hold for any other such algebra.*

Proof. This follows from Theorem 4.6 and Lemma 2.1(β). \square

Corollary 4.8. *Let G be complete. Then (1) holds if, and only if, \mathcal{A} is atomless and the set $\{\mu(a) : \mu \in csa(\mathcal{A}, G), a \in \mathcal{A}\}$ is dense in G .*

Proof. Set $M := csa(\mathcal{A}, G)$. Then $M(1) = \{\mu(a) : \mu \in csa(\mathcal{A}, G), a \in \mathcal{A}\}$, where 1 denotes the maximal element of \mathcal{A} . Thus the statement follows from Theorem 4.6, Lemma 2.1(β) and Lemma 3.2. \square

Remark 4.9. The characterization of (1) shown in the latter corollary fails for incomplete groups.

To see this, consider the algebra \mathcal{A} generated by the collection \mathcal{F} in Example 4.5 and the Borel subsets of $[1, 2]$. Let $G := \mathbb{Q}$, and $M := csa(\mathcal{A}, \mathbb{Q})$. Then $\{\mu(a) : \mu \in csa(\mathcal{A}, \mathbb{Q}), a \in \mathcal{A}\} = \mathbb{Q}$, since $M([0, 1]) = \mathbb{Q}$. The set M however is not dense in $a(\mathcal{A}, \mathbb{Q})$, according to Lemma 2.1(β), because $M([1, 2]) = \{0\}$.

Remark 4.10. Let G be complete and \mathcal{A} atomless. By Corollary 4.8, it clearly follows that, if

$$\{\nu(a) : \nu \in csa(\mathcal{A}, G), a \in \mathcal{A}\} = G, \tag{9}$$

then (1) holds.

Note that (1) is strictly weaker than (9). In fact, under (9), the group G must be arcwise connected. This follows by combining (9) with (6). On the contrary, as observed in Remark 4.2, when (1) holds, G may fail to be arcwise connected.

Now, let us stress that in the setting of merely complete groups G Corollary 4.8 above characterizes the validity of (1) in terms of the denseness of a precise subset of G . Now we go a bit beyond this, by exhibiting its topological meaning. To this end, Corollary 4.7 is crucial, since it allows us to focus on a particular atomless algebra, that is the atomless algebra \mathcal{A}_o generated by the intervals $[\alpha, \beta[$ of $[0, 1[$. Another ingredient is the following sufficient condition for strong continuity of exhaustive G -valued finitely additive functions, which is stated in [9, Theorem 2] in the case that G is a normed vector space. The proof of [9] can be easily transferred to the group-valued setting and we include it here for the sake of completeness.

Lemma 4.11. *Let $\mu \in sa(\mathcal{A}, G)$. If for any 0-neighborhood U in G and $a \in \mathcal{A}$ there is a finite partition $\{d_1, \dots, d_n\}$ of $a \in \mathcal{A}$ such that $\mu(d_i) \in U$ for $i = 1, \dots, n$, then $\mu \in csa(\mathcal{A}, G)$.*

Proof. For any 0-neighborhood U in G , let $\mathcal{A}(U)$ be the set of all $a \in \mathcal{A}$ admitting a finite partition $D \subseteq \mathcal{A}$ such that $\mu_d(\mathcal{A}) \subseteq U$ for each $d \in D$. Assume that μ is not strongly continuous. Then there is a 0-neighborhood U in G such that $\mathcal{A}(U) \neq \mathcal{A}$.

We claim that if $a_0 \in \mathcal{A} \setminus \mathcal{A}(U)$ and V is a 0-neighborhood such that $V + V \subseteq U$, then there exist two disjoint elements a_1, d_1 , both smaller than a_0 , with $a_1 \notin \mathcal{A}(U)$ and $\mu(d_1) \notin V$.

To see this, take $b \in \mathcal{A}$ such that $b \leq a_0$ and $\mu(b) \notin U$. According to the made assumption, then there are finite partitions D_1 and D_2 of b and $a_0 \setminus b$, respectively, such that $\mu(d) \in V$ for all $d \in D := D_1 \cup D_2$. Since $\sup D = a_0 \notin \mathcal{A}(U)$, then there exists an element $a_1 \in D$ such that $a_1 \notin \mathcal{A}(U)$. Clearly, either $a_1 \leq b$ or $a_1 \wedge b = 0$. Moreover, setting $d_1 := b \setminus a_1$ in the first case and $d_1 := b$ in the latter, one has that $\mu(d_1) \notin V$.

Using the claim just proved, we can inductively find sequences $(a_k), (d_k)$ in \mathcal{A} such that $a_k \wedge d_k = 0$, $a_k \vee d_k \leq a_{k-1}$, $a_k \notin \mathcal{A}(U)$ and $\mu(d_k) \notin V$ for all k . This clearly contradicts the exhaustivity of μ , because the d_k 's are disjoint, and ends the proof. \square

As a consequence, one gets the following

Lemma 4.12. *Let $\mu \in a(\mathcal{A}_o, G)$, where \mathcal{A}_o is the algebra generated by the intervals $[\alpha, \beta[$, $0 \leq \alpha < \beta \leq 1$. Then $\mu \in csa(\mathcal{A}_o, G)$ if, and only if, the function $\gamma(t) := \mu([0, t])$, $t \in [0, 1]$, is continuous and satisfies the condition*

$$\left. \begin{aligned} & \sum_{n=1}^{+\infty} (\gamma(\beta_n) - \gamma(\alpha_n)) \quad \text{converges in the completion of } G \\ & \text{for every disjoint sequence of subintervals } [\alpha_n, \beta_n[\text{ of } [0, 1] \end{aligned} \right\}. \quad (10)$$

Proof. First, observe that the exhaustivity of μ is equivalent to condition (10) for γ . Now let γ be continuous in $[0, 1]$. Since γ is actually uniformly continuous, the strong continuity of μ easily follows from Lemma 4.11. Conversely, let μ be strongly continuous. Then for every 0-neighborhood U in G there is a partition $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n = 1$ such that $\mu(I) \in U$ for every subinterval $I \in \mathcal{A}_o$ of one of the intervals $[\alpha_{i-1}, \alpha_i[$. Define $\delta := \min_{1 \leq i \leq n} (\alpha_i - \alpha_{i-1})$; then $\gamma(s) - \gamma(t) \in U + U$ for $s, t \in [0, 1]$ such that $|s - t| < \delta$. That is, γ is continuous in $[0, 1]$. \square

We note in passing that, in the case $G = \mathbb{R}$, a path γ satisfies (10) if, and only if, γ is of bounded variation. When G is a normed group, any path γ of bounded variation fulfils (10).

In view of Lemma 4.12, we introduce the following relevant subset of the subgroup $c_o(G)$ defined by (5) in Theorem 4.1.

Definition 4.13. The set $c_1(G)$ consists of all elements in G which can be joined to 0 by a path fulfilling (10).

We are now able to prove

Theorem 4.14. *Let G be complete. For any atomless algebra \mathcal{A} the following hold:*

- (i) $\overline{c_1(G)} = \overline{\{\nu(a) : \nu \in csa(\mathcal{A}, G)\}}$ for each $a \in \mathcal{A} \setminus \{0\}$;
- (ii) $\overline{csa(\mathcal{A}, G)}^{\tau_p} = a(\mathcal{A}, \overline{c_1(G)})$.

Proof. (i) Lemma 4.12 tells us that $c_1(G) = \{\nu([0, 1]) : \nu \in csa(\mathcal{A}_o, G)\}$, where \mathcal{A}_o is the algebra generated by the family $\{[\alpha, \beta[: 0 \leq \alpha < \beta \leq 1\}$. Hence Theorem 4.6 concludes the proof.

(ii) It follows from (i), and Corollary 4.8 replacing G by $\overline{c_1(G)}$. \square

As a consequence of Theorem 4.14 (i) one obtains

Corollary 4.15. *Let G be complete. Then $c_1(G) = \{0\}$ if, and only if, for any atomless algebra \mathcal{A} $csa(\mathcal{A}, G) = \{0\}$.*

Corollary 4.16. *Let G be complete. Then (1) holds true if, and only if, \mathcal{A} is atomless and $c_1(G)$ is dense in G .*

Proof. (\Rightarrow) It follows by Lemma 3.2, Lemma 2.1(β) and Theorem 4.14(i).

(\Leftarrow) Apply Theorem 4.14(ii). \square

Remark 4.17. In Example 5.13 below, we will exhibit a broad class of complete topological groups G for which (1) fails for any atomless Boolean algebra. These groups are arcwise connected (so $c_o(G)$ is dense in G), but $c_1(G)$ consists of the single point 0.

5. Appendix: Examples of topological groups

Here we present the examples of topological groups announced in Remark 4.2 and Remark 4.17, respectively.

Example 5.1. Fix a prime p , and consider the topological group \mathbb{Z}_p of p -adic integers. Let $G := \mathbb{R} \times \mathbb{Z}_p$, and $H := \{(m, m) : m \in \mathbb{Z}\}$. Then the quotient group G/H is a compact, connected group containing $c_o(G/H)$ as a dense subgroup, but G/H is not arcwise connected.

The fact that G/H is a compact, connected group containing $c_o(G/H)$ as a dense subgroup follows by [8, Theorem 10.13], being G/H a special instance of a p -adic solenoid.

In order to show that G/H fails to be arcwise connected, we use Lemma 5.2 below which furnishes a necessary condition for the arcwise connectedness of a quotient group G/H provided H is a discrete subgroup of G , namely, (11) must hold. Thus it suffices to note that the component C of 0 in $\mathbb{R} \times \mathbb{Z}_p$ is $\mathbb{R} \times \{0\}$, because $(\mathbb{R} \times \mathbb{Z}_p)/(\mathbb{R} \times \{0\})$ -being isomorphic to \mathbb{Z}_p - is totally disconnected; hence $G \neq C + H$, i.e. (11) does not hold.

Lemma 5.2. *Let H be a discrete subgroup of G . If the quotient group G/H is arcwise connected, then*

$$G = C + H = c_o(G) + H, \tag{11}$$

where C is the component of 0 in G and $c_o(G)$ its subgroup defined in (5).

Proof. It is convenient to structure the proof in three steps. Hereafter $\pi : G \rightarrow G/H$ denotes the quotient map.

Step 1. *There exists a 0-neighborhood U in G such that the restriction $\pi|_U$ of the quotient map π to U is one-to-one and its inverse $(\pi|_U)^{-1}$ is continuous.*

Because H is discrete, there exists some 0-neighborhood V in G such that $V \cap H = \{0\}$. Then pick an open 0-neighborhood U in G so that $U - U \subseteq V$. Clearly $\pi|_U$ is one-to-one. Moreover, as π is an open map and U is open, one concludes that $\pi|_U$ is open as well, i.e. $(\pi|_U)^{-1}$ is continuous.

Step 2. *Let $f : [0, 1] \rightarrow G/H$ be a continuous function, and $x \in G$ such that $\pi(x) = f(0)$. Then there exists a (unique) continuous function $F : [0, 1] \rightarrow G$ such that $\pi \circ F = f$ and $F(0) = x$.*

Take a 0-neighborhood U in G as in Step 1. Then the uniform continuity of f assures that there is some $\delta > 0$ such that

$$f(t') - f(t'') \in \pi(U) \quad \text{for all } t', t'' \in [0, 1] \text{ so that } |t' - t''| \leq \delta. \tag{12}$$

Consider now a partition $\{t_0, \dots, t_n\}$ of $[0, 1]$ whose mesh is less than δ . Then, by (12), for each $i \in \{1, \dots, n\}$ the function $F_i : [t_{i-1}, t_i] \rightarrow G$ defined by $F_i(t) := (\pi|_U)^{-1}(f(t) - f(t_{i-1}))$ is continuous and $F_i(t_{i-1}) = 0$.

Then the function $F : [0, 1] \rightarrow G$ defined recursively by

$$F(t) := \begin{cases} x + F_1(t), & t \in [0, t_1] \\ F(t_i) + F_{i+1}(t), & t \in [t_i, t_{i+1}], i \in \{1, \dots, n - 1\}, \end{cases}$$

is continuous and satisfies the required properties. The uniqueness statement follows at once by applying the discreteness of H .

Step 3. $G \subseteq c_o(G) + H$.

Let $y \in G$. Since G/H is arcwise connected, there is a path $f : [0, 1] \rightarrow G/H$ with endpoints $f(0) := \pi(0)$ and $f(1) := \pi(y)$. Then, by Step 2, there is a (unique) path $F : [0, 1] \rightarrow G$ satisfying $\pi \circ F = f$ and $F(0) = 0$. As $F(1) = y + h$ for some $h \in H$, clearly $y + h$ belongs to $c_o(G)$; so $y \in c_o(G) + H$. Therefore $G \subseteq c_o(G) + H$.

Obviously, $c_o(G) + H \subseteq C + H$. By Step 3, one concludes that $G = c_o(G) + H = C + H$. □

The example announced in Remark 4.17 demands some preparation, and it will conclude this section. We firstly recall the basic background from the theory of free Abelian topological groups, that will be needed in what follows. For an exhaustive treatment of these topics, see, e.g., [1, 12] and the references therein.

For a Tychonoff space (X, τ) , the *free Abelian topological group of X* is the Hausdorff topological commutative group $A(X)$ such that X is topologically embedded in $A(X)$ and, for any continuous mapping f of the space X to a topological commutative group H , there exists a unique continuous homomorphism $\tilde{f} : A(X) \rightarrow H$ such that $f = \tilde{f}|_X$.

Every element $g \in A(X)$ has the form $g = \sum_{i=1}^n \epsilon_i x_i$, where each x_i belongs to X and $\epsilon_i = \pm 1$. Such expression is called *word* written by using letters of the alphabet $X \cup (-X)$; the *empty word* is the neutral element 0 of $A(X)$.

A non-empty word is said to be *reduced* provided it does not contain two letters x and $-x$ simultaneously. Note that the reduced word representing an element $g \in A(X) \setminus \{0\}$ is unique up to permutations. Therefore the *length* of $g \in A(X) \setminus \{0\}$ is well-defined by the number $l(g)$ of letters in the reduced word representing it; instead, the number of different letters will be denoted by $\lambda(g)$. Clearly $\lambda(g) \leq l(g)$. For $g = 0$, we write $l(0) := 0$. Actually, $A(X)$ is the union of an increasing chain of subsets $B_n(X) := \{g \in A(X) : l(g) \leq n\}$, $n \in \mathbb{N} \cup \{0\}$, which are closed by [1, Theorem 7.1.13]. Moreover, by [14, Theorem 4], the group $A(X)$ is complete if, and only if, the Tychonoff space (X, τ) is Dieudonné complete, namely, the universal uniformity on X is complete (see, e.g., [6]).

The set $A_o(X) := \{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i : n \in \mathbb{N}; x_i, y_i \in X\}$ is an open (and closed) subgroup of $A(X)$. By [1, Lemma 7.10.2], $A_o(X)$ coincides with the connected component of the neutral element 0 in $A(X)$ whenever (X, τ) is connected; further, $A_o(X)$ is connected if, and only if, X is connected.

Here we firstly show that such an equivalence fails with connectedness replaced by arcwise connectedness.

Lemma 5.3. *Let (X, τ) be a Tychonoff space. If X is arcwise connected, then $A_o(X)$ is arcwise connected - i.e. $c_o(A_o(X)) = A_o(X)$.*

Proof. Take any $g := \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \in A_o(X)$. Being X arcwise connected, for each $i \in \{1, \dots, n\}$, there is a path $\gamma_i : [0, 1] \rightarrow X$ from x_i to y_i . Then $\sum_{i=1}^n \gamma_i - \sum_{i=1}^n y_i$ is a path lying entirely within $A_o(X)$ from g to 0 . Thus $A_o(X) \subseteq c_o(A_o(X))$. \square

Example 5.4. Let $X := X_1 \cup X_2$, where

$$X_1 := \{0\} \times [-1, 1] \quad \text{and} \quad X_2 := \left\{ \left(t, \sin \frac{1}{t} \right) : 0 < t \leq 1 \right\}.$$

Then $A_o(X)$ is arcwise connected whereas X is (a connected and compact subset of \mathbb{R}^2) not arcwise connected.

To see that $A_o(X)$ is arcwise connected, set $x_1 := (0, \sin 1)$ and $x_2 := (1, \sin 1)$; as $A_o(X)$ is generated by $x_1 - x_2$ and those elements $x - y$ with $x, y \in X_1$ or $x, y \in X_2$, then all we need to know is that such elements can be connected with 0 by a path lying within $A_o(X)$.

This is obvious for those elements $x - y$ with $x, y \in X_1$ or $x, y \in X_2$. Now let $\varphi_1(t) := (0, \sin \frac{1}{t})$ and $\varphi_2(t) := (t, \sin \frac{1}{t})$, $t \in]0, 1]$ and define $\gamma : [0, 1] \rightarrow A_o(X)$ by $\gamma(t) := \varphi_1(t) - \varphi_2(t)$ for $0 < t \leq 1$ and $\gamma(0) = 0$. The function γ provides a path from 0 to $x_1 - x_2$. In fact, for any vanishing sequence $(t_n)_{n \in \mathbb{N}}$ in $]0, 1]$ the Euclidean distance between $\varphi_1(t_n)$ and $\varphi_2(t_n)$ converges to 0 . Because X is compact, both \mathbb{R}^2 and $A_o(X)$ induce on X the same uniformity. Thus $\gamma(t_n)$ converge to 0 in $A_o(X)$, that is γ is continuous (at 0).

Our main goal is to prove

Theorem 5.5. *For any Tychonoff space (X, τ)*

$$c_1(A_o(X)) = c_1(A(X)) = \{0\}.$$

One of the two main ingredients in the proof of Theorem 5.5 is the following

Theorem 5.6. *Let (X, τ) be a Tychonoff space. If $\gamma : [0, 1] \rightarrow A(X)$ is a non-constant path in $A(X)$, then γ is expressible in some $[\alpha, \beta] \subseteq [0, 1]$, with $\alpha < \beta$, as*

$$\gamma(x) = \sum_{i=1}^m k_i \gamma_i(x) \quad \text{for } x \in [\alpha, \beta],$$

where $k_i \in \mathbb{Z}$, $\gamma_i : [\alpha, \beta] \rightarrow X$ is continuous, $\gamma_i([\alpha, \beta]) \subseteq V_i \in \tau$, with $V_i \cap V_j = \emptyset$ for $i \neq j$, and $\gamma_1(\alpha) \neq \gamma_1(\beta)$.

Proof. Let $K := \gamma([0, 1])$. Because γ is not constant, we may assume without loss of generality that $0 \notin K$. Since K can be expressed as $\cup_m K \cap B_m(X)$, where all the sets $K \cap B_m(X)$ are closed, then Baire's Theorem provides the existence of at least one of them admitting an interior point in the relative topology of K . Let n be the smallest index such that $K \cap (B_n(X) \setminus B_{n-1}(X))$ contains some non-empty $U \in \tau$.

Then there exists some $[\alpha, \beta] \subseteq [0, 1]$, with $\alpha < \beta$, such that $\gamma([\alpha, \beta]) \subseteq U$, $\gamma|_{[\alpha, \beta]}$ is not constant and $\gamma(\alpha) \neq \gamma(\beta)$. Take $s \in [\alpha, \beta]$ such that $\lambda(\gamma(s)) = \max\{\lambda(\gamma(t)) : t \in [\alpha, \beta]\}$ and consider the word $\gamma(s)$ in its reduced form $\gamma(s) := \sum_{i=1}^n \epsilon_i x_i$. By [12, Corollary 7.1] (or [1, Corollary 7.1.19]), $\gamma(s)$ has an open neighborhood $V := \sum_{i=1}^n \epsilon_i V_i$ in $B_n(X) \setminus B_{n-1}(X)$, where V_i is some open neighborhood of x_i in X and $V_i \cap V_j = \emptyset$ if $x_i \neq x_j$. One may assume that $\gamma([\alpha, \beta]) \subseteq V$. Since any $g \in V$ obeys the estimate $\lambda(g) \geq \lambda(\gamma(s))$, one has that $\lambda(\gamma(t)) = \lambda(\gamma(s))$ for all $t \in [\alpha, \beta]$. Thus writing γ as

$$\gamma = \sum_{i=1}^n \epsilon_i \gamma_i, \quad \text{where } \gamma_i : [\alpha, \beta] \rightarrow V_i, \tag{13}$$

shows that $\gamma_i = \gamma_j$ whenever $x_i = x_j$.

Let us prove that each γ_i is continuous. It is enough to do it for γ_1 . Consider any $t_o \in [\alpha, \beta]$ and an open neighborhood W_i of $\gamma_i(t_o)$ in X such that $W_i \subseteq V_i$. By the continuity of γ in t_o , there exists some $\delta > 0$ such that $\gamma(t) \in W := \sum_{i=1}^n \epsilon_i W_i$ for $|t - t_o| < \delta$. From this it follows that $\gamma_1(t) \in W_1$, i.e. the continuity of γ_1 in t_o . Indeed, if $\gamma(t) = \sum_{i=1}^n \epsilon_i y_i$ with $y_i \in W_i$, by (13) one infers that $y_1 = \gamma_j(t)$ for some $j \in \{1, \dots, n\}$. As $\gamma_j(t) \in W_1 \subseteq V_1$, it follows that $x_j = x_1$. Therefore $\gamma_j = \gamma_1$ and $\gamma_1(t) = y_1 \in W_1$.

Since $\gamma(\alpha) \neq \gamma(\beta)$, up to some permutation, one can assume that $\gamma_1(\alpha) \neq \gamma_1(\beta)$. □

Corollary 5.7. *Let (X, τ) be a Tychonoff space. Then X contains an arc if, and only if, $A(X)$ contains an arc.*

Proof. It follows by the previous theorem and the fact that X topological embeds into $A(X)$. □

Corollary 5.8. *If X is a connected metric space which does not contain any arc, then $A_o(X)$ is a complete and connected Hausdorff topological commutative group which does not contain any arc.*

Proof. Being a metric space, X is Dieudonné complete and therefore $A(X)$ is complete by [14, Theorem 4]. As $A_o(X)$ is a closed subgroup of $A(X)$, then $A_o(X)$ is complete. Moreover $A_o(X)$ is connected, since X is connected (see [1, Lemma 7.10.2]). Finally $A_o(X)$ does not contain any arc, since there is no arc in $A(X)$ by Corollary 5.7. □

Remark 5.9. A special instance of a metric space which satisfies the assumption of Corollary 5.8 is the Knaster-Kuratowski fan (also known as Cantor’s Teepee; see, e.g., [13, Counterexample 129]).

The other main ingredient for proving Theorem 5.5 is Lemma 5.12 below. For it, we need two preliminary results. The first is well-known (see, e.g., [12, 14, 15]).

Lemma 5.10. *Let (X, τ) be a Tychonoff space. If $p : X \rightarrow [0, 1]$ is a continuous pseudometric on X , then the map $\|\cdot\|_p : A(X) \rightarrow [0, +\infty[$ defined by*

$$\|g\|_p := \begin{cases} \inf \left\{ \sum_{i=1}^n p(x_i, y_i) : g = \sum_{i=1}^n (x_i - y_i), x_i, y_i \in X \right\} & \text{for } g \in A_o(X) \\ l(g) & \text{for } g \in A(X) \setminus A_o(X) \end{cases}$$

is a continuous seminorm on $A(X)$.

Further the topology of $A(X)$ is determined by the family $\{\|\cdot\|_p : p \in \mathcal{D}(X)\}$, where $\mathcal{D}(X)$ stands for the family of all continuous pseudometrics on X bounded by 1.

Next we show

Lemma 5.11. *Let (X, τ) be a Tychonoff space. For any sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of X , there exists a continuous pseudometric $p : X \rightarrow [0, 1]$ such that $p(x, y) \geq \frac{1}{n}$ for all distinct $x, y \in F_n$.*

Proof. As (X, τ) is a Tychonoff space, for each n there exists a continuous pseudometric $p_n : X \rightarrow [0, +\infty[$ such that $p_n(x, y) \geq 1$ for all distinct $x, y \in F_n$. Set $p_o := \sum_{n \in \mathbb{N}} \min \{p_n, \frac{1}{3^n}\}$, then p_o is continuous pseudometric on X such that $p_o(x, y) \geq \frac{1}{3^n}$ whenever $x, y \in F_n$ with $x \neq y$. Now take an increasing continuous concave function $f : [0, 1] \rightarrow [0, 1]$ such that $f(0) = 0$ and $f(\frac{1}{3^n}) = \frac{1}{n}$. Because f is subadditive, the function $p := f \circ p_o$ is a continuous pseudometric having the required property. \square

Lemma 5.12. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a Tychonoff space (X, τ) . If $(g_n)_{n \in \mathbb{N}}$ is a sequence in $A(X)$ such that $g_n := \sum_{x \in L_n} x$, where L_n is a finite subset of $X \cup (-X \setminus \{-x_m : m \in \mathbb{N}\})$ and $L_n \ni x_n$, then the series $\sum_{n \in \mathbb{N}} g_n$ does not converge.*

Proof. For any n take an integer $j(n) > n$ such that $\sum_{k=n}^{j(n)} \frac{1}{k} \geq 1$. Setting $F_n := \cup_{k=n}^{j(n)} (L_k \cup -L_k)$, Lemma 5.11 assures the existence of a continuous pseudometric $p : X \rightarrow [0, 1]$ such that $p(x, y) \geq \frac{1}{n}$ for all distinct $x, y \in F_n$.

Now we show that $\|\sum_{k=n}^{j(n)} g_k\|_p \geq 1$ for each n . This is obvious whenever $\sum_{k=n}^{j(n)} g_k \in A(X) \setminus A_o(X)$. Now let $\sum_{k=n}^{j(n)} g_k \in A_o(X)$. Then $\sum_{k=n}^{j(n)} g_k = \sum_{k=n}^N (a_k - b_k)$ with $a_k, b_k \in X$. Note that all a_k, b_k belong to F_n and moreover one may assume that $x_k = a_k$ for $k = n, \dots, j(n)$. Hence

$$\sum_{k=n}^N p(a_k, b_k) \geq \sum_{k=n}^{j(n)} p(x_k, b_k) \geq \sum_{k=n}^{j(n)} \frac{1}{k} \geq 1$$

and consequently $\|\sum_{k=n}^{j(n)} g_k\|_p \geq 1$. □

Proof of Theorem 5.5. Suppose that $c_1(A(X)) \neq \{0\}$. Then there exists a path $\gamma : [0, 1] \rightarrow A(X)$ which is non-constant and satisfies (10). According to Theorem 5.6, there is some $[\alpha, \beta] \subset [0, 1]$, with $\alpha < \beta$, such that $\gamma|_{[\alpha, \beta]} = \sum_{i=1}^m k_i \gamma_i$, where all γ_i are continuous functions whose ranges are mutually disjoint in X , $k_i \in \mathbb{Z}$ and $\gamma_1(\alpha) \neq \gamma_1(\beta)$. One can select inductively two sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$ in $] \alpha, \beta [$ so that $\alpha_n < \beta_n < \alpha_{n+1} < \beta_{n+1}$ and $\gamma_1(\beta), \gamma_1(\alpha_n), \gamma_1(\beta_n)$ are distinct for all n . Now set $x_n := \gamma_1(\beta_n)$ and $g_n := \gamma(\beta_n) - \gamma(\alpha_n)$. By Lemma 5.12, $\sum_{n=1}^{\infty} \gamma(\beta_n) - \gamma(\alpha_n)$ does not converge. But this contradicts (10). So $c_1(A(X)) = \{0\}$. □

Coupling Lemma 5.3 and Theorem 5.5 allows us to exhibit in the next example a broad class of complete topological groups G for which (1) fails for any atomless Boolean algebra, on account of Corollary 4.16. These groups are indeed arcwise connected (so $c_o(G)$ is dense in G), but $c_1(G)$ consists of the single point 0.

Example 5.13. Let (X, τ) be a Tychonoff space. When X is arcwise connected, then $G := A_o(X)$ is arcwise connected and $c_1(G) = \{0\}$. If, in addition, X is Dieudonné complete, then G is complete.

Acknowledgement. This research was partially supported by G.N.A.M.P.A. of Istituto Nazionale di Alta Matematica (Italy).

References

- [1] Arhangel'skii, A. and Tkachenko, M., *Topological Groups and Related Structure*. Amsterdam: Atlantic Press 2008.
- [2] Bhaskara Rao, K. P. S. and Bhaskara Rao, M., Charges on Boolean algebras and almost discrete spaces. *Mathematika* 20 (1973), 214 – 223.
- [3] Bhaskara Rao, K. P. S. and Bhaskara Rao, M., *Theory of Charges. A Study of Finitely Additive Measures*. New York: Academic Press 1983.

- [4] de Lucia, P., *Funzioni Finitamente Additive a Valori in un Gruppo Topologico* (in Italian). Bologna: Pitagora Editrice 1985.
- [5] de Lucia, P. and Wright, J., Group valued measures with the Lyapunoff property. *Rend. Circ. Mat. Palermo (2)* 40 (1991)(3), 442 – 452.
- [6] Engelking, R., *General Topology*. Berlin: Heldermann 1989.
- [7] Fuchs, L., *Infinite Abelian Groups. Vol. I*. New York: Academic Press 1970.
- [8] Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis. Vol. I*. Berlin: Springer 1963.
- [9] Klimkin, V. M. and Svistula, M. G., On the pointwise limit of vector charges with the Saks property. *Math. Notes* 74 (2003)(3–4), 385 – 392.
- [10] Koppelberg, S., *Handbook of Boolean Algebras. Vol. I*. Amsterdam: North-Holland 1989.
- [11] Lipecki, Z., Extensions of additive set functions with values in a topological group. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 22 (1974), 19 – 27.
- [12] Sipachëva, O. V., The topology of a free topological group. *J. Math. Sci. (N. Y.)* 131 (2005)(4), 5765 – 5838.
- [13] Steen, L. A. and Seebach, J. A. Jr., *Counterexamples in Topology*. New York: Holt, Rinehart and Winston 1970.
- [14] Tkachenko, M., On completeness of free Abelian topological groups. *Dokl. Akad. Nauk SSSR* 27 (1983)(2), 341 – 345.
- [15] Uspenskii, V. V., Free topological groups of metrizable spaces. *Math. USSR Izv.* 37 (1991)(3), 657 – 679.
- [16] Weber, H., *FN*-topologies and group-valued measures. In: *Handbook of Measure Theory. Vol. I* (ed.: E. Pap). Amsterdam: North-Holland 2002, pp. 703 – 743.

Received May 17, 2012; revised December 28, 2012