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Existence and Stability of Periodic Planar Standing Waves in Phase-Transitional Elasticity with Strain-Gradient Effects I: General Theory

Jinghua Yao

Abstract. Extending investigations of Antman & Malek-Madani, Schecter & Shearer, Slemrod, Barker & Lewicka & Zumbrun, and others, we investigate phase-transitional elasticity models with strain-gradient effect. We prove the existence of non-constant planar periodic standing waves in these models by variational methods, for deformations of arbitrary dimension and general, physical, viscosity and strain-gradient terms. Previous investigations considered one-dimensional phenomenological models with artificial viscosity/strain gradient effect, for which the existence reduces to a standard (scalar) nonlinear oscillator. For our variational analysis, we require that the mean vector of the unknowns over one period be in the elliptic region with respect to the corresponding pure inviscid elastic model. Previous such results were confined to one-dimensional deformations in models with artificial viscosity–strain-gradient coefficients.

Keywords. Elasticity, strain-gradient effect, periodic wave, Hamiltonian system.
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1. Introduction

The mathematical study of elasticity has been an important topic (see [1–5, 11, 13–15, 17, 22, 31–35, 38, 39], etc., and references therein) due to wide applications. The study of traveling waves of phase-transitional elasticity has been carried out for phenomenological models in [32–35, 38] for one-dimensional shear flow, with classical double well potential and artificial viscosity–capillarity terms (see however the important work [17] for multi-D multiphase elasticity). The treatment of the general, physical, case was cited in [11, Appendix A], as an important direction for further study.

J. Yao: Department of Mathematics, Indiana University, Bloomington, IN 47405, USA; yaoj@indiana.edu

In this paper, continuing the work of Antman and Malek-Madani [4], Slemrod [33–35], Schecter and Shearer [32], and Barker, Lewicka and Zumbrun [11], we study the existence of planar elastic periodic traveling waves, compressible or incompressible, for deformations of arbitrary dimensions, starting from the most general form of the physical equations. It turns out for general elasticity model with strain-gradient effects that, similarly as observed for the phenomenological models studied previously, the periodic traveling waves can only be standing waves and the corresponding ordinary differential equation (ODE) system exhibits Hamiltonian structure.

As we know, for a planar Hamiltonian system, we can use phase-plane analysis to study its closed orbits. This corresponds to the case of one-dimensional deformations, for which the unknowns are scalar in the ODE system (4.5). For higher dimensional Hamiltonian systems, this method does not apply directly. In order to prove the existence of non-constant periodic waves when the unknowns are vectors, we consider the problem under the framework of calculus of variation. However, there are several difficulties to overcome. First, we need to formulate the problem in proper Banach spaces. It turns out that the proper space for our purpose is the periodic Sobolev space with mean zero property. Working in this framework amounts to prescribing the mean of the unknown over one periodic (no real restriction, since each periodic wave has a mean as long as it exists). Second, we need to make sure that the waves we find are not constant waves. We overcome this issue by considering the equations satisfied by the difference between the original unknown and its mean. This makes the 0 element in our working space always a critical point, which helps to eliminate the possibility that the solutions we find are trivial. Third, in the global model of elasticity, we need to consider the assumption $\tau_3 > 0$ (see Section 2 or 3). This kind of condition usually leads to a variational inequality and is related to an obstacle problem. Meanwhile, this inequality condition makes our admissible set (to make the wave physically meaningful) not weakly closed. However, the asymptotic behavior of the elastic potential will help overcome the related problem. We note that in the pure elastic case without strain-gradient effects and viscosity, this restriction on τ_3 imposes significant challenges in the mathematical analysis (see the discussions in [2, 5]).

Besides the Hamiltonian structure of the standing wave equations, here we prove that for the general physical model, there exist non-constant periodic waves no matter whether the unknowns are scalar or not (see Theorem 6.13) under assumptions on the mean vector over one periodic of the wave. In the followup paper [36], for some specific phase-transitional models, we apply the results in present paper and give explicit conditions under which the non-constant oscillatory waves exist. In particular, for the one dimensional models, we use phase-plane analysis to get detailed information on the wave phenomena (existence of periodic, homoclinic, heteroclinic waves), and compare these results with results obtained from our general theory. They match very well.

Comparing our results with others (see the references of this paper), the problems here are interesting enough even only from the modeling point of view, without even finding any waves. In [4], the authors treated the shear flow without the strain-gradient effect and with an isotropic assumption preventing phase transition model (see discussion in [11]). Here we include models with strain-gradient effects and the materials can be anisotropic, which gives rich wave phenomena (see [36]). Antmann, Slemrod and others (see [3, 4, 33–35] and references therein) have previously studied phenomenological 1D phase-transitional models with double-well potentials. Here we justify those types of qualitative models by direct derivation from the physical shear flow model (see [36]).

It would be very interesting to explore by numerics cases that do not fit the hypotheses here (which are sufficient but by not necessary in the non-scalar case) but nonetheless support periodic waves and also to explore either numerically or analytically the spectral stability of these waves. We hope to address these issues in a followup work [12]. Numerical study of existence and stability of shock waves (which, since not necessarily zero-speed, are more plentiful) would be another interesting direction for future study.

2. Elasticity models with strain-gradient effects

In this section, we will proceed following the presentations of [2,5,11,28]. Let Ω be the reference configuration which models an elastic body with constant temperature and density. A typical point in Ω will be denoted by X. We use $\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ to denote the deformation (i.e., the deformed position of the material point X). Consequently, the deformation gradient is given by $F := \nabla_X \xi$, which we regard as an element in $\mathbb{R}^{3\times 3}$.

Adopting the notations above, the equations of isothermal elasticity with strain-gradient effect are given through the following balance of linear momentum

$$\xi_{tt} - \nabla_X \cdot \Big(DW(\nabla\xi) + \mathcal{Z}(\nabla\xi, \nabla\xi_t) - \mathcal{E}(\nabla^2\xi) \Big) = 0. \tag{2.1}$$

We make the following physical constraint on the deformation gradient (see [5, 11] and [2, 28] for the physical background), prohibiting local self-impingement of the material:

$$\det F > 0. \tag{2.2}$$

In (2.1), the operator $\nabla_X \cdot$ stands for the divergence of an approximate field. As in [15, 28], for a matrix-valued vector field, we use the convention that the divergence is taken row-wise. In what follows, we shall also use the matrix norm $|F| = (\mathrm{tr}(F^T F))^{\frac{1}{2}}$, which is induced by the inner product: $F_1 : F_2 := \mathrm{tr}(F_1^T F_2)$.

In view of the second law of thermodynamics (see [5, 27]), the Piola-Kirchhoff stress tensor $DW : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is expressed as the derivative of an elastic energy density $W : \mathbb{R}^{3\times3} \to \overline{\mathbb{R}}_+$. Throughout the paper, we assume as in [2,5,28] the elastic energy density function W is frame-indifference. Let $SO(3)$ be the group of proper rotations in $\mathbb{R}^3$. Then the frame-indifference assumption can be formulated as

$$W(RF) = W(F), \quad \forall F \in \mathbb{R}^{3\times3},\ \forall R \in SO(3). \tag{2.3}$$

Also, the material consistency (to avoid interpenetration of matter, (2.2), [2, 5]) requires the following important assumption:

$$W(F) \to +\infty \quad \text{as } \det F \to 0. \tag{2.4}$$

We emphasize that viscous stress tensor $\mathcal{Z} : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ depends on both the deformation gradient F and the velocity gradient $Q = F_t = \nabla\xi_t = \nabla v$, where $v = \xi_t$. From physical point of view, the stress tensor $\mathcal{Z}$ should also be compatible with principles of continuum mechanics (balance of angular momentum, frame invariance, and the Claussius-Duhem inequality etc). For the related mathematical descriptions and corresponding stress forms see [2, 5, 11] and references therein.

The strain-gradient effect $\mathcal{E}$ is given by

$$\mathcal{E}(\nabla^2\xi) = \nabla_X \cdot D\Psi(\nabla^2\xi) = \left[\sum_{i=1}^{3} \frac{\partial}{\partial X_i}\Big(\frac{\partial}{\partial(\partial_{ij}\zeta^k)}\Psi(\nabla^2\xi)\Big)\right]_{j,k=1,\ldots,3}$$

for some convex density $\Psi : \mathbb{R}^{3\times3\times3} \to \mathbb{R}$, compatible with frame indifference.

The corresponding inviscid part of system (2.1)

$$\xi_{tt} - \nabla_X \cdot \Big(DW(\nabla\xi)\Big) = 0 \tag{2.5}$$

can be written as

$$(F, \tau)_t + \sum_{i=1}^{3} \partial_{X_i}\Big(\tilde{G}_i(F, \tau)\Big) = 0. \tag{2.6}$$

Above, $(F, \tau) : \Omega \to \mathbb{R}^{12}$ represents conserved quantities, while $\tilde{G}_i : \mathbb{R}^{12} \to \mathbb{R}^{12}$ given by

$$-\tilde{G}_i(F, \tau) = \tau^1 e_i \oplus \tau^2 e_i \oplus \tau^3 e_i \oplus \left[\frac{\partial}{\partial F_{ki}} W(F)\right]_{k=1}^{3}, \quad i = 1, \ldots, 3$$

are the fluxes, and e_i denotes the i-th coordinate vector in $\mathbb{R}^3$.

The convex density Ψ contributes to equation (2.1) the term

$$\nabla_X \cdot \Big(\mathcal{E}(\nabla^2\xi)\Big) = \nabla_X \cdot \{\nabla_X \cdot D\Psi(\nabla^2\xi)\}. \tag{2.7}$$

In view of the orders of differentiation and convexity of Ψ, we may assume that

$$\Psi \geq 0; \quad \Psi(0) = 0; \quad D\Psi(0) = 0; \quad \delta Id \leq D^2\Psi(\cdot) \leq MId$$

where δ, M are two positive real numbers and Id is an element in the space $\mathcal{L}(\mathbb{R}^{3\times3\times3}; \mathbb{R}^{3\times3\times3})$. The mapping relations (ignoring physical constraints) are

$$\begin{aligned} \Psi &: \mathbb{R}^{3\times3\times3} \to \mathbb{R}_+ \\ D\Psi &: \mathbb{R}^{3\times3\times3} \to \mathbb{R}^{3\times3\times3} \\ D^2\Psi &: \mathbb{R}^{3\times3\times3} \to \mathcal{L}(\mathbb{R}^{3\times3\times3}; \mathbb{R}^{3\times3\times3}) \end{aligned}$$

When the operator $\nabla_X\cdot$ reduces to the operator ∂_x where x is a one dimension variable, (2.7) takes the form $\partial_x\{\partial_x D\Psi(\partial_x^2\xi)\}$. If we identify ξ_x as τ, then $\partial_x^2\xi = \tau_x$ and (2.7) becomes $\partial_x\{\partial_x D\Psi(\tau_x)\} = \partial_x\{D^2\Psi(\tau_x)\tau_{xx}\}$. Note that $D^2\Psi : \mathbb{R}^3 \to \mathcal{L}(\mathbb{R}^3; \mathbb{R}^3)$ when $\nabla_X\cdot$ reduces to ∂_x. So we assume that $D^2\Psi(\cdot)$ as matrix function satisfy the assumption $\delta Id \leq D^2\Psi(\cdot) \leq MId$ as operators.

3. Equations and specific models

In this paper, we focus on the interesting subclass of planar solutions, which are solutions in the full 3D space that depend only on a single coordinate direction; that is, we investigate deformations ξ given by

$$\xi(X) = X + U(z), \quad X = (x, y, z), \quad U = (U_1, U_2, U_3) \in \mathbb{R}^3.$$

Corresponding to the above deformation or displacement ξ, the deformation gradient with respect to X is

$$F = \begin{pmatrix} 1 & 0 & U_{1,z} \\ 0 & 1 & U_{2,z} \\ 0 & 0 & 1 + U_{3,z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix}. \tag{3.1}$$

We shall denote $V = (\tau, u) = (\tau_1, \tau_2, \tau_3, u_1, u_2, u_3)$, where $\tau_1 = U_{1,z}$, $\tau_2 = U_{2,z}$, $\tau_3 = 1 + U_{3,z}$ and $u_1 = U_{1,t}, u_2 = U_{2,t}, u_3 = U_{3,t}$ with the physical constraint $\tau_3 > 0$, corresponding to $\det F > 0$ in the region of physical feasibility of V.

Writing $W(\tau) = W\left(\begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix}\right)$, we see that for all F as in (2.1) there holds

$$\nabla_X \cdot (DW(F)) = (D_\tau W(\tau))_z.$$

That is, the planar equations inherit a vector-valued variational structure echoing the matrix valued variational structure (note that the left hand side is the divergence of $DW(F)$).

We study models (traveling wave ODEs, Hamiltonian ODEs, existence of standing waves) for general elastic potential energy and give a rather general abstract existence result in the following sections.

The system. As a convention, we shall use $x \in \mathbb{R}^1$ as the space variable instead of z. So we have the following system

$$\begin{cases} \tau_t - u_x = 0 \\ u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(\tau_x)\tau_{xx})_x. \end{cases} \tag{3.2}$$

with $\sigma := -D_\tau W(\tau)$, $d(\cdot) := D^2\Psi(\cdot)$, and $b(\tau)$ positive definite matrix function.

We are interested in the existence of periodic traveling waves of the above system, which involves a third order term because of the strain-gradient effect.

4. Traveling wave ODE system

We seek traveling wave solution of the system (3.2), $(\tau(x,t), u(x,t)) := (\tau(x-st), u(x-st))$, where $s \in \mathbb{R}$ is the wave speed. Let us denote in the following $'$ as differentiation with respect to $x - st$. For convenience, we still use x to represent $x - st$ (Indeed, we will show a bit later that in fact $s = 0$ is necessary for the existence of periodic or homoclinic waves; see equation (5.6)). With further investigation in mind, we write the related equations for the general class of elastic models with strain-gradient effects. Now from system (3.2), we have the ODE system

$$\begin{cases} -s\tau' - u' = 0 \\ -su' + \sigma(\tau)' = (b(\tau)u')' - (d(\tau')\tau'')'. \end{cases} \tag{4.1}$$

Plugging the first equation into the second in the above system, we obtain the following second-order ODE in τ:

$$s^2\tau' + \sigma(\tau)' = -(b(\tau)s\tau')' - (d(\tau')\tau'')'. \tag{4.2}$$

In view of $d(\cdot) = D^2\Psi(\cdot)$, we readily see:

$$s^2\tau' + \sigma(\tau)' = -(b(\tau)s\tau')' - (D^2\Psi(\tau')\tau'')'. \tag{4.3}$$

Choosing a specific space point, say x_0, we integrate once to get:

$$s^2\tau + \sigma(\tau) + q = -sb(\tau)\tau' - D\Psi(\tau')' \tag{4.4}$$

Here q is an integral constant vector. Relating this with the elastic potential function W, we have

$$-DW(\tau) + s^2\tau + q = -sb(\tau)\tau' - D\Psi(\tau')' \tag{4.5}$$

Note carefully that the integral constant vector is given by

$$q = \{DW(\tau) - s^2\tau - sb(\tau)\tau' - D\Psi(\tau')'\}\Big|_{x=x_0}. \tag{4.6}$$

5. Hamiltonian structure

Defining $G(P) := \langle P, D\Psi(P)\rangle - \Psi(P)$, we see that $\frac{dG}{dP} = \langle P, D^2\Psi\rangle$. Here $P \in \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \to \mathbb{R}$ (for our purpose $n = 1, 2, 3$), $G : \mathbb{R}^n \to \mathbb{R}$ a different scalar potential type function. Now we are ready to state a structural property about the traveling wave ODE system (4.5).

Proposition 5.1. *When $s = 0$, the system (4.5) is a Hamiltonian system with factor $\Big(D^2\Psi(\tau')\Big)^{-1}$, preserving the Hamiltonian*

$$H(\tau, \tau') = -W(\tau) + q\tau + G(\tau') \equiv \textit{constant.}$$

Proof. When $s = 0$, the traveling wave ODE (4.5) becomes:

$$-dW(\tau) + q = -D\Psi(\tau')' \tag{5.1}$$

and the constant $q = \{DW(\tau) - D\Psi(\tau')'\}\Big|_{x=x_0}$. In view of the positive-definiteness of $D^2\Psi(\cdot)$, we may write the ODE as a first order system by regarding τ, τ' as independent variables:

$$\begin{aligned} \tau' &= [D^2\Psi(\tau')]^{-1} D^2\Psi(\tau')\tau' \\ \tau'' &= -[D^2\Psi(\tau')]^{-1}(-DW(\tau) + q) \end{aligned} \tag{5.2}$$

Now, consider the energy surface given by:

$$H(\tau, \tau') := -W(\tau) + q\tau + G(\tau'). \tag{5.3}$$

We see that

$$\begin{aligned} \frac{\partial}{\partial \tau'} H(\tau, \tau') &= \frac{dG(\tau')}{d\tau'} = D^2\Psi(\tau')\tau' \\ \frac{\partial}{\partial \tau} H(\tau, \tau') &= -DW(\tau) + q. \end{aligned} \tag{5.4}$$

Comparing (5.2), (5.4), we see that the traveling wave ODE is a Hamiltonian system with factor $\gamma := [D^2\Psi(\tau')]^{-1}$. Thus, (4.5) preserves the Hamiltonian H. We can see this also by explicit computation, writing $\zeta = x - st$:

$$\begin{aligned} \frac{d}{d\zeta} H(\tau, \tau') =& \frac{\partial}{\partial \tau} H(\tau, \tau')\tau' + \frac{\partial}{\partial \tau'} H(\tau, \tau')\tau'' \\ =& \gamma \frac{\partial}{\partial \tau} H(\tau, \tau') \frac{\partial}{\partial \tau'} H(\tau, \tau') + \gamma \frac{\partial}{\partial \tau'} H(\tau, \tau') \left\{ -\frac{\partial}{\partial \tau} H(\tau, \tau') \right\} \\ =& 0. \end{aligned}$$

□

From the above structural information, we easily get a necessary condition for the existence of periodic or homoclinic waves, extending results of [26] in a one-dimensional model case.

Theorem 5.2. *For* (4.5) *with* $s \gtreqless 0$, *there holds* $\frac{dH}{d\zeta} \lesseqgtr 0$, *where*

$$H(\tau, \tau') := -W(\tau) + \frac{s^2}{2}|\tau|^2 + q\tau + G(\tau'), \tag{5.5}$$

so that no homoclinic or periodic orbits can occur unless $s = 0$.

Proof. Considering the evolution of $\frac{d}{d\zeta}H(\tau, \tau')$ along the flow of traveling wave ODE system (4.5), we have

$$\begin{aligned}
\frac{d}{d\zeta}H(\tau, \tau') =& \frac{\partial}{\partial \tau}H(\tau, \tau')\tau' + \frac{\partial}{\partial \tau'}H(\tau, \tau')\tau'' \\
=& \langle -D_\tau W(\tau) + q + s^2\tau, \tau' \rangle + \langle DG(\tau'), \tau'' \rangle \\
=& \langle -D_\tau W(\tau) + q + s^2\tau, \tau' \rangle + \langle D^2\Psi(\tau')\tau', \tau'' \rangle \\
=& \langle -D_\tau W(\tau) + q + s^2\tau, \tau' \rangle + \langle D^2\Psi(\tau')\tau'', \tau' \rangle \\
=& \langle -D_\tau W(\tau) + q + s^2\tau + D\Psi(\tau')', \tau' \rangle \\
=& \langle -sb(\tau)\tau', \tau' \rangle.
\end{aligned}$$

The conclusion thus follows from the positive definiteness of $b(\tau)$. □

The Hamiltonian system. In the following, we will consider the case $\Psi(P) = \frac{|P|^2}{2}$ as a mathematically natural first step. From the above analysis, we see that necessarily $s = 0$, i.e., all traveling periodic waves are standing. The traveling wave ODE system reduces to the following form with an integral constant q

$$\begin{cases} -\tau'' = -D_\tau W(\tau) + q \\ \quad q = \{D_\tau W(\tau) - \tau''\}\Big|_{x=x_0}. \end{cases} \tag{5.6}$$

If we take the Hamiltonian point of view, the corresponding Hamiltonian for the above system is

$$H(\tau, \tau') = \frac{1}{2}|\tau'(x)|^2 + V(\tau, \tau'),$$

where $V(\tau, \tau') := q \cdot \tau(x) - W(\tau(x))$. The periodic solutions of the system are confined to the surface $H(\tau, \tau') \equiv$ constant.

6. Calculus of Variations

In this section, we formulate the problem in the framework of Calculus of Variations and give the proof of the existence result.

6.1. Space structure. As a first step, we recall the notions of Sobolev spaces involving periodicity and introduce the space structure we are going to use (see [24]). For fixed real number $T > 0$, let C_T^∞ be the space of infinitely differentiable T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$ (for our purpose $n = 1, 2, 3.$).

Lemma 6.1. *Let $u, v \in L^1(0, T; \mathbb{R}^n)$. If, for every $f \in C_T^\infty$,*

$$\int_0^T (u(t), f'(t))dt = -\int_0^T (v(t), f(t))dt,$$

then

$$\int_0^T v(s)ds = 0$$

and there exists a constant vector c in $\mathbb{R}^N$ such that

$$u(t) = \int_0^t v(s)ds + c \quad a.e. \text{ on } [0, T].$$

Proof. For the mean zero property, we could consider the specific test function $f = e_j$. For the integral formulation, we can use the Fubini Theorem and Fourier expansion of f to conclude ([24]). □

The function $v := u'$ is called the *weak derivative* of u. Consequently, we have

$$u(t) = \int_0^t u'(l)dl + c,$$

which implies the following:

$$u(0) = u(T) = c; \quad u(t) = u(s) + \int_s^t u'(l)dl.$$

Define the Hilbert space H_T^1 as usual (hence reflexive Banach space) with the following inner product and corresponding norm: for $u, v \in H_T^1$,

$$\langle u, v\rangle := \int_0^T (u, v) + (u', v')ds; \quad \|u\|^2 := \int_0^T |u|^2 + |u'|^2 ds.$$

Next, we collect some facts for later use.

Proposition 6.2 (Compact Sobolev embedding property)**.** *The embedding $H_T^1 \subset\subset C[0, T]$ is compact.*

Proposition 6.3. *If $u \in H_T^1$ and $\int_0^T u(t)\,dt = 0$, then we have the Wirtinger inequality*

$$\int_0^T |u(t)|^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T |u'(t)|^2\,dt$$

and a Sobolev inequality

$$|u|_\infty^2 \le \frac{T}{12}\int_0^T |u'(t)|^2\,dt.$$

The compact Sobolev embedding property will give us the required weak lower semi-continuity property for the nonlinear functionals. The Wirtinger inequality supplies us equivalent norms in related Sobolev spaces with mean zero property (see [24] for complete proofs).

6.2. Variational formulation of the problems. Now for a given $T > 0$, we consider problem (5.6) in H_T^1

$$\begin{cases} -\tau'' = -D_\tau W(\tau) + q = -D_\tau(W(\tau) - q\cdot\tau) \\ \qquad\qquad \tau(0) - \tau(T) = 0; \quad \tau'(0) - \tau'(T) = 0. \end{cases}$$

Let us first consider the cases and formulations without the physical restriction $\tau_3 > 0$. Assume that:

$$\bar\tau := \frac{1}{T}\int_0^T \tau(x)dx = m.$$

Here $m \in \mathbb{R}^n$, $n = 1, 2, 3$ and we will use bar to represent mean over one period similarly. Hence, we consider the following problem

$$\begin{cases} \tau''(x) = DW(\tau) - q \\ \tau(0) = \tau(T); \quad \tau'(0) = \tau'(T); \quad \dfrac{1}{T}\displaystyle\int_0^T \tau(x)\,dx = m. \end{cases} \tag{6.1}$$

If we seek periodic solutions, q can be determined by integrating the equations above over one period; that is,

$$q = \frac{1}{T}\int_0^T DW(\tau(x))\,dx.$$

Define $v(x) = \tau(x) - m$. We see easily that $\frac{1}{T}\int_0^T v(x)\,dx = 0$, and $v(x)$ satisfies the system of equations:

$$\begin{cases} v''(x) = DW(v + m) - q \\ v(0) = v(T); \quad v'(0) = v'(T); \quad \dfrac{1}{T}\displaystyle\int_0^T v(x)\,dx = 0. \end{cases} \tag{6.2}$$

For convenience, we rewrite the above system as

$$\begin{cases} v''(x) = DW(v+m) - DW(m) + DW(m) - q \\ v(0) = v(T); \quad v'(0) = v'(T); \quad \dfrac{1}{T}\displaystyle\int_0^T v(x)\,dx = 0. \end{cases} \tag{6.3}$$

Define $\tilde{W}(v) = W(v+m) - DW(m)\cdot v$ and $\tilde{q} = q - DW(m)$. We get the following problem

$$\begin{cases} v''(x) = D\tilde{W}(v) - \tilde{q} \\ v(0) = v(T); \quad v'(0) = v'(T); \quad \dfrac{1}{T}\displaystyle\int_0^T v(x)\,dx = 0. \end{cases} \tag{6.4}$$

Here $\tilde{q}$ is determined by integration: $\frac{1}{T}\int_0^T D\tilde{W}(v)\,dx = \tilde{q}$.

Remark 6.4. We require $v_3 > -m_3$ on $[0,T]$ for models involving τ_3 direction in view of the physical assumption (2.2).

Define $F(v) = W(v+m) - W(m) - DW(m)\cdot v$ and introduce the functional

$$\mathcal{I}(v) = \int_0^T \frac{1}{2}|v'|^2\,dx + \int_0^T F(v)\,dx \tag{6.5}$$

on the space

$$H^1_{T,0} := \left\{ v \in H^1_T;\ \bar{v} = \frac{1}{T}\int_0^T v\,dx = 0 \right\}.$$

Proposition 6.5. 0 *is always a critical point of the functional* $\mathcal{I}$ *defined above on* $H^1_{T,0}$.

Proof. It is easy to verify that for $\phi \in H^1_{T,0}$, there holds

$$I'(v)\phi = \int_0^T v'\cdot\phi' + D\tilde{W}(v)\cdot\phi\,dx.$$

Taking $v = 0$ and noticing that $D\tilde{W}(0) = 0$, we get the desired result. □

Remark 6.6. By our formulation, we make 0 always a critical point and it corresponds to the constant solution. This geometric property supplies us a nice way to exclude the possibility that the periodic solution we find is constant, i.e., to help prove that the periodic waves we find are oscillatory.

Proposition 6.7. *Without physical restriction on* τ_3*, the critical point of* $\mathcal{I}$ *corresponds to the solution of* (5.6).

Proof. This can be regarded as a simple consequence of Corollary 1.1 in [24]. For completeness, we write the details here. First, assume that v solves

$$\begin{cases} v''(x) = D\tilde{W}(v) - \tilde{q} \\ v(0) = v(T); \quad v'(0) = v'(T); \quad \dfrac{1}{T}\displaystyle\int_0^T v(x)\,dx = 0. \end{cases} \tag{6.6}$$

Multiplying the equation by $\phi \in H^1_{T,0}$ and integrating, we get

$$\int_0^T v'\phi' + D\tilde{W}(v)\cdot\phi\,dx = 0,$$

i.e., v is a critical point of $\mathcal{I}$.

Next, we assume that v is a critical point and $\phi \in H^1_T$. Then $\phi - \bar{\phi} \in H^1_{T,0}$. Hence we have $\int_0^T v'\cdot(\phi-\bar{\phi})' + \int_0^T D\tilde{W}(v)\cdot(\phi-\bar{\phi})\,dx = 0$, i.e.,

$$\int_0^T v'\cdot\phi' + \int_0^T D\tilde{W}(v)\cdot\phi - \int_0^T D\tilde{W}(v)\cdot\bar{\phi}\,dx = 0.$$

Noting that $\bar{\phi} = \frac{1}{T}\int_0^T \phi\,dx$, we find that the left-hand side expression above is:

$$\int_0^T v'\cdot\phi' + \int_0^T D\tilde{W}(v)\cdot\phi - \int_0^T D\tilde{W}(v)\cdot\left(\frac{1}{T}\int_0^T \phi\,dx\right)\,dx = 0$$

Noticing that $\frac{1}{T}\int_0^T D\tilde{W}(v)\,dx = \tilde{q}$, we get $\int_0^T v'\cdot\phi' + \int_0^T (D\tilde{W}(v)-\tilde{q})\cdot\phi\,dx = 0$, which implies $v'' = D\tilde{W}(v) - \tilde{q}$. □

Remark 6.8. If we consider models involving the restriction $v_3 > -m_3$, we need to consider a variational problem with this constraint, which will make the admissible set not weakly closed.

In order to deal with the integral constant q, we may restrict the admissible sets (or choose proper function space) on which we consider the functional or use Lagrange multiplier to recover it by adding restriction functional on the original space on which the functional is defined.

In the following, we give some propositions on general nonlinear functionals. These propositions and further materials can be found in [16, 25, 37] and the references therein.

Proposition 6.9. *Let $\mathcal{X}$ be a Banach space, I a real functional defined on $\mathcal{X}$ and U be a sequentially weakly compact set in $\mathcal{X}$. If I is weakly lower semi-continuous, then I attains its minimum on U, i.e., there is $x_0 \in U$, such that $I(x_0) = \inf_{x\in U} I(x)$.*

Proof. Let $c := \inf_{x\in U} I(x)$. By definition of inf, there exists $\{x_n\} \subset U$ such that $I(x_n) \to c$. In view that U is sequentially weakly compact, $\{x_n\}$ admits a weakly convergent subsequence, still denoted by $\{x_n\}$. Denote $x_0 \in \mathcal{X}$ the corresponding weak limit. Since U is weakly closed, we know $x_0 \in U$. Noticing that weakly lower semi-continuity of I, we have $c = \lim_n I(x_n) \geq I(x_0)$. By the definition of c, we in turn know $I(x_0) = c > -\infty$, which completes the proof. □

It is well-known that a bounded weakly closed set in a reflexive Banach space is weakly compact. In particular, a bounded closed convex set in reflexive Banach space is weakly compact since weakly close and close in norm are equivalent for convex sets. Hence we have the following corollaries:

Corollary 6.10. *Let U be a bounded weakly closed set in a reflexive Banach space $\mathcal{X}$ and I be a weakly lower semi-continuous real functional on $\mathcal{X}$. Then there exists $x_0 \in U$ such that $I(x_0) = \inf_{x\in U} \mathcal{I}$.*

Definition 6.11. A real functional I on a Banach space $\mathcal{X}$ is said to be *coercive* if

$$\lim_{|x|_{\mathcal{X}} \to +\infty} I(x) = +\infty.$$

Corollary 6.12. *Any coercive weakly lower semi-continuous real functional I defined on a reflexive Banach space $\mathcal{X}$ admits a global minimizer.*

6.3. A general existence result. In this part, we first give a general result for models with the physical assumption $\tau_3 > 0$, i.e, $v_3 > -m_3$. We will assume the following conditions on the potential W

(A1) $W \in C^2$ and $W(\tau) \to +\infty$ as $\tau_3 \to 0^+$. For $\tau_3 \leq 0$, define $W(\tau) = +\infty$;

(A2) There exist a positive constant C such that $W(\tau) \geq \frac{C}{\tau_3^2}$ for $\tau \in R^n$ $(n = 1, 2, 3)$;

(A3) There exists a constant vector $m \in \mathbb{R}^3_+ := \{m \in \mathbb{R}^3; m_3 > 0\}$ such that $\sigma\{D^2W(m)\} \cap \mathbb{R}^1_- \neq \emptyset$. Here $\sigma\{D^2W(m)\}$ is the spectrum set of $D^2W(m)$.

Assumption (A2) implies in particular that the potential is bounded from below. Assumption (A3) amounts to saying that there is a point where the potential is non-convex. From the physical point of view, this is quite reasonable.

Remark 6.13. A simple kind of potential function is that for an isentropic polytropic gas, for which $dW(\tau) = c\tau_3^{-\gamma}$, $\gamma > 1$. This yields $W(\tau) = c_2\tau_3^{1-\gamma}$, with $0 < 1-\gamma < 1$ for γ in the typical range $1 < \gamma < 2$ suggested by statistical mechanics [6], hence blowup as $\tau_3 \to 0$ at rate slower than $c\tau_3^{-2}$. Indeed, a point charge model with inverse square law yields in the continuum limit $W(\tau) \sim \tau_3^{-\frac{2}{3}}$ for dimension 3, consistent with a monatomic gas law $\gamma = \frac{5}{3}$.

Thus, in the simple gas-dynamical setting, (A2) requires a near-range repulsion stronger than inverse square. Alternatively, one may assume not point charges but particles of finite radius, as is often done in the literature, in which case $W(\tau) = \infty$ for $\tau_3 \leq \alpha$, $\alpha > 0$, also satisfying (A2). However, in this case, a much simpler argument would suffice to yield $\tau_3 \geq \alpha$ a.e.

Theorem 6.14. *Assume* (A1), (A2) *and* (A3). *If* $\left(\frac{2\pi}{T}\right)^2 < \lambda(m)$, *then we have a physical nonconstant periodic wave solution for the problem* (3.2) *for which the mean over one period of* τ *is* m. *Here* $-\lambda(m)$ *is the smallest eigenvalue of* $D^2W(m)$.

In the following lemmas of this section, we assume that (A1), (A2) and (A3) hold. Define two subsets of $H^1_{T,0}$ by

$$\mathcal{A}_1 := \{v \in H^1_{T,0};\, v_3 > -m_3\}, \quad \mathcal{A}_2 := \{v \in H^1_{T,0};\, v_3 \geq -m_3\}.$$

Remark 6.15. The admissible set $\mathcal{A}_1$ is not weakly closed in $H^1_{T,0}$.

Lemma 6.16. *Under assumptions* (A1)–(A3), $\mathcal{I}$ *is a coercive functional on* $H^1_{T,0}$.

Proof. By the definition of $\mathcal{I}$, we just need to consider the part $\int_0^T F(v)\,dx$. By assumption (A2), we have

$$\begin{aligned}
\int_0^T F(v)\,dx &= \int_0^T W(v+m) - W(m) - DW(m)\cdot v\,dx \\
&= \int_0^T W(v+m) - W(m)\,dx \\
&\geq -W(m)T \\
&> -\infty.
\end{aligned}$$

□

By the above lemma, we see that for sufficient large R the minimizers of $\mathcal{I}$ on $\mathcal{A}_i$ are restricted to the sets $\bar{\mathcal{A}}_i := \mathcal{A}_i \cap B_{H^1_{T,0}}[0,R]$ for $i = 1,2$ where $B_{H^1_{T,0}}[0,R]$ is the closed ball with center 0 and radius R in $H^1_{T,0}$. Define $S_i := \{v \in \mathcal{A}_i;\, \mathcal{I}(v) = \inf_{\tilde{v}\in\mathcal{A}_i} \mathcal{I}(\tilde{v})\}$. Obviously, we have $S_i := \{v \in \bar{\mathcal{A}}_i;\, \mathcal{I}(v) = \inf_{\tilde{v}\in\mathcal{A}_i} \mathcal{I}(\tilde{v})\}$.

Lemma 6.17. $\bar{\mathcal{A}}_2$ *is a weakly compact set in* $H^1_{T,0}$.

Proof. $\bar{\mathcal{A}}_2$ is bounded by its definition. Since H^1_T is reflexive, we know $\bar{\mathcal{A}}_2$ is weakly sequentially compact. Also, $\bar{\mathcal{A}}_2$ is convex. Indeed, we can use the definition of convexity of a set to check this easily. An appeal to Sobolev embedding theorem yields that $\bar{\mathcal{A}}_2$ is closed in norm topology of H^T_1. For a convex set, closeness in norm topology and weak topology coincides, hence we have that $\bar{\mathcal{A}}_2$ is weakly closed. Putting this information together, we have shown that $\bar{\mathcal{A}}_2$ is weakly compact. □

Lemma 6.18. *$\mathcal{I}$ is a weakly lower semi-continuous functional on $H^1_{T,0}$.*

Proof. Let $v^n \to v$ weakly in $H^1_{T,0}$. By Sobolev imbedding, we have $v^n \to v$ uniformly in $[0,T]$. Hence we have $\int_0^T F(v^n)\,dx \to \int_0^T F(v)\,dx$. Because of the mean zero property, $\int_0^T |v'|^2\,dx$ is of norm form, hence it is a weakly lower semi-continuous functional. □

Lemma 6.19. *There hold $S_2 \neq \emptyset$ and $v_3 \geq -m_3 + \epsilon$ for $v \in S_2$ under the assumption of Theorem* 6.14. *Here ϵ is a positive constant.*

Proof. By Proposition 6.8, $S_2 \neq \emptyset$. Note that $0 \in \mathcal{A}_2$, $\mathcal{I}(0) = 0$ and hence $\mathcal{I}(v) \leq 0$. Hence we will have $v_3 \geq -m_3 + \epsilon$. Indeed, suppose there were $x_0 \in [0,T]$ such that $v_3(x_0) = -m_3$. Then by Sobolev embedding there would be a positive constant K such that $|v_3(x)+m_3| = |(v_3(x)+m_3)-(v_3(x_0)+m_3)| \leq K|x-x_0|^{\frac{1}{2}}$ for $x \in [0,T]$. By assumption (A2), we would have $I(v) = \int_0^T(\frac{1}{2})|v'|^2\,dx + \int_0^T W(v+m)-W(m)\,dx \geq \int_0^T CK|x-x_0|^{-1}dx - \int_0^T W(m)dx = +\infty$, a contradiction. □

Lemma 6.20. *There holds $0 \notin S_1 = S_2$ under the assumption of Theorem* 6.14.

Proof. Consider the second variation. An easy computation shows that for v, ϕ in $H^1_{T,0}$

$$\mathcal{I}''(v) : (\phi \otimes \phi) = \int_0^T |\phi'|^2\,dx + \int_0^T D^2W(v+m) : (\phi \otimes \phi)\,dx.$$

To show $0 \notin S_2$, consider

$$\mathcal{I}''(0) : (\phi \otimes \phi) = \int_0^T |\phi'|^2\,dx + \int_0^T D^2W(m) : (\phi \otimes \phi)\,dx.$$

Let $\tilde{\phi}(x) = \eta \sin\left(\frac{2\pi x}{T}\right)$ for $0 < \eta < m_3$ and $v_0 \in \mathbb{R}^3$ be a unit eigenvector corresponding to $-\lambda(m)$. We see that $\phi(x) := \tilde{\phi}(x)v_0 \in \mathcal{A}_2$. Since 0 is a critical point of $\mathcal{I}$ on $H^1_{T,0}$ and

$$\begin{aligned}\mathcal{I}''(0) : (\phi v_0 \otimes \phi v_0) &= \int_0^T \eta^2 \left(\frac{2\pi}{T}\right)^2 \left(\cos\left(\frac{2\pi x}{T}\right)\right)^2 dx - \lambda(m) \int_0^T \eta^2 \left(\sin\left(\frac{2\pi x}{T}\right)\right)^2 dx \\ &= \frac{\eta^2 T}{2} \left\{ \left(\frac{2\pi}{T}\right)^2 - \lambda(m) \right\} \\ &< 0.\end{aligned}$$

Hence we see that $0 \notin S_2$ and $S_1 = S_2$ is obvious. □

Proof of Theorem 6.14. Combining Lemmas 6.16–6.20, we finish the proof of Theorem 6.14. □

Remark 6.21. The condition $\left(\frac{2\pi}{T}\right)^2 < \lambda(m)$ in Theorem 6.14 on the period T, is readily seen by Fourier analysis to be the sharp criterion for stability of the constant solution $\tau \equiv m$, $u \equiv 0$. Equivalently, it is the Hopf bifurcation condition as period is increased, marking the minimum period of bifurcating periodic waves. Thus, it is natural, and no real restriction. On the other hand, there may well exist minimizers at whose mean m W is convex; this condition is sufficient but certainly not necessary. Likewise, there exist saddle-point solutions not detected by the direct approach.

6.4. Relation to standard results, and directions for further study. In the scalar case $\tau \in \mathbb{R}^1$, the condition that $D^2W(m)$ have a negative eigenvalue is equivalent to convexity of the Hamiltonian H at the equilibrium $(m, 0)$, under which assumption there are many results on existence of periodic solutions of all amplitudes; see, for example, [30] and later elaborations. Likewise in the vectorial case $\tau \in \mathbb{R}^d$, $d > 1$, if $D^2W(m) < 0$, then we may appeal to standard theory to obtain existence of periodic solutions by a variety of means; indeed, the convexity condition may be substantially relaxed for solutions in the large, as described in [30], and replaced by global conditions ensuring, roughly, star-shaped level sets of the Hamiltonian. On the other hand, review of the potentials considered here reveals that, typically, it is a single eigenvalue of D^2W that becomes negative and not all eigenvalues, and so these methods cannot be directly applied.

It is an interesting question to what extent such standard methods could be adapted to the situation of a Hamiltonian potential (in our case $-W$) with a single convex mode. Existence of small amplitude periodic waves at least is treatable by Hopf bifurcation analysis. The question is to what extent if any one can make global conclusions beyond what we have done here, in particular, to relax for large solutions the nonconvexity condition on W at m. Finally, it would be interesting to find natural and readily verifiable conditions for existence of saddle-point solutions in this context.

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