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Solvability of Fractional Integral Equations on an Unbounded Interval through the Theory of Volterra-Stieltjes Integral Equations

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Abstract. In this paper we study the existence of solutions of a nonlinear quadratic Volterra-Stieltjes integral equation in the space of real functions being continuous and bounded on the interval of nonnegative numbers. Moreover, we also investigate the solvability of the equation in question in the classes of functions being asymptotically stable or having limits at infinity, for example. The main tool used in our considerations is the technique of measures of noncompactness constructed in a special way. It is shown that results obtained in the paper are applicable to the class of fractional integral equations and Volterra-Chandrasekhar integral equations, among others.

Keywords. Function of bounded variation, Riemann-Stieltjes integral, measure of noncompactness, fractional integral equation, Volterra-Stieltjes integral equation, attractivity, asymptotic stability

Mathematics Subject Classification (2010). Primary 45G10, secondary 47H08

1. Introduction

The paper is devoted to the study of solutions of the nonlinear integral equation of Volterra-Stieltjes type having the form

$$x(t) = (F_1x)(t) + (F_2x)(t) \int_0^t u(t, \tau, (Tx)(\tau)) d_\tau g(t, \tau),$$

where $t \geq 0$ and F_1, F_2 are superposition operators defined on the function space $BC(\mathbb{R}_+)$. The precise definitions will be given later.

Our aim is to show the solvability of the equation in question under some reasonable and handy assumptions. Moreover, we will also investigate some important properties of solutions of this equation such as asymptotic stability and the existence of a common limit at infinity for all solutions belonging to a ball in

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the space $BC(\mathbb{R}_+)$. In our investigations we will use measures of noncompactness, the theory of functions of bounded variation and the Riemann-Stieltjes integral with a kernel depending on two variables. The main result of the paper is contained in Theorem 3.7. That theorem covers, as particular cases, the classical Volterra integral equation, the integral equation of fractional order and the Volterra counterpart of the famous integral equation of Chandrasekhar type on an unbounded interval (see Section 4). It is worth pointing out that integral equations of fractional order play nowadays very important role and create a wide branch of the theory of differential and integral equations and the so-called fractional calculus. These equations are closely related to the Riemann-Liouville integral of fractional order. That integral plays a very important role in unification and generalization of the concept of n-order differentiation and n-fold integration (cf. [1,23–27,29,32]). It is worthwhile mentioning that the Riemann-Liouville integral of fractional order provides also a very useful example of a semi-group of linear bounded operators (see [21]).

Recently, integral equations of fractional order find a lot of applications in physics, mechanics, engineering, electrochemistry and economics, among other (see [18–20, 23, 26, 27, 30–32] for instance). Integral equations of Chandrasekhar type mentioned above can be very often encountered in several applications as well (cf. [8, 11, 14] and references therein). It is worth emphasizing that integral equations of fractional order are studied in several papers (cf. [4, 6, 7, 9, 10, 12, 15–17, 28]) but only a few papers investigate those equations on an unbounded interval [7, 10, 28].

Finally, let us remark that this paper generalizes the results obtained in the paper [10] (cf. also [7, 11, 28]).

2. Preliminaries

Assume that $g(t,\tau) = g$ is a real function defined on a subset $A \subset \mathbb{R}^2$. The symbol $\bigvee_{\tau=p}^{q} g(t,\tau)$ stands for the variation of the function $\tau \to g(t,\tau)$ on the interval $[p,q]$ which is contained in the domain of this function, where the variable t is fixed.

In what follows we will used the Riemann-Stieltjes integral of the form

$$\int_a^b x(\tau) d_\tau g(t,\tau),$$

where the symbol d_τ indicates the integral with respect to the variable τ, where t is fixed. Let us mention that in some situation lower and upper limit of the integration can also depend upon the variable t.

Now, we provide some classical results connected with measures of noncompactness.

Assume that $(E, ||\cdot||)$ is a real Banach space. Denote by $B(x,r)$ the closed ball centered at x and with radius r. Instead $B(0,r)$ we will write B_r. If X is a subset of E then the symbols $\overline{X}$ and Conv$X$ denote the closure and the convex closed hull of the set X, respectively. Further, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E. The symbol $\mathfrak{N}_E$ stands for the subfamily of $\mathfrak{M}_E$ consisting of all relatively compact sets. We will accept the following definition of a measure of noncompactness [5].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, +\infty)$ will be called a measure of noncompactness in the space E if it satisfies the following conditions:

1^o The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.

2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3^o $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv}X)$.

4^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

5^o If (X_n) is a sequence of closed sets belonging to $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described in 1^0 is called *the kernel of the measure of noncompactness* μ. The key role in our further considerations will be played by the following fixed point theorem of Darbo type [5].

Theorem 2.2. *Let Ω be a nonempty, bounded, closed, and convex subset of the space E and let $Q : \Omega \to \Omega$ be a continuous transformation. Assume that there exists a constant $k \in [0,1)$ such that $\mu(QX) \leq k\mu(X)$ for any nonempty subset X of Ω. Then Q has at least one fixed point in the set Ω. Moreover, the set Fix Q of all fixed points of Q belonging to Ω is a member of the family kerμ.*

Our considerations in this paper will be placed in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous and bounded on the interval $\mathbb{R}_+$ with the standard supremum norm.

Now, we define some quantities, which will be employed in our further considerations. To this end, take a nonempty and bounded subset X of the space $BC(\mathbb{R}_+)$. Fix $\varepsilon > 0$, $T > 0$, and take $x \in X$. Denote by $\omega^T(x,\varepsilon)$ *the modulus of continuity* of the function x on the interval $[0,T]$, defined by the formula

$$\omega^T(x,\varepsilon) = \sup\{|x(s) - x(t)| : s,t \in [0,T],\ |s-t| \leq \varepsilon\}.$$

Next, let us define

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon) \quad \text{and} \quad \omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

Moreover, we define the following quantities:

$$\beta_T(x) = \sup\left\{|x(s) - x(t)| : s \geq T,\ t \geq T\right\},$$
$$\beta(X) = \lim_{T\to\infty}\left\{\sup\{\beta_T(x) :\ x \in X\}\right\},$$
$$\tilde{\beta}(X) = \lim_{T\to\infty}\left\{\sup_{x\in X}\{\sup[|x(t)| : t \geq T]\}\right\},$$

and

$$\gamma(X) = \limsup_{t\to\infty} \operatorname{diam} X(t)$$

where we denoted $\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}$. It can be shown that the quantities

$$\mu(X) = \omega_0(X) + \beta(X), \tag{2.1}$$

$$\tilde{\mu}(X) = \omega_0(X) + \tilde{\beta}(X) \tag{2.2}$$

are measures of noncompactness in the space $BC(\mathbb{R}_+)$. Moreover, the kernel $\ker\mu$ consists of all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$, have finite limits at infinity and tend to those limits uniformly with respect to the set X, i.e. for each $\varepsilon > 0$ there exists $T > 0$ such that $|x(s) - x(t)| < \varepsilon$ for all $s, t \geq T$ and for all $x \in X$. The description of the kernel $\ker\tilde{\mu}$ is similar. In the sequel we will also use the so-called *superposition* (or *Nemytskii*) *operator* (see [3]).

To define the operator in question suppose $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given function. For any function $x(t) = x : \mathbb{R}_+ \to \mathbb{R}$, we can define the function Fx by putting $(Fx)(t) = f(t, x(t))$, $t \in \mathbb{R}_+$. The operator F defined in such a way is called the superposition operator generated by the function f.

Finally, we pay our attention to the concept of the *asymptotic stability* (sometimes we say also: *local uniform attractivity*) of solutions of an operator equation. To this end, assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$ and Q is an operator acting from Ω into $BC(\mathbb{R}_+)$. Consider the operator equation of the form

$$x(t) = (Qx)(t), \quad t \geq 0. \tag{2.3}$$

Definition 2.3. We say that a solution x of Equation (2.3) is asymptotically stable if there exists a ball $B(x_0, r)$ $(r > 0)$ in the space $BC(\mathbb{R}_+)$ such that $x \in B(x_0, r) \cap \Omega$ and for any $\varepsilon > 0$ there exists $T > 0$ such that $|x(t) - y(t)| \leq \varepsilon$ for each solution $y \in B(x_0, r) \cap \Omega$ of Equation (2.3) and for any $t \geq T$.

3. Main result

In this section we will investigate the nonlinear quadratic Volterra-Stieltjes integral equation which has the form

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) \int_0^t u(t, \tau, (Tx)(\tau)) d_\tau g(t, \tau), \quad t \geq 0. \tag{3.1}$$

We look for solutions of this equation in the space $BC(\mathbb{R}_+)$. In our study we will assume that the following assumptions are satisfied:

(i) The functions $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous and there exist nondecreasing functions $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f_i(t, x) - f_i(t, y)| \leq k_i(r)|x - y| \quad (i = 1, 2)$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in [-r, r]$, where $r \geq 0$ is an arbitrary fixed number. Moreover, the function $t \to f_i(t, 0)$ belongs to $BC(\mathbb{R}_+)$ for $i = 1, 2$.

Observe that on the basis of the above assumption we may define the finite constants $\overline{F_1}$, $\overline{F_2}$ by putting

$$\overline{F_i} = \sup\{|f_i(t, 0)| : t \in \mathbb{R}_+\} \quad (i = 1, 2).$$

(ii) The equality

$$\lim_{T \to \infty} \{\sup\{|f_1(t, x) - f_1(s, x)| : t, s \geq T, \ |x| \leq r\}\} = 0$$

holds for each $r > 0$.

For further purposes denote by Δ and Δ_T the following triangles

$$\Delta_T = \{(t, \tau) \in \mathbb{R}^2 : 0 \leq \tau \leq t \leq T\}, \quad \Delta = \{(t, \tau) \in \mathbb{R}^2 : 0 \leq \tau \leq t\},$$

where $T > 0$ is arbitrarily fixed number.

(iii) The function $u(t, \tau, x) = u : \Delta \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exists a continuous function $n(t, \tau) = n : \Delta \to \mathbb{R}_+$ and a nondecreasing and continuous at zero function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(0) = 0$ such that

$$|u(t, \tau, x) - u(t, \tau, y)| \leq n(t, \tau)\phi(|x - y|)$$

for all $(t, \tau) \in \Delta$ and $x, y \in \mathbb{R}$.

(iv) The function $g(t, \tau) = g : \Delta \to \mathbb{R}$ is continuous with respect to the variable τ on the interval $[0, t]$, where $t \geq 0$ is fixed.

(v) For any $t \in \mathbb{R}_+$ the function $\tau \to g(t, \tau)$ is of bounded variation on the interval $[0, t]$.

(vi) For each $\varepsilon > 0$ and $T > 0$ there exists $\delta > 0$ such that for all $t, s \in [0, T]$ and $|s - t| \leq \delta$ the following inequality holds

$$\bigvee_{\tau=0}^{\min\{t,s\}} [g(s,\tau) - g(t,\tau)] \leq \varepsilon.$$

Now, we present a few properties of the function g, which will be employed in the sequel.

Lemma 3.1. *We have the following statements:*

(a) *Assume that conditions* (iv) *and* (v) *are fulfilled. Then for each* $s \in [0, +\infty)$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $t \in [0, +\infty)$ *and* $t \leq s \leq t + \delta$ *the following inequality is satisfied*

$$\bigvee_{\tau=t}^{s} g(s,\tau) \leq \varepsilon.$$

(b) *Let us fix arbitrarily* $T > 0$ *and assume that conditions* (iv)–(vi) *are fulfilled. Then for each* $t \in [0, T]$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $s \in [0, T]$ *and* $t \leq s \leq t + \delta$ *the following inequality is satisfied*

$$\bigvee_{\tau=t}^{s} g(s,\tau) \leq \varepsilon.$$

Proof. Part (a). Fix arbitrarily $s \in [0, +\infty)$ and $\varepsilon > 0$. Consider the function h defined on the interval $[0, s]$ by the formula

$$h(p) = \bigvee_{\tau=0}^{p} g(s,\tau).$$

Clearly, the function h is continuous at the point s. Hence we deduce that there exists $\delta > 0$ such that if $t \in [0, +\infty)$, $t \leq s \leq t+\delta$ then we have $|h(s)-h(t)| \leq \varepsilon$. On the other hand

$$\begin{aligned} |h(s) - h(t)| &= \left| \bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(s,\tau) \right| \\ &= \left| \bigvee_{\tau=0}^{t} g(s,\tau) + \bigvee_{\tau=t}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(s,\tau) \right| \\ &= \bigvee_{\tau=t}^{s} g(s,\tau) \\ &\leq \varepsilon. \end{aligned}$$

Part (b). Suppose the assertion is false. Then we could find $t_0 \in [0,T]$, $\{t_n\}_{n=1}^{\infty} \subset [0,T]$, $\varepsilon > 0$ such that $t_n \downarrow t_0$ and $\bigvee_{\tau=t_0}^{t_n} g(t_n,\tau) > \varepsilon$. Let $\delta > 0$ be chosen for $\frac{1}{4}\varepsilon$ according to assumption (vi). Without loss of generality we may assume that $t_1 - t_0 < \delta$. Let us construct a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$, for which

$$\bigvee_{\tau=t_{n_{k+1}}}^{t_{n_k}} g(t_1,\tau) \geq \frac{1}{2}\varepsilon, \quad k \geq 1. \tag{3.2}$$

For $k = 1$ we put $n_1 = 1$. We can choose index n_2 in such a way that

$$\bigvee_{\tau=t_{n_2}}^{t_1} g(t_1,\tau) \geq \frac{1}{2}\varepsilon.$$

Now, let us assume that $k \geq 1$ and we have already chosen, $t_{n_1}, t_{n_2}, \ldots, t_{n_k}, t_{n_{k+1}}$. By selecting δ we have $\bigvee_{\tau=0}^{t_{n_{k+1}}}[g(t_{n_{k+1}},\tau) - g(t_1,\tau)] \leq \frac{1}{4}\varepsilon$. It is easy to see that

$$\bigvee_{\tau=t_0}^{t_{n_{k+1}}} g(t_1,\tau) \geq \bigvee_{\tau=t_0}^{t_{n_{k+1}}} g(t_{n_{k+1}},\tau) - \frac{1}{4}\varepsilon \geq \frac{3}{4}\varepsilon.$$

Index n_{k+2} can be chosen so that $\bigvee_{\tau=t_{n_{k+2}}}^{t_{n_{k+1}}} g(t_1,\tau) \geq \frac{1}{2}\varepsilon$, which completes the inductive proof of the existence of a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ fulfilling (3.2). We obtain a contradiction, because the function $\tau \to g(t_1,\tau)$ fulfilling (3.2) can not have bounded variation on the interval $[t_0,t_1]$. □

Lemma 3.2. *Let us fix arbitrarily $T > 0$ and assume that conditions* (iv)–(vi) *are satisfied. Then*

(a) *The function*

$$[0,T] \ni t \to \bigvee_{\tau=0}^{t} g(t,\tau) \in \mathbb{R}$$

is continuous on $[0,T]$.

(b) *For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, s \in [0,T]$ if $|t-s| < \delta$ then*

$$\bigvee_{\tau=t}^{s} g(s,\tau) \leq \varepsilon \quad \textit{for } t \leq s \quad \textit{and} \quad \bigvee_{\tau=s}^{t} g(t,\tau) \leq \varepsilon \quad \textit{for } s \leq t.$$

Proof. Part (a). Let us fix $s \in [0,T]$. We will prove the continuity of this function at any such point. To do this, fix $\varepsilon > 0$. We estimate the expression

$$\left| \bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau) \right|.$$

First, let us assume that $t \le s$. Then we have

$$\begin{aligned}\left|\bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| &= \left|\bigvee_{\tau=0}^{t} g(s,\tau) + \bigvee_{\tau=t}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| \\ &\le \left|\bigvee_{\tau=0}^{t} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| + \bigvee_{\tau=t}^{s} g(s,\tau) \\ &\le \bigvee_{\tau=0}^{t} [g(s,\tau) - g(t,\tau)] + \bigvee_{\tau=t}^{s} g(s,\tau).\end{aligned}$$

Following a similar reasoning in case $t > s$, we get $\left|\bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| \le \bigvee_{\tau=0}^{s}[g(s,\tau) - g(t,\tau)] + \bigvee_{\tau=s}^{t} g(t,\tau)$. Using Lemma 3.1 we obtain the existence of $\delta > 0$ such that if $|t-s| < \delta$ then

$$\left|\bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| < \varepsilon.$$

Part (b). It suffices to consider the case $t \le s$. Let us fix $\varepsilon > 0$. Then

$$\begin{aligned}\bigvee_{\tau=t}^{s} g(s,\tau) &= \left|\bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(s,\tau)\right| \\ &\le \left|\bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| + \left|\bigvee_{\tau=0}^{t} g(t,\tau) - \bigvee_{\tau=0}^{t} g(s,\tau)\right| \\ &\le \left|\bigvee_{\tau=0}^{s} g(s,\tau) - \bigvee_{\tau=0}^{t} g(t,\tau)\right| + \bigvee_{\tau=0}^{t} [g(t,\tau) - g(s,\tau)].\end{aligned}$$

The existence of δ is ensured by part (a). □

Now, we define the function $G(t,\tau) = G : \Delta \to \mathbb{R}$ by putting

$$G(t,\tau) = \bigvee_{p=0}^{\tau} g(t,p).$$

Notice that the function $\tau \to G(t,\tau)$ is well defined and nondecreasing on the interval $[0,t]$, for any fixed $t \ge 0$. Below we show a connection between the functions g and G.

Lemma 3.3. *Under assumption* (v) *the inequality*

$$\bigvee_{p=0}^{t} [G(s,p) - G(t,p)] \le \bigvee_{p=0}^{t} [g(s,p) - g(t,p)]$$

holds for all $t, s \ge 0$, $t < s$.

Proof. Let $0 = a_0 < a_1 < \cdots < a_n = t$ be a partition of the interval $[0,t]$. Then we have

$$\sum_{i=1}^{n} |[G(s,a_i) - G(t,a_i)] - [G(s,a_{i-1}) - G(t,a_{i-1})]|$$
$$= \sum_{i=1}^{n} \left| \left[\bigvee_{p=0}^{a_i} g(s,p) - \bigvee_{p=0}^{a_i} g(t,p) \right] - \left[\bigvee_{p=0}^{a_{i-1}} g(s,p) - \bigvee_{p=0}^{a_{i-1}} g(t,p) \right] \right|$$
$$= \sum_{i=1}^{n} \left| \bigvee_{p=a_{i-1}}^{a_i} g(s,p) - \bigvee_{p=a_{i-1}}^{a_i} g(t,p) \right|$$
$$\leq \sum_{i=1}^{n} \bigvee_{p=a_{i-1}}^{a_i} [g(s,p) - g(t,p)]$$
$$= \bigvee_{p=0}^{t} [g(s,p) - g(t,p)].$$

Thus the inequality follows. □

In what follows, let us denote by $\overline{n}(t)$ and $\overline{u}(t)$ the functions defined on $\mathbb{R}_+$ in the following way:

$$\overline{n}(t) = \int_0^t n(t,\tau) d_\tau G(t,\tau), \quad \overline{u}(t) = \int_0^t |u(t,\tau,0)| d_\tau G(t,\tau).$$

We will need the following property of these functions.

Lemma 3.4. *Assume that conditions* (iii)–(vi) *are satisfied. Then the functions* $\overline{n}$ *and* $\overline{u}$ *are continuous on the interval* $\mathbb{R}_+$.

Proof. Obviously both functions are well defined on $\mathbb{R}_+$. We see that it is sufficient to prove our lemma for the function $\overline{n}$. To this end, fix arbitrarily $T > 0$, $\varepsilon > 0$ and $t, s \in [0,T]$ such that $|t-s| \leq \varepsilon$. Without loss of generality we can assume that $t < s$. Then we obtain

$$\begin{aligned}
|\overline{n}(s) - \overline{n}(t)| \leq & \left| \int_0^s n(s,\tau) d_\tau G(s,\tau) - \int_0^t n(s,\tau) d_\tau G(s,\tau) \right| \\
& + \left| \int_0^t n(s,\tau) d_\tau G(s,\tau) - \int_0^t n(t,\tau) d_\tau G(s,\tau) \right| \\
& + \left| \int_0^t n(t,\tau) d_\tau G(s,\tau) - \int_0^t n(t,\tau) d_\tau G(t,\tau) \right| \\
\leq & \int_t^s |n(s,\tau)| d_\tau \left(\bigvee_{p=t}^{\tau} G(s,p) \right) + \int_0^t |n(s,\tau) - n(t,\tau)| d_\tau \left(\bigvee_{p=0}^{\tau} G(s,p) \right) \\
& + \int_0^t |n(t,\tau)| d_\tau \left(\bigvee_{p=0}^{\tau} [G(s,p) - G(t,p)] \right).
\end{aligned}$$

Thus, using Lemma 3.3 we have

$$|\overline{n}(s)-\overline{n}(t)| \le n_T \bigvee_{p=t}^{s} G(s,p) + \omega_1^T(n,\varepsilon) \bigvee_{p=0}^{t} G(s,p) + n_T \bigvee_{p=0}^{t} [G(s,p)-G(t,p)]$$

$$\le n_T \bigvee_{p=t}^{s} g(s,p) + \omega_1^T(n,\varepsilon) \bigvee_{p=0}^{s} g(s,p) + n_T \bigvee_{p=0}^{t} [g(s,p)-g(t,p)],$$

where

$$n_T = \sup\{n(t,\tau) : (t,\tau) \in \Delta_T\},$$
$$\omega_1^T(n,\varepsilon) = \sup\{|n(s,\tau)-n(t,\tau)| : (s,\tau),(t,\tau) \in \Delta_T,\ |t-s| \le \varepsilon\}.$$

In view of Lemma 3.2 and uniform continuity of the function n on the set Δ_T we obtain our assertion. □

Now, we can formulate our next assumptions:

(vii) The functions $\overline{n}$ and $\overline{u}$ vanish at infinity, i.e.

$$\lim_{t\to\infty} \overline{n}(t) = \lim_{t\to\infty} \overline{u}(t) = 0.$$

Observe that in view of Lemma 3.4 the constants $\overline{N}$ and $\overline{U}$ defined as follows

$$\overline{N} = \sup\{\overline{n}(t) : t \ge 0\}, \quad \overline{U} = \sup\{\overline{u}(t) : t \ge 0\}$$

are finite.

(viii) The operator $T : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ is continuous and there exists a nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $||Tx|| \le \Psi(||x||)$ for any $x \in BC(\mathbb{R}_+)$.

(ix) There exists a positive real number r_0 which satisfies the inequalities

$$rk_1(r) + \overline{F_1} + rk_2(r)\phi(\Psi(r))\overline{N} + rk_2(r)\overline{U} + \overline{F_2}\phi(\Psi(r))\overline{N} + \overline{F_2 U} \le r$$
$$k_1(r) + k_2(r)(\phi(\Psi(r))\overline{N} + \overline{U}) < 1.$$

Remark 3.5. Observe that if r_0 is a positive solution of the first inequality from assumption (ix) and if one of the terms $\overline{F_2}\phi(\Psi(r_0))\overline{N}$, $\overline{F_2 U}$, $\overline{F_1}$ does not vanish, then the second inequality from assumption (ix) is automatically satisfied.

Now, let us consider the operators $F_i (i = 1,2)$, U, V defined on the space $BC(\mathbb{R}_+)$ by the formulas:

$$(F_i x)(t) = f_i(t, x(t)) \quad (i = 1,2)$$
$$(Ux)(t) = \int_0^t u(t,\tau,(Tx)(\tau))d_\tau g(t,\tau)$$
$$(Vx)(t) = (F_1x)(t) + (F_2x)(t)(Ux)(t).$$

In the next lemma we prove some important properties of the operator V.

Lemma 3.6. *Let assumptions* (i)–(ix) *hold. Then the operator* V *acts from the space* $BC(\mathbb{R}_+)$ *into* $BC(\mathbb{R}_+)$. *Moreover, the operator* V *transforms continuously the ball* B_{r_0} *into itself, where* r_0 *is the number appearing in assumption* (ix).

Proof. We first show that for any function $x \in BC(\mathbb{R}_+)$ the function Vx is continuous on $\mathbb{R}_+$. To do this, fix $T > 0$, $\varepsilon > 0$. Next, assume that $t, s \in [0, T]$ are such that $|t - s| \leq \varepsilon$. Without restriction of generality we can assume that $t < s$. Then we get

$$
\begin{aligned}
&|(Ux)(s)-(Ux)(t)| \\
&\leq \left|\int_0^s u(s,\tau,(Tx)(\tau))d_\tau g(s,\tau)-\int_0^t u(s,\tau,(Tx)(\tau))d_\tau g(s,\tau)\right| \\
&\quad+\left|\int_0^t u(s,\tau,(Tx)(\tau))d_\tau g(s,\tau)-\int_0^t u(t,\tau,(Tx)(\tau))d_\tau g(s,\tau)\right| \\
&\quad+\left|\int_0^t u(t,\tau,(Tx)(\tau))d_\tau g(s,\tau)-\int_0^t u(t,\tau,(Tx)(\tau))d_\tau g(t,\tau)\right| \\
&\leq \int_t^s |u(s,\tau,(Tx)(\tau))| d_\tau\left(\bigvee_{p=t}^{\tau} g(s,p)\right) \\
&\quad+\int_0^t |u(s,\tau,(Tx)(\tau))-u(t,\tau,(Tx)(\tau))| d_\tau\left(\bigvee_{p=0}^{\tau} g(s,p)\right) \\
&\quad+\int_0^t |u(t,\tau,(Tx)(\tau))| d_\tau\left(\bigvee_{p=0}^{\tau} [g(s,p)-g(t,p)]\right) \\
&\leq \int_t^s [|u(s,\tau,(Tx)(\tau))-u(s,\tau,0)|+|u(s,\tau,0)|] d_\tau\left(\bigvee_{p=t}^{\tau} g(s,p)\right) \\
&\quad+\int_0^t \omega_1^T(u,\varepsilon;\Psi(||x||)) d_\tau\left(\bigvee_{p=0}^{\tau} g(s,p)\right) \\
&\quad+\int_0^t [|u(t,\tau,(Tx)(\tau))-u(t,\tau,0)|+|u(t,\tau,0)|] d_\tau\left(\bigvee_{p=0}^{\tau} [g(s,p)-g(t,p)]\right) \\
&\leq (n_T\phi(\Psi(||x||))+u_T)\bigvee_{p=t}^{s} g(s,p)+\omega_1^T(u,\varepsilon;\Psi(||x||))\bigvee_{p=0}^{s} g(s,p) \\
&\quad+(n_T\phi(\Psi(||x||))+u_T)\bigvee_{p=0}^{t} [g(s,p)-g(t,p)],
\end{aligned}
\tag{3.3}
$$

where

$$\omega_1^T(u,\varepsilon;a)=\sup\left\{|u(s,\tau,y)-u(t,\tau,y)|:\begin{array}{c}(s,\tau),(t,\tau)\in\Delta_T,\\ |s-t|\le\varepsilon,\ |y|\le a\end{array}\right\}$$

$$u_T=\sup\{|u(t,\tau,0)|:(t,\tau)\in\Delta_T\}.$$

Using Lemma 3.2 and uniform continuity of the function u on the set $\Delta_T\times[-\Psi(||x||),\Psi(||x||)]$ we obtain continuity of the function Ux on the interval $[0,T]$. This yields the continuity of Ux on $\mathbb{R}_+$. Obviously the functions F_1x and F_2x are contiuous on $\mathbb{R}_+$. Combining this facts we have that the function Vx is continuous on $\mathbb{R}_+$.

Now, we show that for any function $x\in BC(\mathbb{R}_+)$, the function Vx is bounded on $\mathbb{R}_+$. For this purpose, fix $t\in\mathbb{R}_+$. Then we get

$$\begin{aligned}&|(Vx)(t)|\\ &\le|f_1(t,x(t))-f_1(t,0)|+|f_1(t,0)|+[|f_2(t,x(t))-f_2(t,0)|\\ &\quad+|f_2(t,0)|]\left|\int_0^t u(t,\tau,(Tx)(\tau))d_\tau g(t,\tau)\right|\\ &\le k_1(||x||)|x(t)|+\overline{F_1}\\ &\quad+[k_2(||x||)|x(t)|+\overline{F_2}]\int_0^t[|u(t,\tau,(Tx)(\tau))-u(t,\tau,0)|+|u(t,\tau,0)|]d_\tau\left(\bigvee_{p=0}^{\tau}g(t,p)\right).\end{aligned}$$

Since

$$|u(t,\tau,(Tx)(\tau))-u(t,\tau,0)|\le n(t,\tau)\phi(|(Tx)(\tau)|)\le n(t,\tau)\phi(||Tx||),\tag{3.4}$$

we have the following estimation

$$\begin{aligned}&|(Vx)(t)|\\ &\le k_1(||x||)|x(t)|+\overline{F_1}\\ &\quad+[k_2(||x||)||x||+\overline{F_2}]\left[\phi(||Tx||)\int_0^t n(t,\tau)d_\tau G(t,\tau)+\int_0^t|u(t,\tau,0)|d_\tau G(t,\tau)\right]\\ &\le k_1(||x||)|x(t)|+\overline{F_1}+k_2(||x||)||x||\phi(||Tx||)\overline{n}(t)\\ &\quad+k_2(||x||)||x||\overline{u}(t)+\overline{F_2}\phi(||Tx||)\overline{n}(t)+\overline{F_2}\overline{u}(t).\end{aligned}\tag{3.5}$$

From the above inequality we infer that the function Vx is bounded on $\mathbb{R}_+$.

Apart from this we observe that estimation (3.5) yields

$$\begin{aligned}||Vx||\le{}&k_1(||x||)||x||+\overline{F_1}+k_2(||x||)||x||\phi(\Psi(||x||))\overline{N}\\ &+\ k_2(||x||)||x||\overline{U}+\overline{F_2}\phi(\Psi(||x||))\overline{N}+\overline{F_2}\overline{U}.\end{aligned}$$

From this we see that $V(B_{r_0})\subset B_{r_0}$.

In what follows we show that the operator V is continuous on the ball B_{r_0}. To do this, fix $\varepsilon > 0$ and $x_0 \in B_{r_0}$. We can find $\delta > 0$ such that for an arbitrary $x \in B_{r_0}$, if $||x - x_0|| \le \delta$ we have $||Tx - Tx_0|| \le \varepsilon$. Hence, for arbitrarily fixed $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
&|(Vx)(t) - (Vx_0)(t)|\\
&\le |f_1(t, x(t)) - f_1(t, x_0(t))|\\
&\quad + \left| f_2(t, x(t)) \int_0^t u(t,\tau,(Tx)(\tau)) d_\tau g(t,\tau) - f_2(t, x_0(t)) \int_0^t u(t,\tau,(Tx)(\tau)) d_\tau g(t,\tau) \right|\\
&\quad + \left| f_2(t, x_0(t)) \int_0^t u(t,\tau,(Tx)(\tau)) d_\tau g(t,\tau) - f_2(t, x_0(t)) \int_0^t u(t,\tau,(Tx_0)(\tau)) d_\tau g(t,\tau) \right|\\
&\le k_1(r_0)|x(t) - x_0(t)| + |f_2(t, x(t)) - f_2(t, x_0(t))|\\
&\quad \times \int_0^t [|u(t,\tau,(Tx)(\tau)) - u(t,\tau,0)| + |u(t,\tau,0)|] d_\tau \left(\bigvee_{p=0}^{\tau} g(t,p) \right)\\
&\quad + [|f_2(t, x_0(t)) - f_2(t,0)| + |f_2(t,0)|]\\
&\quad \times \int_0^t |u(t,\tau,(Tx)(\tau)) - u(t,\tau,(Tx_0)(\tau))| d_\tau \left(\bigvee_{p=0}^{\tau} g(t,p) \right).
\end{aligned}$$

Using (3.4) we have $|u(t, \tau, (Tx)(\tau)) - u(t, \tau, 0)| \le n(t, \tau)\phi(\Psi(||x||)) \le n(t, \tau)\phi(\Psi(r_0))$. Since

$$\begin{aligned}
|u(t, \tau, (Tx)(\tau)) - u(t, \tau, (Tx_0)(\tau))| &\le n(t, \tau)\phi(|(Tx)(\tau) - (Tx_0)(\tau)|)\\
&\le n(t, \tau)\phi(||Tx - Tx_0||)
\end{aligned}$$

we obtain

$$\begin{aligned}
&|(Vx)(t) - (Vx_0)(t)|\\
&\le k_1(r_0)|x(t) - x_0(t)| + k_2(r_0)|x(t) - x_0(t)|\\
&\quad \times \left[\phi(\Psi(r_0)) \int_0^t n(t, \tau) d_\tau G(t, \tau) + \int_0^t |u(t, \tau, 0)| d_\tau G(t, \tau) \right]\\
&\quad + (k_2(r_0)r_0 + \overline{F_2})\phi(||Tx - Tx_0||) \int_0^t n(t, \tau) d_\tau G(t, \tau).
\end{aligned} \tag{3.6}$$

Thus

$$\begin{aligned}
|(Vx)(t) - (Vx_0)(t)| \le k_1(r_0)||x - x_0|| + k_2(r_0)||x - x_0||[\phi(\Psi(r_0))\overline{N} + \overline{U}]\\
+ \ (k_2(r_0)r_0 + \overline{F_2})\phi(||Tx - Tx_0||)\overline{N}.
\end{aligned}$$

From the above estimation we derive the desired continuity of operator V. $\square$

We can now formulate our main result.

Theorem 3.7. *Suppose that assumptions* (i)–(ix) *are fulfilled. Then*

(a) *Equation* (3.1) *has at least one solution in the space* $BC(\mathbb{R}_+)$ *(more precisely in the ball* B_{r_0}*, where* r_0 *is the number appearing in assumption* (ix)*). Moreover, all solutions of Equation* (3.1) *from that ball are locally equicontinuous, asypmptotically stable, have a common finite limit at infinity and they tend to this limit uniformly, i.e. if* g *is this common limit, then for every* $\varepsilon > 0$ *there exists* $T > 0$ *such that* $|x(t) - g| < \varepsilon$ *for every* $t \geq T$ *and for every solution* x *from the ball* B_{r_0}*.*

(b) *If additionally* $\overline{F_1} = 0$*, then the limit at infinity of each solution of Equation* (3.1) *belonging to the ball* B_{r_0} *is equal to zero.*

Proof. We will study behaviour of the operator V with respect to the measure of noncompactness μ defined by formula (2.1). To this end, take a nonempty subset X of the ball B_{r_0}. Fix arbitrarily $\varepsilon > 0$, $T > 0$ and $x \in X$. Next, choose arbitrary numbers $t, s \in [0, T]$ such that $|t - s| \leq \varepsilon$. We can assume that $t < s$. Then using (3.3), we obtain

$$\begin{aligned}
&|(Vx)(s) - (Vx)(t)| \\
&\leq |(F_1x)(s) - (F_1x)(t)| + |(F_2x)(s)(Ux)(s) - (F_2x)(t)(Ux)(s)| \\
&\quad + |(F_2x)(t)(Ux)(s) - (F_2x)(t)(Ux)(t)| \\
&\leq |f_1(s, x(s)) - f_1(s, x(t))| + |f_1(s, x(t)) - f_1(t, x(t))| \\
&\quad + |(Ux)(s)|[|f_2(s, x(s)) - f_2(s, x(t))| + |f_2(s, x(t)) - f_2(t, x(t))|] \\
&\quad + [|f_2(t, x(t)) - f_2(t, 0)| + |f_2(t, 0)|]|(Ux)(s) - (Ux)(t)| \\
&\leq k_1(r_0)|x(s) - x(t)| + \omega_1^T(f_1, \varepsilon; r_0) + [k_2(r_0)|x(s) - x(t)| + \omega_1^T(f_2, \varepsilon; r_0)] \\
&\quad \times \int_0^s [|u(s, \tau, (Tx)(\tau)) - u(s, \tau, 0)| + |u(s, \tau, 0)|] d_\tau\left(\bigvee_{p=0}^{\tau} g(s, p)\right) \\
&\quad + (r_0 k_2(r_0) + \overline{F_2}) W(\varepsilon),
\end{aligned}$$

where

$$\omega_1^T(f_i, \varepsilon; a) = \sup\{|f_i(s, x) - f_i(t, x)| : t, s \in [0, T],\ |t - s| \leq \varepsilon,\ |x| \leq a\}$$

for $i = 1, 2$,

$$\begin{aligned}
W(\varepsilon) = (n_T \phi(\Psi(r_0)) + u_T) \bigvee_{p=t}^{s} g(s, p) + \omega_1^T(u, \varepsilon; \Psi(r_0)) \bigvee_{p=0}^{s} g(s, p) \\
+ (n_T \phi(\Psi(r_0)) + u_T) \bigvee_{p=0}^{t} [g(s, p) - g(t, p)].
\end{aligned}$$

On account of (3.4) we get

$$
\begin{aligned}
&|(Vx)(s)-(Vx)(t)| \\
&\le k_1(r_0)\omega^T(x,\varepsilon)+\omega_1^T(f_1,\varepsilon;r_0)+[k_2(r_0)\omega^T(x,\varepsilon)+\omega_1^T(f_2,\varepsilon;r_0)] \\
&\quad\times\left[\phi(\Psi(r_0))\int_0^s n(s,\tau)d_\tau G(s,\tau)+\int_0^s |u(s,\tau,0)|d_\tau G(s,\tau)\right] \\
&\quad+(r_0k_2(r_0)+\overline{F_2})W(\varepsilon) \\
&\le k_1(r_0)\omega^T(x,\varepsilon)+\omega_1^T(f_1,\varepsilon;r_0)+[k_2(r_0)\omega^T(x,\varepsilon) \\
&\quad+\omega_1^T(f_2,\varepsilon;r_0)][\phi(\Psi(r_0))\overline{N}+\overline{U}]+(r_0k_2(r_0)+\overline{F_2})W(\varepsilon).
\end{aligned}
$$

Thus $\omega^T(VX,\varepsilon)\le k_1(r_0)\omega^T(X,\varepsilon)+\omega_1^T(f_1,\varepsilon;r_0)+[k_2(r_0)\omega^T(X,\varepsilon)+\omega_1^T(f_2,\varepsilon;r_0)]\times[\phi(\Psi(r_0))\overline{N}+\overline{U}]+(r_0k_2(r_0)+\overline{F_2})W(\varepsilon)$. According to Lemma 3.2, uniform continuity of the function f_i on the set $[0,T]\times[-r_0,r_0](i=1,2)$ and uniform continuity of the function u on the set $\Delta_T\times[-\Phi(r_0),\Phi(r_0)]$ we derive the following inequality $\omega_0^T(VX)\le k_1(r_0)\omega_0^T(X)+k_2(r_0)(\phi(\Psi(r_0))\overline{N}+\overline{U})\omega_0^T(X)$ and consequently

$$
\omega_0(VX)\le[k_1(r_0)+k_2(r_0)(\phi(\Psi(r_0))\overline{N}+\overline{U})]\omega_0(X). \tag{3.7}
$$

In the next step of our proof, similarly as before, let us take a nonempty set $X\subset B_{r_0}$ and a number $T>0$. Then, for arbitrarily fixed $x\in X$ and for arbitrary numbers t,s such that $t\ge T$, $s\ge T$, we obtain

$$
\begin{aligned}
&|(Vx)(s)-(Vx)(t)| \\
&\le|f_1(s,x(s))-f_1(s,x(t))|+|f_1(s,x(t))-f_1(t,x(t))| \\
&\quad+\left|f_2(s,x(s))\int_0^s u(s,\tau,(Tx)(\tau))d_\tau g(s,\tau)\right|+\left|f_2(t,x(t))\int_0^t u(t,\tau,(Tx)(\tau))d_\tau g(t,\tau)\right| \\
&\le k_1(r_0)|x(s)-x(t)|+|f_1(s,x(t))-f_1(t,x(t))|+[|f_2(s,x(s))-f_2(s,0)| \\
&\quad+|f_2(s,0)|]\int_0^s[|u(s,\tau,(Tx)(\tau))-u(s,\tau,0)|+|u(s,\tau,0)|]d_\tau\left(\bigvee_{p=0}^{\tau}g(s,p)\right) \\
&\quad+[|f_2(t,x(t))-f_2(t,0)|+|f_2(t,0)|] \\
&\quad\times\int_0^t[|u(t,\tau,(Tx)(\tau))-u(t,\tau,0)|+|u(t,\tau,0)|]d_\tau\left(\bigvee_{p=0}^{\tau}g(t,\tau)\right) \\
&\le k_1(r_0)|x(s)-x(t)|+|f_1(s,x(t))-f_1(t,x(t))| \\
&\quad+(r_0k_2(r_0)+\overline{F_2})\left[\phi(\Psi(r_0))\int_0^s n(s,\tau)d_\tau G(s,\tau)+\int_0^s|u(s,\tau,0)|d_\tau G(s,\tau)\right] \\
&\quad+(r_0k_2(r_0)+\overline{F_2})\left[\phi(\Psi(r_0))\int_0^t n(t,\tau)d_\tau G(t,\tau)+\int_0^t|u(t,\tau,0)|d_\tau G(t,\tau)\right] \\
&\le k_1(r_0)|x(s)-x(t)|+|f_1(s,x(t))-f_1(t,x(t))| \\
&\quad+(r_0k_2(r_0)+\overline{F_2})[\phi(\Psi(r_0))(\overline{n}(s)+\overline{n}(t))+\overline{u}(s)+\overline{u}(t)].
\end{aligned}
$$

Hence we get

$$\beta_T(Vx) \le k_1(r_0)\beta_T(x) + \sup\{|f_1(s,x) - f_1(t,x)| \colon t \ge T,\ s \ge T,\ |x| \le r_0\} \\ + (r_0 k_2(r_0) + \overline{F_2})[2\phi(\Psi(r_0)) \sup\{\overline{n}(t) \colon t \ge T\} + 2\sup\{\overline{u}(t) \colon t \ge T\}]$$

and finally

$$\beta(VX) \le k_1(r_0)\beta(X). \tag{3.8}$$

Linking (3.7) and (3.8) we obtain $\mu(VX) \le [k_1(r_0) + k_2(r_0)(\phi(\Psi(r_0))\overline{N} + \overline{U})]\mu(X)$. According to Theorem 2.2 we infer that the operator V has at least one fixed point in the ball B_{r_0}. This means that Equation (3.1) has at least one solution in B_{r_0}.

Moreover, let us observe that on the base of Theorem 2.2 and in view of description of the kernel $\ker\mu$ (cf. Section 2) we obtain that all solutions of Equation (3.1) belonging to B_{r_0} are locally equicontinuous and have a finite limit at infinity. At this point of our proof we do not know yet if these limits are equal.

Now, we proceed to the study of asymptotic stability of solutions of Equation (3.1). To this end, fix a nonempty subset X of the ball B_{r_0}. Next, take $x, y \in X$ and $t \ge 0$. We estimate the value of $|(Vx)(t) - (Vy)(t)|$. We can use (3.6) replacing the function x_0 by the function y. Then we have

$$|(Vx)(t) - (Vy)(t)| \le k_1(r_0)|x(t) - y(t)| + k_2(r_0)|x(t) - y(t)| \\ \times \left[\phi(\Psi(r_0)) \int_0^t n(t,\tau) d_\tau G(t,\tau) + \int_0^t |u(t,\tau,0)| d_\tau G(t,\tau)\right] \\ + (k_2(r_0) r_0 + \overline{F_2})\phi(||Tx - Ty||) \int_0^t n(t,\tau) d_\tau G(t,\tau).$$

Thus

$$|(Vx)(t) - (Vy)(t)| \le k_1(r_0)|x(t) - y(t)| + 2r_0 k_2(r_0)[\phi(\Psi(r_0))\overline{n}(t) + \overline{u}(t)] \\ + (k_2(r_0) r_0 + \overline{F_2})\phi(2\Psi(r_0))\overline{n}(t).$$

Hence we get $\mathrm{diam}(VX)(t) \le k_1(r_0)\mathrm{diam}X(t) + 2r_0 k_2(r_0)[\phi(\Psi(r_0))\overline{n}(t) + \overline{u}(t)] + (k_2(r_0)r_0 + \overline{F_2})\phi(2\Psi(r_0))\overline{n}(t)$ and consequently

$$\gamma(VX) \le k_1(r_0)\gamma(X).$$

Now, let us consider the set $X_0 = \mathrm{Fix}V \cap B_{r_0}$. We already know that it is nonempty. Since $V(X_0) = X_0$ and $k_1(r_0) < 1$ we obtain

$$\gamma(X_0) \le k_1(r_0)\gamma(X_0).$$

This gives $\gamma(X_0) = 0$, which leads to the asymptotic stability of all solutions of Equation (3.1) belonging to B_{r_0}.

This implies that all solutions of Equation (3.1) belonging to B_{r_0} have a common finite limit at infinity. Indeed, assume that $g_x = \lim_{t\to\infty} x(t)$ and $g_y = \lim_{t\to\infty} y(t)$, where x and y are arbitrary solutions of Equation (3.1) in B_{r_0}. Then, for a given $\varepsilon > 0$, there exists $T > 0$ such that $|x(t) - g_x| < \frac{\varepsilon}{3}$, $|y(t) - g_y| < \frac{\varepsilon}{3}$, and $|x(t) - y(t)| < \frac{\varepsilon}{3}$ for $t \geq T$. Hence we get

$$|g_x - g_y| \leq |g_x - x(t)| + |x(t) - y(t)| + |g_y - y(t)| < \varepsilon \quad \text{for } t \geq T,$$

and as a consequence, $g_x = g_y$.

Moreover, according to the description of the kernel $\ker \mu$, all solutions of Equation (3.1) from the ball B_{r_0} tend to the common limit uniformly in the sense described in part (a) of the assertion. The proof of part (a) is complete.

We now prove assertion (b). Let $\emptyset \neq X \subset B_{r_0}$, $T > 0$, $x \in X$ and $t \geq T$. Owing to (3.4) we have $|(Vx)(t)| \leq k_1(r_0)|x(t)| + k_2(r_0)r_0\phi(\Psi(r_0))\overline{n}(t) + k_2(r_0)r_0\overline{u}(t) + \overline{F_2}\phi(\Psi(r_0))\overline{n}(t) + \overline{F_2}\overline{u}(t)$. Thus

$$\tilde{\beta}(VX) \leq k_1(r_0)\tilde{\beta}(X).$$

If $\tilde{\mu}$ is a measure of noncompactnes given by (2.2), then $\tilde{\mu}(VX) \leq [k_1(r_0) + k_2(r_0)(\phi(\Psi(r_0))\overline{N} + \overline{U})]\tilde{\mu}(X)$. Applying Theorem 2.2 and keeping in mind the description of $\ker \tilde{\mu}$ we complete the proof. □

4. Applications and an example

Let us notice that a crucial role in proving Theorem 3.7 was played by assumption (vi). Now, we formulate some conditions being handy in applications and guaranteeing that this assumption is satisfied.

Consider the following conditions:

(A) The function $g(t,\tau) = g : \Delta \to \mathbb{R}$ is continuous on the triangle Δ.

(B) For arbitrarily fixed $t, s \geq 0$ such that $t < s$ the function $\tau \to g(s,\tau) - g(t,\tau)$ is nondecreasing on the interval $[0,t]$.

(C) For arbitrarily fixed $t, s \geq 0$ such that $t < s$ the function $\tau \to g(s,\tau) - g(t,\tau)$ is nonincreasing on the interval $[0,t]$.

The result announced previously is presented in the below given lemma. The proof is standard so we omit it (cf. [8, 11]).

Lemma 4.1. *Assume that the function $g = g(t,\tau)$ satisfies conditions* (A), (B) *or* (A), (C). *Then g satisfies assumption* (vi).

Now, we present an application of Theorem 3.7 in the situation of the classical equations on an unbounded interval.

Let us consider the equation

$$x(t) = f_1(t, x(t)) + \frac{\tilde{f}_2(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,\tau,(Tx)(\tau))}{(t-\tau)^{1-\alpha}} d\tau, \ t \geq 0, \tag{4.1}$$

where Γ denotes the Euler gamma function and $\alpha > 0$. It is the well known *integral equation of fractional order*.

If we take (on the set Δ) the function g defined by

$$g(t,\tau) = \frac{1}{\alpha}\left[t^\alpha - (t-\tau)^\alpha\right],$$

then it is easy to check that Equation (4.1) is a special case of Equation (3.1). Clearly, assumption (ix) in this situation takes the form

(ix*) There exists a positive real number r_0 which satisfies the inequalities

$$rk_1(r)+\overline{F_1}+\frac{1}{\Gamma(\alpha)}(r\tilde{k_2}(r)\phi(\Psi(r))\overline{N}+r\tilde{k_2}(r)\overline{U}+\tilde{F_2}\phi(\Psi(r))\overline{N}+\tilde{F_2}\overline{U})\leq r$$

$$k_1(r)+\frac{1}{\Gamma(\alpha)}\tilde{k_2}(r)(\phi(\Psi(r))\overline{N}+\overline{U})<1$$

where $\tilde{F_2} = \sup\{|\tilde{f_2}(t,0)| : t \in \mathbb{R}_+\}$ and $\tilde{k_2}$ is a function chosen for $\tilde{f_2}$ based on assumption (i).

Using Lemma 4.1 and the standard methods of differential calculus we can show that the function g satisfies assumptions (iv)–(vi). Obviously, when $\alpha = 1$ Equation (4.1) reduces to the classical *nonlinear quadratic Volterra integral equation*. Let us mention that a direct approach (without using the theory of functions of bounded variation and the Riemann-Stieltjes integral) to the study of integral equations of fractional order on an unbounded interval can be found in [10].

Now, let us consider the equation

$$x(t) = f_1(t,x(t)) + f_2(t,x(t))\int_0^t \frac{t}{t+\tau}u(t,\tau,(Tx)(\tau))d\tau, \quad t \geq 0.$$

It is *the Volterra counterpart of the quadratic integral equation of Chandrasekhar type*. This equation is also a special case of Equation (3.1), in which

$$g(t,\tau) = \begin{cases} t\ln(1+\frac{\tau}{t}), & (t,\tau) \in \Delta \setminus \{(0,0)\} \\ 0, & t = \tau = 0\ . \end{cases}$$

Using, as before, Lemma 4.1 and the standard methods of differential calculus we can show that this function satisfies assumptions (iv)–(vi).

Remark 4.2. If we put $f_2(t,x) \equiv 0$ in Equation (3.1) we obtain the classical functional equation of the first order on an unbounded interval.

Now, we provide an example illustrating Theorem 3.7.

Example 4.3. Consider the following integral equation

$$x(t) = ax(t) + b\tanh t + \frac{1}{\Gamma(\frac{1}{3})}\frac{t}{t+1}x^2(t)\int_0^t \frac{\max\left\{\ln\left(1+\sqrt{|x(u)|}\right) : u\in[0,\tau]\right\}}{(1+t^2+\tau^2)\sqrt[3]{(t-\tau)^2}}d\tau \tag{4.2}$$

where $t \geq 0$, $a \geq 0$, $b \geq 0$.

Obviously this equation is a special case of Equation (4.1) if we put $\alpha = \frac{1}{3}$ and

$$\begin{aligned} f_1(t,x) &= ax + b\tanh t \\ \tilde{f}_2(t,x) &= \frac{t}{t+1}x^2 \\ u(t,\tau,x) &= \frac{x}{1+t^2+\tau^2} \\ (Tx)(t) &= \max\left\{\ln\left(1+\sqrt{|x(u)|}\right) : u\in[0,t]\right\}. \end{aligned}$$

It is easy to check that assumptions (i)–(viii) of Theorem 3.7 are satisfied and $k_1(r) = a$, $\overline{F_1} = b$, $\tilde{k}_2(r) = 2r$, $\tilde{F}_2 = 0$, $\phi(r) = r$, $\Psi(r) = \sqrt{r}$, $\overline{N} \leq 2$, $\overline{U} = 0$. Using standard estimation $\Gamma(\alpha) > 0.8856$ for $\alpha > 0$ and taking sufficiently small r_0, a and b (e.g. $r_0 = \frac{1}{4}$, $a = \frac{1}{6}$, $b = \frac{1}{20}$ (*or* $b = 0$)) we check that assumption (ix*) is also satisfied. Therefore in case of Equation (4.2) we can apply Theorem 3.7.

It is worthwhile mentioning that the integral equation (4.2) belongs to the important class of integral equations called the *equations with supremum* (or *with maximum*). Equations of this type has been recently investigated in the papers [2, 6, 13, 22], for example.

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