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Existence and Nonexistence of Global Solutions for Higher-Order Nonlinear Viscoelastic Equations

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Abstract. The initial-boundary value problem for some nonlinear higher-order viscoelastic equation with damping and source terms in a bounded domain is studied. The existence of global solutions for this problem is established by constructing a stable set in $H_0^l(\Omega)$ and the decay of energy based on a difference inequality due to M. Nakao is obtained. Meanwhile, under suitable conditions on relaxation function $g(\cdot)$ and the positive initial energy, it is proved that the solution blows up in the finite time and the lifespan estimates of solutions are also given.

Keywords. Nonlinear higher-order viscoelastic equation, initial-boundary value problem, global existence and nonexistence, decay estimate of solution

Mathematics Subject Classification (2010). Primary 35L75, secondary 35B40, 35G20

1. Introduction

In this paper, we are concerned with the following initial-boundary value problem of higher-order nonlinear viscoelastic equation

$$\begin{cases} u_{tt} + (-\Delta)^l u - \displaystyle\int_0^t g(t-s)(-\Delta)^l u(s)ds + a|u_t|^{m-2}u_t + (-\Delta)^k u_t \\ \qquad = b|u|^{p-2}u, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x) \in H_0^l(\Omega), \quad u_t(x,0) = u_1(x) \in L^2(\Omega), \\ \dfrac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,2,\dots,l-1, \ x \in \partial\Omega, \ t \geq 0, \end{cases} \tag{1.1}$$

where $l,\ k \geq 1$ is a natural number, $p > 2,\ m \geq 2$ and $a,\ b > 0$ are real numbers, Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$ so that

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divergence theorem can be applied, ν is unit outward normal on $\partial\Omega$, and $\frac{\partial^i u}{\partial \nu^i}$ denotes the i-order normal derivation of u.

When $l = 1$, $g(\cdot) \neq 0$ and in the absence of the strong dissipation $(-\Delta)^k u_t$, the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (x,t) \in \Omega \times R^+, \tag{1.2}$$

was investigated by S. A. Messaoudi in [9], where the author proved that any weak solution with negative initial energy blows up in finite time if $p > m$ and $\int_0^{+\infty} g(s)ds \leq \frac{p-2}{p-2+\frac{1}{p}}$, meanwhile the solution continues to exist globally for any initial data in the appropriate space if $m \geq p$. In [10], a blow-up result was obtained for positive initial energy under suitable conditions on $g(\cdot), m$ and p. Y. J. Wang [22] studied equation (1.2) and established a blow-up result with positive initial energy.

As the nonlinear dissipative term $a|u_t|^{m-2}u_t$ in (1.2) is replaced by $a(x)u_t$, M. M. Cavalcanti et al. [5] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + b|u|^{p-2}u = 0, \quad (x,t) \in \Omega \times R^+,$$

for $p > 2, g(\cdot)$ is a positive function, and $a(x) : \Omega \to R^+$ is a function which may be null on a part of Ω. Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t)$, $t \geq 0$, they obtained an exponential rate of decay.

In the case of $g(\cdot) = 0$, but without the strong damping term $(-\Delta)^k u_t$, the problem (1.1) become the following initial-boundary problem

$$\begin{cases} u_{tt} + (-\Delta)^l u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x) \in H_0^l(\Omega), \quad u_t(x,0) = u_1(x) \in L^2(\Omega), \\ \dfrac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,2,\ldots,l-1, \ x \in \partial\Omega, \ t \geq 0. \end{cases} \tag{1.3}$$

M. Nakao [12] has used Galerkin method to present the existence and uniqueness of the bounded solutions, periodic and almost periodic solutions to the problem (1.3) as the dissipative term is a linear function μu_t. M. Nakao and H. Kuwahara [14] studied the decay estimates of global solutions for the problem (1.3) with the degenerate dissipative term $a(x)u_t$ by using a difference inequality.

In the absence of the nonlinear dissipative term ($a = 0$), P. Brenner and W. von Wahl [2] proved the existence and uniqueness of classical solutions to (1.3) in Hilbert space. H. Pecher [16] investigated the existence and uniqueness of Cauchy problem for the equation in (1.3) by use of the potential well method due to L. Payne and D. H. Sattinger [15] and D. H. Sattinger [18]. B. X.

Wang [21] showed that the scattering operators map a band in H^s into H^s if the nonlinearities have critical or subcritical powers in H^s. C. X. Miao [11] obtained the scattering theory at low energy using time-space estimates and nonlinear estimates. Meanwhile, he also gave the global existence and uniqueness of solutions under the condition of low energy.

More recently, Y. J. Ye [26] dealt with the existence and asymptotic behavior of global solutions for (1.3). At the same time, A. B. Aliev and B. H. Lichaei [1] consider the Cauchy problem for equation in (1.3), and they found the existence and nonexistence criteria of global solutions using the L^p-L^q estimate for the corresponding linear problem and also established the asymptotic behavior of solutions and their derivatives as $t \to +\infty$.

In this paper, we prove the global existence for the problem (1.1) by constructing a stable set in $H_0^l(\Omega)$ and the decay of energy by applying a difference inequality due to M. Nakao. Meanwhile, under suitable conditions on relaxation function $g(\cdot)$ and the positive initial energy, we obtain the blow-up result and the lifespan estimates of solution are also given.

We adopt the usual notation and convention. Let $H^l(\Omega)$ denote the Sobolev space with the usual scalar products and norm. $H_0^l(\Omega)$ denotes the closure in $H^l(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|\cdot\|_s$ the Lebesgue space $L^s(\Omega)$ norm and $\|\cdot\|$ denotes $L^2(\Omega)$ norm, we write equivalent norm $\|D^l\cdot\|$ instead of $H_0^l(\Omega)$ norm $\|\cdot\|_{H_0^l(\Omega)}$, where D denotes the gradient operator, that is $D\cdot = \nabla\cdot = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$. Moreover, $D^l\cdot = \Delta^j\cdot$ if $l = 2j$ and $D^l\cdot = D\Delta^j\cdot$ if $l = 2j+1$. In addition, C_i $(i = 1, 2, \ldots)$ denotes various positive constants which depend on the known constants and may be different at each appearance.

This paper is organized as follows: In the next section, we give some preliminaries. In Section 3, we study the existence of global solutions of problem (1.1). The Section 4 is devote to the study of the energy decay estimate of global solutions. Then in the last section, we are devoted to the proof of global nonexistence of solution to the problem (1.1), but without strong dissipative term $(-\Delta)^k u_t$ in the equation of (1.1).

2. Preliminaries

To prove our main results, we make the following assumptions.

(A1) $g : R^+ \to R^+$ is a bounded C^1 function which satisfies

$$g(s) > 0, \quad g'(s) \le 0, \quad \beta = 1 - \int_0^{+\infty} g(s)ds > 0,$$

and there exist positive constants η_1 and η_2 such that

$$-\eta_1 g(t) \le g'(t) \le -\eta_2 g(t).$$

(A2) $0 \le k \le l$ are natural number, and $2 \le p < +\infty$ if $n \le 2l$ and $2 \le p \le \frac{2n}{n-2l}$ if $n > 2l$.

Now, we define the following functionals:

$$J(t) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|D^l u(t)\|^2 + \frac{1}{2}(g \circ D^l u)(t) - \frac{b}{p}\|u(t)\|_p^p, \tag{2.1}$$

$$K(t) = \left(1 - \int_0^t g(s)ds\right)\|D^l u(t)\|^2 + (g \circ D^l u)(t) - b\|u(t)\|_p^p, \tag{2.2}$$

for $u \in H_0^l(\Omega)$, where

$$(g \circ D^l u)(t) = \int_0^t g(t-s)\|D^l u(s) - D^l u(t)\|^2 ds.$$

Then we introduce the stable set W by

$$W = \{u \in H_0^l(\Omega) : \ K(t) > 0\} \cup \{0\}. \tag{2.3}$$

We denote the total energy related to the problem (1.1) by

$$\begin{aligned} E(t) &= \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|D^l u(t)\|^2 + \frac{1}{2}(g \circ D^l u)(t) - \frac{b}{p}\|u(t)\|_p^p \\ &= \frac{1}{2}||u_t(t)||^2 + J(t) \end{aligned} \tag{2.4}$$

for $u \in H_0^l(\Omega)$, $t \ge 0$ and $E(0) = \frac{1}{2}||u_1||^2 + J(0)$ is the initial total energy.

For latter applications, we list up some lemmas.

Lemma 2.1. *Let s be a number with $2 \le s < +\infty$ if $n \le 2l$ and $2 \le s \le \frac{2n}{n-2l}$ if $n > 2l$. Then there is a constant B_1 depending on Ω and s such that*

$$\|u\|_s \le B_1 \|D^l u\|, \quad \forall u \in H_0^l(\Omega).$$

Lemma 2.2. (Young inequality) *Let X, Y and ε are positive constants and $p,\ q \ge 1,\ \frac{1}{p} + \frac{1}{q} = 1$. Then one has the inequality*

$$XY \le \frac{\varepsilon^p X^p}{p} + \frac{Y^q}{q\varepsilon^q}.$$

Lemma 2.3. *Supposed that* (A1) *holds and that $u(t)$ is a solution of the problem* (1.1)*, then $E(t)$ is a non-increasing function for $t > 0$ and*

$$E'(t) = \frac{1}{2}(g' \circ D^l u)(t) - \frac{1}{2}g(t)\|D^l u(t)\|^2 - \|D^k u_t(t)\|^2 - a\|u_t(t)\|_m^m \le 0, \tag{2.5}$$

where

$$(g' \circ D^l u)(t) = \int_0^t g'(t-s)\|D^l u(s) - D^l u(t)\|^2 ds.$$

Proof. Multiplying the equation in (1.1) by u_t, and integrating over $\Omega \times [0,t]$. Then we get from integrating by parts that

$$E(t) = E(0)+\int_0^t \left[\frac{1}{2}(g' \circ D^l u)(s) - \frac{1}{2}g(s)\|D^l u\|^2 - \|D^k u_t\|^2 - a\|u_t\|_m^m\right] ds \quad (2.6)$$

for $t \geq 0$. Being the primitive of an integrable function, $E(t)$ is absolutely continuous and equality (2.5) is satisfied. □

We conclude this section by stating a local existence result of the problem (1.1), which can be established by combination of the arguments in [4,5,7,8,17]. The readers are also referred to the references [3, 6, 24, 25].

Theorem 2.1. (Local existence) *Assume that* (A1) *and* (A2) *hold, if* $(u_0, u_1) \in H_0^l(\Omega) \times L^2(\Omega)$. *Then there exists* $T > 0$ *such that the problem* (1.1) *has a unique local solution* $u(t)$ *which satisfies* $u \in C([0,T); H_0^l(\Omega))$ *and* $u_t \in C([0,T); L^2(\Omega)) \cap L^m(\Omega \times [0,T)) \cap L^2([0,T); H_0^k(\Omega))$. *Moreover, at least one of the following statements holds true:*

$$\begin{cases} (1) \quad \|u_t(t)\|^2 + \|D^l u(t)\|^2 \to \infty \ \text{as} \ t \to T^-, \\ (2) \quad T = +\infty. \end{cases} \quad (2.7)$$

3. Global solutions

The following theorem shows that the solution obtained in Theorem 2.1 is a global solution if $p \leq m$.

Theorem 3.1. *Let* (A1) *and* (A2) *hold and* $p \leq m$. *Then the local solution furnished in Theorem* 2.1 *is a global solution and* T *may be taken arbitrarily large.*

Proof. Let u be a solution to the problem (1.1) defined on $[0,T]$ which is obtained in Theorem 2.1. We define

$$\begin{aligned} E_1(t) &= \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}\beta\|D^l u(t)\|^2 + \frac{b}{p}\|u(t)\|_p^p, \\ E_2(t) &= E(t) + \frac{2b}{p}\|u(t)\|_p^p. \end{aligned} \quad (3.1)$$

Our aim is to prove that the following inequality holds:

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\beta\|D^l u\|^2 + \frac{b}{p}\|u\|_p^p + \int_0^t (\|D^k u_t\|^2 + a\|u_t\|_m^m) ds \leq C_T, \quad (3.2)$$

for $t \in [0,T]$, where C_T depends on $\|D^l u_0\|^2$ and $\|u_1\|^2$.

It follows from (2.4) and (3.1) that

$$E_2(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2 + \frac{1}{2}(g \circ D^l u)(t) + \frac{b}{p}\|u\|_p^p, \quad (3.3)$$

which implies that

$$\frac{b}{p}\|u\|_p^p \leq E_2(t). \quad (3.4)$$

We obtain from (2.5) and (3.1) that

$$\begin{aligned} &E_2(t)+\frac{1}{2}\int_0^t g(s)\|D^l u\|^2 ds - \frac{1}{2}\int_0^t (g' \circ D^l u)(s)ds + \int_0^t (\|D^k u_t\|^2 + a\|u_t\|_m^m)ds \\ &\leq E_2(0) + 2b\int_0^t \int_\Omega |u|^{p-1}|u_t|dxds. \end{aligned} \quad (3.5)$$

We are going to estimate the last term in (3.5) in the following. Putting $Q_t = \Omega \times [0, t]$ and

$$Q_1 = \{(x, s) \in Q_t : \ |u(x, s)| \leq 1\}, \quad Q_2 = \{(x, s) \in Q_t : \ |u(x, s)| \geq 1\}.$$

Then

$$I = \int_0^t \int_\Omega |u|^{p-1}|u_t|dxds = \int_{Q_1} |u|^{p-1}|u_t|dxds + \int_{Q_2} |u|^{p-1}|u_t|dxds = I_1 + I_2. \quad (3.6)$$

Next we deal with I_1 and I_2 in (3.6). It is easy to see from Lemma 2.2, (3.1) and (3.3) that

$$I_1 \leq \int_{Q_1} |u_t|dxds \leq \delta|Q_t| + C_\delta \int_{Q_1} |u_t|^2 dxds \leq \delta|Q_t| + 2C_\delta \int_0^t E_2(s)ds, \quad (3.7)$$

for some $\delta > 0$ and in which $|Q_t|$ denotes the Lebesgue measure of Q_t.

Let $\alpha = \frac{m-p}{m}$, then by $p \leq m$, we have $\alpha \geq 0$ and $(p+\alpha-1)\frac{m}{m-1} = p$. Since $|u| \geq 1$ on Q_2, then we get from Lemma 2.2 and (3.4) that

$$\begin{aligned} I_2 &\leq \int_{Q_2} |u|^{p+\alpha-1}|u_t|dxds \\ &\leq \varepsilon \int_{Q_2} |u_t|^m dxds + C_\varepsilon \int_{Q_2} |u|^p dxds \\ &\leq \varepsilon \int_0^t \|u_t\|_m^m ds + \frac{pC_\varepsilon}{b} \int_0^t E_2(s)ds, \end{aligned} \quad (3.8)$$

for any $\varepsilon > 0$. We obtain from (3.6), (3.7) and (3.8) that

$$I \leq \delta|Q_t| + \varepsilon \int_0^t \|u_t\|_m^m ds + (2C_\delta + \frac{pC_\varepsilon}{b}) \int_0^t E_2(s)ds. \quad (3.9)$$

By choosing $\varepsilon > 0$ small enough, then it follows from (3.5) and (3.9) that

$$\begin{aligned}&E_2(t) + \frac{1}{2}\int_0^t g(s)\|D^l u\|^2 ds - \frac{1}{2}\int_0^t (g' \circ D^l u)(s) ds \\ &+ C_\varepsilon \int_0^t \|u_t(s)\|_m^m ds + \int_0^t \|D^k u_t(s)\|^2 ds \\ &\le E_2(0) + C\delta|Q_t| + C_{\varepsilon,\delta}\int_0^t E_2(s) ds,\end{aligned} \tag{3.10}$$

for some positive constants $C_\varepsilon > 0$ and $C_{\varepsilon,\delta} > 0$. It follows from (3.10) and Gronwall's inequality that

$$E_2(t) \le (E_2(0) + C\delta|Q_t|)e^{Ct}, \tag{3.11}$$

in which $C > 0$ is some positive constant.

Finally, we infer from (3.10) and (3.11) that

$$\begin{aligned}&E_2(t) + \frac{1}{2}\int_0^t g(s)\|D^l u\|^2 ds - \frac{1}{2}\int_0^t (g' \circ D^l u)(s) ds \\ &+ C_\varepsilon \int_0^t \|u_t(s)\|_m^m ds + \int_0^t \|D^k u_t(s)\|^2 ds \\ &\le C_T(E_2(0) + C\delta|Q_T|),\end{aligned} \tag{3.12}$$

for all $0 < t \le T$, where T is arbitrary. Then (3.2) follows from (3.3) and (3.12). Therefore, the conclusion in Theorem 3.1 is valid according to (3.2) and the standard continuation argument. Thus, the proof of Theorem 3.1 is now complete. □

To study and prove the second result on the global existence of solution to the problem (1.1), we need the following Lemmas:

Lemma 3.1. *Supposed that* (A1) *and* (A2) *hold, then*

$$\frac{p-2}{2p}\left[\left(1 - \int_0^t g(s) ds\right)\|D^l u\|^2 + (g \circ D^l u)(t)\right] \le J(t), \tag{3.13}$$

for $u \in W$.

Proof. By the definition of $K(t)$ and $J(t)$, we have the following identity

$$pJ(t) = K(t) + \frac{p-2}{2}\left[\left(1 - \int_0^t g(s) ds\right)\|D^l u\|^2 + (g \circ D^l u)(t)\right]. \tag{3.14}$$

Since $u \in W$, so we get $K(t) > 0$. Therefore, we obtain from (3.14) that (3.13) is valid. □

Lemma 3.2. *Let* (A1) *and* (A2) *hold, if* $u_0 \in W$ *and* $u_1 \in L^2(\Omega)$ *such that*

$$\theta = \frac{bB_1^p}{\beta}\left[\frac{2p}{(p-2)\beta}E(0)\right]^{\frac{p-2}{2}} < 1, \tag{3.15}$$

then $u(t) \in W$*, for each* $t \in [0, T)$.

Proof. Since $u_0 \in W$, so $K(0) > 0$. Then it follows from the continuity of $u(t)$ that

$$K(t) \geq 0, \tag{3.16}$$

for some interval near $t=0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (3.16) holds on $[0, t_{\max})$.

We have from (2.4) and (3.13) that

$$\frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2 + (g \circ D^l u)(t)\right] \leq E(t), \tag{3.17}$$

and it follows from Lemma 2.3 that

$$\left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2 + (g \circ D^l u)(t) \leq \frac{2p}{p-2}E(0). \tag{3.18}$$

We get from (A1) and (3.18) that

$$\beta\|D^l u\|^2 \leq \left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2 \leq \frac{2p}{p-2}E(0), \tag{3.19}$$

for all $t \in [0, t_{max})$.

By exploiting (A1), Lemma 2.1, (3.15) and (3.19), we easily arrive at $b\|u(t)\|_p^p \leq bB_1^p\|D^l u(t)\|^p \leq u(t)\|^{p-2}\beta\|D^l u(t)\|^2 \leq \frac{bB_1^p}{\beta}\left[\frac{2p}{(p-2)\beta}E(0)\right]^{\frac{p-2}{2}}\beta\|D^l u(t)\|^2$ $\leq \theta\beta\|D^l u(t)\|^2 \leq \theta\left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2 < \left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2$, for all $t \in [0, t_{\max})$. Therefore, $K(t) = \left(1 - \int_0^t g(s)ds\right)\|D^l u\|^2 + (g \circ D^l u)(t) - b\|u\|_p^p > 0$ on $t \in [0, t_{\max})$. By repeating this procedure, and using the fact that

$$\lim_{t \to t_{\max}} \frac{bB_1^p}{\beta}\left[\frac{2p}{(p-2)\beta}E(t)\right]^{\frac{p-2}{2}} \leq \theta < 1,$$

$t_{\max}$ is extended to T. Thus, we conclude that $u(t) \in W$ on $[0, T)$. □

The following result is concerned with the existence of global solution without the restricted condition $p \leq m$. The result reads as follows:

Theorem 3.2. *Assumed that* (A1) *and* (A2) *hold.* $u(t)$ *is a local solution obtained in Theorem* 2.1*. If* $u_0 \in W$ *and* $u_1 \in L^2(\Omega)$ *satisfy* (3.15)*, then the solution* $u(t)$ *is a global and bounded solution of the problem* (1.1).

Proof. It suffices to show that $\|D^l u(t)\|^2 + \|u_t(t)\|^2$ is bounded uniformly with respect to t.

Under the hypotheses in Theorem 3.2, we get from Lemma 3.2 that $u(t)\in W$ on $[0,T)$. So the formula (3.13) in Lemma 3.1 holds on $[0,T)$.

Therefore, we have from Lemma 3.1 that

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2 + \frac{p-2}{2p}\left[\beta\|D^l u\|^2 + (g\circ D^l u)(t)\right]\\ &\le \frac{1}{2}\|u_t\|^2 + \frac{p-2}{2p}\left[\left(1-\int_0^t g(s)ds\right)\|D^l u\|^2 + (g\circ D^l u)(t)\right]\\ &\le \frac{1}{2}\|u_t\|^2 + J(t) = E(t)\\ &\le E(0).\end{aligned} \tag{3.20}$$

Hence, we get $\|u_t(t)\|^2 + \|D^l u(t)\|^2 \le \max\left(2, \frac{2p}{p-2}\right)E(0) < +\infty$.

The above inequality and the continuation principle [7,19] lead to the global existence of the solution, that is, $T = +\infty$. Thus, the solution $u(x,t)$ is a global solution of the problem (1.1). □

4. Energy decay of global solution

The following lemmas play an important role in studying the energy decay estimate of global solutions for the problem (1.1).

Lemma 4.1. [13] *Suppose that $\varphi(t)$ is a non-increasing and nonnegative function on $[0,T]$, $T>1$, such that*

$$\varphi(t)^{1+r} \le \omega_0[\varphi(t) - \varphi(t+1)], \quad \text{on } [0,T],$$

where ω_0 is a positive constant and r is a nonnegative constant. Then $\varphi(t)$ has the following decay properties

(i) *If $r>0$, then*

$$\varphi(t) \le \left(\varphi(0)^{-r} + \omega_0^{-1} r[t-1]^+\right)^{-\frac{1}{r}} \quad \text{on } [0,T],$$

where $[t-1]^+ = \max\{t-1,\ 0\}$.

(ii) *If $r=0$, then*

$$\varphi(t) \le \varphi(0)e^{-\vartheta[t-1]^+} \quad \text{on } [0,T],$$

where $\vartheta = \ln\frac{\omega_0}{\omega_0-1}$, $\omega_0>1$.

Lemma 4.2. *Let u satisfy the assumptions of Lemma 3.2. Then there exists $0<\theta_1<1$ such that*

$$\left(1-\int_0^t g(s)ds\right)\|D^l u(t)\|^2 \le \frac{1}{\theta_1}K(t), \quad t\in[0,T], \tag{4.1}$$

where $\theta_1 = 1-\theta$.

Proof. From (3.20), we get that $b\|u(t)\|_p^p \le \theta\left(1-\int_0^t g(s)ds\right)\|D^l u(t)\|^2$, $t\in[0,T]$. Let $\theta = 1-\theta_1$, then

$$b\|u(t)\|_p^p \le (1-\theta_1)\left(1-\int_0^t g(s)ds\right)\|D^l u(t)\|^2 + (g\circ D^l u)(t), \quad t\in[0,T]. \tag{4.2}$$

From (2.2) and (4.2) we have that (4.1) is valid. □

Theorem 4.1. *Let the assumptions of Theorem* 3.2 *hold, and* $2\le m<+\infty$ *if* $n\le 2l$ *and* $2\le m\le \frac{2n}{n-2l}$ *if* $n>2l$. *If* $u_0\in W$ *and* $u_1\in L^2(\Omega)$ *satisfy* (3.16), *then the global solution* $u\in W$ *for the problem (1.1) have the following decay properties:*

(i) *If* $m=2$, *then*

$$E(t)\le E(0)e^{-\vartheta[t-1]^+}.$$

(ii) *If* $m>2$, *then*

$$E(t)\le \left(E(0)^{-\frac{m-2}{2}}+\hbar[t-1]^+\right)^{-\frac{2}{m-2}}.$$

where ϑ *and* $\hbar$ *are positive constants which will be determined later.*

Proof. Multiplying equation in (1.1) by u_t and integrating over $\Omega\times[t,t+1]$, we get

$$\begin{aligned}&\int_t^{t+1}(a\|u_t(s)\|_m^m+\|D^k u_t(s)\|_2^2)ds\\&-\frac{1}{2}\int_t^{t+1}(g'\circ D^l u)(s)ds+\frac{1}{2}\int_t^{t+1}g(s)\|D^l u(s)\|^2ds\\&=E(t)-E(t+1).\end{aligned}\tag{4.3}$$

Thus, there exists $t_1\in\left[t,t+\frac{1}{4}\right]$, $t_2\in\left[t+\frac{3}{4},t+1\right]$ such that

$$\begin{aligned}&4[a\|u_t(t_i)\|_m^m+\|D^k u_t(t_i)\|_2^2]-2(g'\circ D^l u)(t_i)+2g(t_i)\|D^l u(t_i)\|^2\\&=E(t)-E(t+1),\quad i=1,2.\end{aligned}\tag{4.4}$$

On the other hand, we multiply the equation in (1.1) by u and integrate over $\Omega\times[t_1,t_2]$. We obtain

$$\begin{aligned}&\int_{t_1}^{t_2}K(s)ds\\&=\int_{t_1}^{t_2}\|u_t\|^2ds+(u_t(t_1),u(t_1))-(u_t(t_2),u(t_2))\\&\quad-\int_{t_1}^{t_2}\int_\Omega|u_t|^{m-2}u_t u\,dxds-\int_{t_1}^{t_2}\int_\Omega D^k u\cdot D^k u_t\,dxds\\&\quad+\int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)D^l u(t)[D^l u(s)-D^l u(t)]dsdxdt+\int_{t_1}^{t_2}(g\circ D^l u)(s)ds.\end{aligned}\tag{4.5}$$

From Hölder inequality, (A2), (4.3) and (3.20) we have that

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_\Omega D^k u\cdot D^k u_t dxds\right| &\le \int_{t_1}^{t_2}\|D^k u\|\cdot\|D^k u_t\|ds\\ &\le B_2\int_{t_1}^{t_2}\|D^l u\|\cdot\|D^k u_t\|ds\\ &\le \frac{1}{2}C_2[E(t)-E(t+1)]^{\frac{1}{2}}\sup\nolimits_{t\le s\le t+1}E(s)^{\frac{1}{2}},\end{aligned}\tag{4.6}$$

where $C_2=(\frac{2pB_2^2}{\beta(p-2)})^{\frac{1}{2}}$ and B_2 is a Sobolev constant from $H_0^l(\Omega)$ to $H_0^k(\Omega)$.

From (4.3) and Hölder inequality, we have

$$\int_{t_1}^{t_2}\|u_t\|^2ds\le C_3\left(\int_{t_1}^{t_2}\|u_t\|_m^m ds\right)^{\frac{2}{m}}\le C_3(E(t)-E(t+1))^{\frac{2}{m}},\tag{4.7}$$

where $C_3=(2|\Omega|)^{\frac{m-2}{m}}$ and $|\Omega|$ denotes the Lebesgue measure of the bounded domain Ω.

We get from (3.20), (4.4), Lemma 2.1 and Hölder inequality that

$$\begin{aligned}|(u_t(t_i),u(t_i))| &\le \|u_t(t_i)\|\|u(t_i)\|\\ &\le B_1|\Omega|^{\frac{m-2}{2m}}\|u_t(t_i)\|_m\|D^l u(t_i)\|\\ &\le C_5(E(t)-E(t+1))^{\frac{1}{m}}\sup_{t\le s\le t+1}E(s)^{\frac{1}{2}},\quad i=1,2,\end{aligned}\tag{4.8}$$

where $C_5=\frac{B_1|\Omega|^{\frac{m-2}{2m}}}{4a}\big(\frac{2p}{\beta(p-2)}\big)^{\frac{1}{2}}$.

From Hölder inequality and Lemma 2.1, (3.20) and (4.3), we get

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_\Omega|u_t|^{m-2}u_t u dxds\right| &\le \int_{t_1}^{t_2}\|u_t\|_m^{m-1}\|u\|_m ds\\ &\le B_1\int_{t_1}^{t_2}\|u_t\|_m^{m-1}\|D^l u\|ds\\ &\le C_6(E(t)-E(t+1))^{\frac{m-1}{m}}\sup_{t\le s\le t+1}E(s)^{\frac{1}{2}},\end{aligned}\tag{4.9}$$

in which $C_6=\big(\frac{2pB_1^2}{\beta(p-2)}\big)^{\frac{1}{2}}a^{-\frac{m-1}{m}}$.

By using Lemma 2.2, we have

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)D^l u(t)[D^l u(s)-D^l u(t)]dsdxdt\\ &\le \sigma\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^l u(t)\|^2dsdt+\frac{1}{4\sigma}\int_{t_1}^{t_2}(g\circ D^l u)(s)ds,\end{aligned}\tag{4.10}$$

where σ is some positive constant to be chosen later.

Therefore, from (4.5)–(4.10) we get that

$$\begin{aligned}
&\int_{t_1}^{t_2} K(t)ds \\
&\le C_7\Bigg[(E(t)-E(t+1))^{\frac{1}{2}} \sup_{t\le s\le t+1} E(s)^{\frac{1}{2}} + (E(t)-E(t+1))^{\frac{2}{m}} \\
&\quad + (E(t)-E(t+1))^{\frac{m-1}{m}} \sup_{t\le s\le t+1} E(s)^{\frac{1}{2}} + (E(t)-E(t+1))^{\frac{1}{m}} \sup_{t\le s\le t+1} E(s)^{\frac{1}{2}}\Bigg] \\
&\quad + \sigma \int_{t_1}^{t_2}\int_0^t g(t-s)\|D^l u(t)\|^2 dsdt + \left(\frac{1}{4\sigma}+1\right)\int_{t_1}^{t_2}(g\circ D^l u)(s)ds,
\end{aligned} \tag{4.11}$$

in which $C_7 = \max\{\frac{1}{2}C_2,\ C_3,\ C_5,\ C_6\}$.

On the other hand, we obtain from (A1) and (4.3) that

$$\int_{t_1}^{t_2} (g\circ D^l u)(s)ds \le -\frac{1}{\eta_2}\int_{t_1}^{t_2} (g'\circ D^l u)(s)ds \le \frac{2}{\eta_2}(E(t)-E(t+1)). \tag{4.12}$$

By (A1) and Lemma 4.2, we have

$$\begin{aligned}
\int_{t_1}^{t_2}\int_0^t g(t-s)\|D^l u(t)\|^2 dsdt &\le -\frac{1}{\eta_2}\int_{t_1}^{t_2}\int_0^t g'(t-s)\|D^l u(t)\|^2 dsdt \\
&= \frac{1}{\eta_2}\int_{t_1}^{t_2} [g(0)-g(t)]\|D^l u(t)\|^2 dt \\
&\le \frac{g(0)}{\eta_2}\int_{t_1}^{t_2} \|D^l u(t)\|^2 dt \\
&\le \frac{g(0)}{\beta\theta_1\eta_2}\int_{t_1}^{t_2} K(t)dt.
\end{aligned} \tag{4.13}$$

Choosing σ such that $\frac{\sigma g(0)}{\beta\theta_1\eta_2} = \frac{1}{2}$, then we get from (4.11), (4.12) and (4.13) that

$$\begin{aligned}
\int_{t_1}^{t_2} K(t)ds \le C_8\Bigg[&(E(t)-E(t+1))^{\frac{1}{2}} \sup_{t\le s\le t+1} E(s)^{\frac{1}{2}} + (E(t)-E(t+1))^{\frac{2}{m}} \\
&+ (E(t)-E(t+1))^{\frac{m-1}{m}} \sup_{t\le s\le t+1} E(s)^{\frac{1}{2}} \\
&+ (E(t)-E(t+1))^{\frac{1}{m}} \sup_{t\le s\le t+1} E(s)^{\frac{1}{2}} + (E(t)-E(t+1))\Bigg],
\end{aligned} \tag{4.14}$$

where $C_8 = \max\{2C_7,\ \frac{4}{\eta_2}(\frac{1}{4\sigma}+1)\}$. It follows from (2.1), (2.2) and Lemma 4.2 that

$$\begin{aligned}
J(t) &= \frac{p-2}{2p}\left(1-\int_0^t g(s)ds\right)\|D^l u(t)\|^2 + \frac{p-2}{2p}(g\circ D^l u)(t) + \frac{1}{p}K(t) \\
&\le \frac{p-2}{2p}(g\circ D^l u)(t) + \frac{p-2+2\theta_1}{2p\theta_1}K(t).
\end{aligned} \tag{4.15}$$

We have from (2.4) and (4.15) that

$$E(t)=\frac{1}{2}\|u_t\|^2+J(t)\le\frac{1}{2}\|u_t\|^2+\frac{p-2}{2p}(g\circ D^lu)(t)+\frac{p-2+2\theta_1}{2p\theta_1}K(t). \quad (4.16)$$

By integrating (4.16) over $[t_1,t_2]$, we obtain from (4.7), (4.12) and (4.14) that

$$\begin{aligned}\int_{t_1}^{t_2}E(t)ds\le C_9\Big[&(E(t)-E(t+1))^{\frac{1}{2}}\sup_{t\le s\le t+1}E(s)^{\frac{1}{2}}+2(E(t)-E(t+1))^{\frac{2}{m}}\\&+(E(t)-E(t+1))^{\frac{m-1}{m}}\sup_{t\le s\le t+1}E(s)^{\frac{1}{2}}\\&+(E(t)-E(t+1))^{\frac{1}{m}}\sup_{t\le s\le t+1}E(s)^{\frac{1}{2}}+2(E(t)-E(t+1))\Big],\end{aligned} \quad (4.17)$$

where $C_9=\max\{\frac{1}{2}C_3,\ \frac{p-2+2\theta_1}{2p\theta_1}C_8,\ \frac{p-2}{p\eta_2}\}$.

Integrating both sides of (2.5) over $[t,t_2]$, we obtain

$$\begin{aligned}E(t)=E(t_2)+&\int_t^{t_2}\|D^ku_t(s)\|^2ds+a\int_t^{t_2}\|u_t(s)\|_m^mds\\&-\frac{1}{2}\int_t^{t_2}(g'\circ D^lu)(s)ds+\frac{1}{2}\int_t^{t_2}g(s)\|D^lu(s)\|^2ds.\end{aligned} \quad (4.18)$$

Since $t_2-t_1\ge\frac{1}{2}$, we get

$$E(t_2)\le 2\int_{t_1}^{t_2}E(s)ds. \quad (4.19)$$

We conclude from (4.3), (4.18) and (4.19) that

$$E(t)\le 2\int_{t_1}^{t_2}E(s)ds+(E(t)-E(t+1)). \quad (4.20)$$

We have from (4.17) and (4.20) that

$$\begin{aligned}E(t)\le 2C_9[&(E(t)-E(t+1))^{\frac{1}{2}}\\&+(E(t)-E(t+1))^{\frac{m-1}{m}}+(E(t)-E(t+1))^{\frac{1}{m}}]E(t)^{\frac{1}{2}}\\&+C_{10}[(E(t)-E(t+1))^{\frac{2}{m}}+(E(t)-E(t+1))],\end{aligned} \quad (4.21)$$

where $C_{10}=\max\{4C_9,\ 1\}$.

Therefore, we obtain from Lemma 2.2 in which we let $\varepsilon=1,p=q=2$ that

$$E(t)\le C_{11}[(E(t)-E(t+1))*(E(t)-E(t+1))^{\frac{2(m-1)}{m}}+(E(t)-E(t+1))^{\frac{2}{m}}], \quad (4.22)$$

where $C_{11}=\max\{2C_9,\ C_{10}\}>1$.

When $m = 2$, we have

$$E(t) \leq 3C_{11}[(E(t) - E(t+1))]. \tag{4.23}$$

Applying Lemma 4.1 to (4.23), we get $E(t) \leq E(0)e^{-\vartheta[t-1]^+}$, where $\vartheta = \ln \frac{3C_{11}}{3C_{11}-1}$.

When $m > 2$, we obtain from (4.22) that

$$E(t) \leq C_{12}[E(t) - E(t+1)]^{\frac{2}{m}}, \tag{4.24}$$

where $C_{12} = C_{11}[1+E(0)^{\frac{m}{2}}+E(0)^{m-1}]$. Obviously, we find that $\lim_{E(0)\to 0} C_{12} = C_{11}$.

We have from (4.24) that

$$E(t)^{\frac{m}{2}} \leq C_{13}[E(t) - E(t+1)], \tag{4.25}$$

where $C_{13} = C_{12}^{\frac{m}{2}}$. Consequently, we obtain from (4.25) and Lemma 4.1 that $E(t) \leq \left(E(0)^{-\frac{m-2}{2}} + \hbar[t-1]^+\right)^{-\frac{2}{m-2}}$, where $\hbar = \frac{m-2}{2C_{13}}$.

Thus, we complete the proof of Theorem 4.1. □

5. Blow-up of solution

In this section, we shall discuss the blow-up property for the problem (1.1) in which the equation don't contain the strong dissipative term $(-\Delta)^k u_t$. That is the following initial-boundary value problem:

$$\begin{cases} u_{tt} + (-\Delta)^l u - \displaystyle\int_0^t g(t-s)(-\Delta)^l u(s)ds + a|u_t|^{m-2}u_t \\ \quad = b|u|^{p-2}u, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x) \in H_0^l(\Omega), \quad u_t(x,0) = u_1(x) \in L^2(\Omega), \\ \dfrac{\partial^i u}{\partial \nu^i} = 0, \quad i = 0,1,2,\ldots,l-1, \; x \in \partial\Omega, t \geq 0. \end{cases} \tag{5.1}$$

In order to state and prove our results, we make an extra assumption on $g(\cdot)$:

$$\int_0^{+\infty} g(s)ds < \min\left(\frac{2(p-2)}{2p-3}, \quad \frac{2p(h-E(0))}{(2p-3)\lambda_1^2}\right), \tag{5.2}$$

where h and λ_1 are some positive constants given latter.

We observe from (2.4) that

$$E(t) \geq \frac{1}{2}[\beta\|D^l u(t)\|^2 + (g \circ D^l u)(t)] - \frac{b}{p}\|u(t)\|_p^p, \tag{5.3}$$

for $u \in H_0^l(\Omega), t \geq 0$.

By (A2) and Lemma 2.1, we get that

$$\|u\|_p \le B_1 \|D^l u\|, \tag{5.4}$$

where B_1 is the optimal Sobolev's constant from $H_0^l(\Omega)$ to $L^p(\Omega)$.

We have from (5.3) and (5.4) that

$$\begin{aligned} E(t) &\ge \frac{1}{2}[\beta\|D^l u(t)\|^2 + (g\circ D^l u)(t)] - \frac{bB_1^p}{p\beta^{\frac{p}{2}}}(\beta\|D^l u(t)\|^2)^{\frac{p}{2}} \\ &\ge \frac{1}{2}[\beta\|D^l u(t)\|^2 + (g\circ D^l u)(t)] - \frac{bB_1^p}{p\beta^{\frac{p}{2}}}[\beta\|D^l u(t)\|^2 + (g\circ D^l u)(t)]^{\frac{p}{2}} \\ &= Q\Big(\sqrt{\beta\|D^l u(t)\|^2 + (g\circ D^l u)(t)}\Big), \end{aligned} \tag{5.5}$$

where

$$Q(\lambda) = \frac{1}{2}\lambda^2 - \frac{bB_1^p}{p\beta^{\frac{p}{2}}}\lambda^p.$$

Therefore, we get that $Q'(\lambda) = \lambda - \frac{bB_1^p}{\sqrt{\beta^p}}\lambda^{p-1}$, $Q''(\lambda) = 1 - \frac{(p-1)bB_1^p}{\sqrt{\beta^p}}\lambda^{p-2}$.

Let $Q'(\lambda) = 0$, which implies that $\lambda_1 = \left(\frac{bB_1^p}{\sqrt{\beta^p}}\right)^{\frac{1}{2-p}}$. As $\lambda = \lambda_1$, an elementary calculation shows that $Q''(\lambda) = 2 - p < 0$. Thus, $Q(\lambda)$ has the maximum at λ_1 and the maximum value is

$$h = Q(\lambda_1) = \frac{p-2}{2p}\Big(\frac{bB_1^p}{\sqrt{\beta^p}}\Big)^{\frac{2}{2-p}} = \frac{p-2}{2p}\lambda_1^2. \tag{5.6}$$

Applying the idea of E. Vitillaro [20] and S. T. Wu [23], we have the following lemma.

Lemma 5.1. *Assume that* (A1) *holds and that* $u_0 \in H_0^l(\Omega)$, $u_1 \in L^2(\Omega)$. *Let* u *be a solution of* (1.1) *with the initial data energy satisfying* $0 < E(0) < h$ *and* $\beta\|D^l u_0\| > \lambda_1$, *then there exists* $\lambda_2 > \lambda_1$ *such that*

$$\beta\|D^l u(t)\|^2 + (g\circ D^l u)(t) \ge \lambda_2^2, \tag{5.7}$$

for $t > 0$.

The blow-up result of solution for the problem (5.1) reads as follows:

Theorem 5.1. *Assume that* (A1), (A2) *and* (5.2) *hold and that* $u_0 \in H_0^l(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Under the condition* $p > m$, *if* $0 < E(0) < h$ *and* $\beta\|D^l u_0\| > \lambda_1$, *then the local solution of the problem* (5.1), *which is obtained in Theorem* 2.1, *blows up at a finite time in the sense of* (2.7). *We remark that the lifespan* T^* *is estimated by* $0 < T^* \le \frac{G(0)^{\frac{\gamma}{1-\gamma}}}{\gamma C_{14}}$, *where* $G(t)$ *and* C_{14} *are given in* (5.26) *and* (5.38) *respectively, and* γ *is some positive constant given in the following proof.*

Proof. Let

$$H(t) = d - E(t), \quad t \geq 0, \tag{5.8}$$

where $d = \frac{E(0)+h}{2}$. We see from (2.5) in Lemma 2.3 that $H'(t) \geq 0$. Thus we obtain

$$H(t) \geq H(0) = d - E(0) > 0, \quad t \geq 0. \tag{5.9}$$

Let

$$F(t) = \int_\Omega u u_t dx. \tag{5.10}$$

By differentiating both sides of (5.10) on t, we get from equation in (5.1) that

$$\begin{aligned} F'(t) &= \|u_t\|^2 - \|D^l u\|^2 + b\|u\|_p^p \\ &- a\int_\Omega |u_t|^{m-2} u_t u dx + \int_\Omega \int_0^t g(t-s) D^l u(s) D^l u(t) ds. \end{aligned} \tag{5.11}$$

We observe from Lemma 2.2 with $p = q = 2, \varepsilon = \sqrt{2}$ that

$$\begin{aligned} &\int_\Omega \int_0^t g(t-s) D^l u(s) D^l u(t) ds dx \\ &= \int_\Omega \int_0^t g(t-s) D^l u(t)[D^l u(s) - D^l u(t)] ds dx + \int_0^t g(t-s) ds \|D^l u(t)\|^2 \\ &\geq -(g \circ D^l u))(t) + \frac{3}{4}\int_0^t g(s) ds \|D^l u(t)\|^2, \end{aligned} \tag{5.12}$$

We have from (2.4), (5.11) and (5.12) that

$$\begin{aligned} F'(t) \geq &\frac{p+2}{2}\|u_t\|^2 + \frac{p-2}{2}(g \circ D^l u)(t) + a_1\beta\|D^l u(t)\|^2 \\ &- a\int_\Omega |u_t|^{m-2} u_t u dx + pH(t) - pd, \end{aligned} \tag{5.13}$$

where $a_1 = \frac{1}{\beta}\big(\frac{p-2}{2} - \frac{2p-3}{4}\int_0^{+\infty} g(s)ds\big)$. We have $a_1 > 0$ by (5.2). Thus, by $\frac{p-2}{2} > a_1$, we deduce that

$$\begin{aligned} F'(t) \geq &\frac{p+2}{2}\|u_t\|^2 + a_1[\beta\|D^l u(t)\|^2 + (g \circ D^l u)(t)] \\ &- a\int_\Omega |u_t|^{m-2} u_t u dx + pH(t) - pd. \end{aligned} \tag{5.14}$$

We obtain from Lemma 5.1 that

$$\begin{aligned} &a_1[\beta\|D^l u(t)\|^2 + (g \circ D^l u)(t)] - pd \\ &= a_1\frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2}[\beta\|D^l u(t)\|^2 + (g \circ D^l u)(t)] \\ &\quad + a_1\lambda_1^2 \cdot \frac{\beta\|D^l u(t)\|^2 + (g \circ D^l u)(t)}{\lambda_2^2} - pd \\ &\geq a_1\frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2}[\beta\|D^l u(t)\|^2 + (g \circ D^l u)(t)] + a_1\lambda_1^2 - pd. \end{aligned} \tag{5.15}$$

By Lemma 5.1, we have that

$$a_1\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}>0, \tag{5.16}$$

and by (5.2), (5.6) and (5.9), we see that

$$\begin{aligned} a_1\lambda_1^2-pd &= \frac{1}{\beta}\left(\frac{p-2}{2}-\frac{2p-3}{4}\int_0^{+\infty}g(s)ds\right)\lambda_1^2-pd \\ &> \left(\frac{p-2}{2}-\frac{2p-3}{4}\int_0^{+\infty}g(s)ds\right)\lambda_1^2-pd \\ &= \frac{p(h-E(0))}{2}-\frac{(2p-3)\lambda_1^2}{4}\int_0^{+\infty}g(s)ds \\ &> 0. \end{aligned} \tag{5.17}$$

Combining (5.14)–(5.17), we see that

$$\begin{aligned} F'(t) &\geq \frac{p+2}{2}\|u_t\|^2+a_1\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}[\beta\|D^lu(t)\|^2+(g\circ D^lu)(t)] \\ &\quad -a\int_\Omega |u_t|^{m-2}u_tudx+pH(t). \end{aligned} \tag{5.18}$$

On the other hand, we have from Hölder inequality that

$$a\left|\int_\Omega |u_t|^{m-2}u_tudx\right|\leq C_{14}\|u\|_p^{1-\frac{p}{m}}\|u\|_p^{\frac{p}{m}}\|u_t\|_m^{m-1}, \tag{5.19}$$

where $C_{14}=a|\Omega|^{\frac{p-m}{mp}}$ in which $|\Omega|$ denotes the Lebesgue measure of Ω.

By $d=\frac{E(0)+h}{2}$, we see that $d<h$. Therefore, we get from (5.3), (5.8) and Lemma 5.1 that

$$H(t)\leq d-\frac{1}{2}[\beta\|D^lu(t)\|^2+(g\circ D^lu)(t)]+\frac{b}{p}\|u\|_p^p\leq h-\frac{1}{2}\lambda_1^2+\frac{b}{p}\|u\|_p^p. \tag{5.20}$$

By (5.6), we have

$$h-\frac{1}{2}\lambda_1^2=-\frac{2}{p-2}h<0, \tag{5.21}$$

so, we have from (5.9), (5.20) and (5.21) that

$$0<H(0)\leq H(t)\leq \frac{b}{p}\|u\|_p^p,\quad t\geq 0. \tag{5.22}$$

We obtain from (5.19) and (5.22) that

$$a\left|\int_\Omega |u_t|^{m-2}u_tudx\right|\leq C_3H(t)^{\frac{1}{p}-\frac{1}{m}}\|u\|_p^{\frac{p}{m}}\|u_t\|_m^{m-1}, \tag{5.23}$$

where $C_{15} = (\frac{p}{b})^{\frac{m-p}{m}} C_{14}$.

We get from (5.8), Lemma 2.2, Lemma 2.3 and (5.23) that

$$a\left|\int_{\Omega} |u_t|^{m-2} u_t u dx\right| \le C_{15}[\varepsilon^m \|u\|_p^p + \varepsilon^{-\frac{m}{m-1}} H'(t)]H(t)^{-\alpha}, \tag{5.24}$$

where $\alpha = \frac{1}{m} - \frac{1}{p}$, $\varepsilon > 0$. Let $0 < \gamma < \alpha$, then we have from (5.9) and (5.24) that

$$a\left|\int_{\Omega} |u_t|^{m-2} u_t u dx\right| \le C_{15}[\varepsilon^m H(0)^{-\alpha} \|u\|_p^p + \varepsilon^{-\frac{m}{m-1}} H(0)^{\gamma-\alpha} H(t)^{-\gamma} H'(t)]. \tag{5.25}$$

Now, we define $G(t)$ as follows.

$$G(t) = H(t)^{1-\gamma} + \sigma F(t), \quad t \ge 0, \tag{5.26}$$

where σ is a positive constant to be determined later. By differentiating (5.26), we see from (5.18) and (5.25) that

$$\begin{aligned} G'(t) &= (1-\gamma)H(t)^{-\gamma}H'(t) + \sigma F'(t) \\ &\ge \left[1 - \gamma - \sigma C_{15} \epsilon^{-\frac{m}{m-1}} H(0)^{\gamma-\alpha}\right] H(t)^{-\gamma} H'(t) \\ &\quad + a_1 \sigma \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2} \left[\beta \|D^l u(t)\|^2 + (g \circ D^l u)(t)\right] \\ &\quad + \sigma \left[\frac{p+2}{2} \|u_t\|^2 + pH(t)\right] - C_{15} \sigma \varepsilon^m H(0)^{-\alpha} \|u\|_p^p. \end{aligned} \tag{5.27}$$

Letting $\kappa = \min\left\{\frac{p}{2},\ a_1 \beta \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2}\right\}$ and decomposing $\sigma p H(t)$ in (5.27) by

$$\sigma p H(t) = 2\kappa \sigma H(t) + \sigma(p - 2\kappa) H(t). \tag{5.28}$$

Combining (5.8), (5.27) and (5.28), we obtain that

$$\begin{aligned} G'(t) \ge& [1-\gamma-\sigma C_{15} \epsilon^{-\frac{m}{m-1}} H(0)^{\gamma-\alpha}] H(t)^{-\gamma} H'(t) \\ &+ \sigma\left(\frac{p+2}{2} - \kappa\right) \|u_t\|^2 + \sigma(p-2\kappa) H(t) + \left[\frac{2\kappa b}{p} - C_{15} \varepsilon^m H(0)^{-\alpha}\right] \sigma \|u\|_p^p \\ &+ \left[a_1 \beta \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2} - \kappa\right] \sigma \|D^l u\|^2 + \left[a_1 \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2} - \kappa\right] \sigma (g \circ D^l u)(t). \end{aligned} \tag{5.29}$$

Choosing $\varepsilon > 0$ small enough such that $\frac{2\kappa b}{p} - C_{15} \varepsilon^m H(0)^{-\alpha} \ge \frac{\kappa b}{2p}$ and $0 < \sigma < \frac{1-\gamma}{C_{15}} \varepsilon^{\frac{m}{m-1}} H(0)^{\alpha-\gamma}$, then we have from (5.29) that

$$G'(t) \ge C_{16} \sigma \left[\|u_t\|^2 + \|D^l u\|^2 + \|u\|_p^p + H(t) + (g \circ D^l u)(t)\right], \tag{5.30}$$

where $C_{16} = \min\left\{\frac{p+2}{2} - \kappa,\ p - 2\kappa,\ \frac{\kappa b}{2p},\ a_1\beta\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2} - \kappa\right\}$. Therefore, $G(t)$ is a nondecreasing function for $t \geq 0$. Letting σ in (5.26) be small enough such that $G(0) > 0$. Thus, we obtain that $G(t) \geq G(0) > 0$ for $t \geq 0$.

Since $0 < \gamma < \alpha < 1$, it is evident that $1 < \frac{1}{1-\gamma} < \frac{1}{1-\alpha}$. We deduce from (5.10) and (5.26) that

$$G(t)^{\frac{1}{1-\gamma}} \leq 2^{\frac{1}{1-\gamma}-1}\left[H(t) + \left(\sigma\int_\Omega uu_t dx\right)^{\frac{1}{1-\gamma}}\right]. \tag{5.31}$$

On the other hand, for $p>2$, we have from Hölder inequality and Lemma 2.2 that

$$\left(\sigma\int_\Omega uu_t dx\right)^{\frac{1}{1-\gamma}} \leq C_{17}\|u_t\|^{\frac{1}{1-\gamma}}\|u\|_p^{\frac{1}{1-\gamma}} \leq C_{18}\left(\|u\|_p^{\frac{\mu}{1-\gamma}} + \|u_t\|^{\frac{\nu}{1-\gamma}}\right), \tag{5.32}$$

where $C_{17} = \sigma^{\frac{1}{1-\gamma}}|\Omega|^{\frac{p-1}{(1-\gamma)p}}$, $\frac{1}{\mu} + \frac{1}{\nu} = 1$, and $C_{18} = C_{18}(C_{17}, \mu, \nu) > 0$.

Let $0 < \gamma < \min\{\alpha, \frac{1}{2} - \frac{1}{p}\}$, $\nu = 2(1-\gamma)$, then $\frac{\mu}{1-\gamma} = \frac{2}{1-2\gamma} < p$. It follows from (5.22) that

$$\left(\frac{b}{pH(0)}\right)^{\frac{1}{p}}\|u\|_p \geq 1. \tag{5.33}$$

Thus, we get from (5.33) that

$$\|u\|_p^{\frac{\mu}{1-\gamma}} = \|u\|_p^{\frac{2}{1-2\gamma}} = \|u\|_p^{\frac{2}{1-2\gamma}-p}\|u\|_p^p \leq \left(\frac{b}{pH(0)}\right)^{1-\frac{2}{p(1-2\gamma)}}\|u\|_p^p. \tag{5.34}$$

We obtain from (5.32) and (5.34) that

$$\left(\sigma\int_\Omega uu_t dx\right)^{\frac{1}{1-\gamma}} \leq C_{19}(\|u_t\|^2 + \|u\|_p^p), \tag{5.35}$$

where $C_{19} = C_{18}\max\left\{1,\ (\frac{b}{pH(0)})^{1-\frac{2}{p(1-2\gamma)}}\right\}$.

Combining (5.31) and (5.35), we find that

$$G(t)^{\frac{1}{1-\gamma}} \leq C_{20}\left[\|u_t\|^2 + \|u\|_p^p + H(t)\right], \tag{5.36}$$

where $C_{20} = 2^{\frac{1}{1-\gamma}-1}\max\{1,\ C_{19}\}$.

We obtain from (5.30) and (5.36) that

$$G'(t) \geq C_{21}G(t)^{\frac{1}{1-\gamma}}, \quad t \geq 0, \tag{5.37}$$

where $C_{21} = \frac{C_{16}\sigma}{C_{20}}$. Integrating both sides of (5.37) over $[0,t]$ yields that

$$G(t) \geq \left(G(0)^{\frac{\gamma}{\gamma-1}} - \frac{C_{21}\gamma}{1-\gamma} t \right)^{-\frac{1-\gamma}{\gamma}}.$$

Noting that $G(0) > 0$, then there exists $T^* = T_{\max} = \frac{(1-\gamma)G(0)^{\frac{\gamma}{\gamma-1}}}{C_{21}\gamma}$ such that $G(t) \to +\infty$ as $t \to +\infty$. Namely, the solution of the problem (5.1) blows up in finite time. □

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