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Self-Similarity in the Collection of ω-Limit Sets

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Abstract. Let ω be the map which takes (x, f) in $I \times C(I \times I)$ to the ω-limit set $\omega(x, f)$ with $\mathcal{L}$ the map taking f in $C(I, I)$ to the family of ω-limit sets $\{\omega(x, f) : x \in I\}$. We study $\mathcal{R}(\omega) = \{\omega(x, f) : (x, f) \in I \times C(I, I)\}$, the range of ω, and $\mathcal{R}(\mathcal{L}) = \{\mathcal{L}(f) : f \in C(I, I)\}$, the range of $\mathcal{L}$. In particular, $\mathcal{R}(\omega)$ and its complement are both dense, $\mathcal{R}(\omega)$ is path-connected, and $\mathcal{R}(\omega)$ is the disjoint union of a dense G_δ set and a first category F_σ set. We see that $\mathcal{R}(\mathcal{L})$ is porous and path-connected, and its closure contains $\mathcal{K} = \{F \subseteq [0, 1] : F \text{ is closed}\}$. Moreover, each of the sets $\mathcal{R}(\omega)$ and $\mathcal{R}(\mathcal{L})$ demonstrates a self-similar structure.

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1. Introduction

Fundamental to the notion of chaos is the idea that points arbitrarily close together can generate trajectories or ω-limit sets that are far apart. In particular, if f is a continuous self-map of $I = [0, 1]$, and $x \in I$, then $\gamma(x, f) = \{x, f(x), f(f(x)), \ldots\}$ is the trajectory of x generated by f, with the collection of subsequential limits of $\gamma(x, f)$ being the ω-limit set $\omega(x, f)$. Equivalently,

$$\omega(x, f) = \cap_{m \geq 0} \overline{\cup_{n \geq m} f^n(x)}.$$

These ω-limit sets are the focus of our analysis, and we begin with a brief overview of some of their properties. Immediate consequences of the definition are that

(1) $\omega(x, f)$ is closed, and
(2) $f(\omega(x, f)) = \omega(x, f)$, that is $\omega(x, f)$ is strongly invariant,

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for all (x, f) in $I \times C(I, I)$.

While all ω-limit sets are closed, only certain classes of closed sets are actually generated as ω-limit sets. We make frequent use of the following characterization of ω-limit sets found in [1].

Theorem 1.1. *Let $F \subseteq [0,1]$ be closed. Then F is an ω-limit set for some f in $C(I, I)$ if and only if F is either nowhere dense, or a finite union of nondegenerate closed intervals.*

As the following shows, the preponderance of ω-limit sets are nowhere dense and perfect.

Theorem 1.2. ([2, Theorem 5]) *For a residual set of points (x, f) in $I \times C(I, I)$, $\omega(x, f)$ is nowhere dense and perfect.*

The main results of [1] are replicated in [11] with a much simpler analysis, and [12] characterizes those ω-limit sets generated by functions with zero topological entropy. Several articles, including [13] and [14], continue the study of typical behavior initiated with [2]. In [13] one finds that if M is the Cantor space or an n-dimensional manifold, then there is a residual set of points (x, f) in $M \times C(M, M)$ all of which generate as their ω-limit set a particular, unique type of adding machine.

Bruckner and Ceder provide in [10] a very interesting study of the map $\omega_f : I \to \mathcal{K}$ given by $x \mapsto \omega(x, f)$, where $f \in C(I, I)$ is fixed. The authors establish a notion of chaos strictly intermediate to positive topological entropy and the existence of an uncountable scrambled set. One also finds a comprehensive analysis of the behavior of a continuous function on its symple systems. Bruckner and Ceder's work foreshadows some of what is found in this article.

As mentioned earlier, $\omega(x, f)$ is necessarily closed whenever (x, f) is in $I \times C(I, I)$. Much less obvious is that $\Lambda(f) = \cup_{x \in I} \omega(x, f)$ is closed in $[0,1]$ whenever $f \in C(I, I)$, and that $\mathcal{L}(f) = \{\omega(x, f) : x \in I\}$ is closed with respect to the Hausdorff metric [4, 18].

In what follows, we consider two maps which deal directly with the ω-limit sets generated by continuous self-maps of $[0,1]$:

$$\omega : I \times C(I, I) \to \mathcal{K} \quad \text{given by} \quad (x, f) \mapsto \omega(x, f)$$

and

$$\mathcal{L} : C(I, I) \to \mathcal{K}^\star \quad \text{given by} \quad f \mapsto \mathcal{L}(f).$$

Here, $\mathcal{K}$ is the metric space composed of the class of nonempty closed sets in I endowed with the Hausdorff metric $\mathcal{H}$ given by $\mathcal{H}(E, F) = \inf\{\delta > 0 : E \subset B_\delta(F), F \subset B_\delta(E)\}$. This space is compact [9].

The metric space $\mathcal{K}^\star$ consists of the nonempty closed subsets of $\mathcal{K}$. Thus, $K \in \mathcal{K}^\star$ if K is a nonempty family of nonempty closed sets in I such that K

is closed in $\mathcal{K}$ with respect to $\mathcal{H}$. We endow $\mathcal{K}^\star$ with the metric $\mathcal{H}^\star$ so that K_1 and K_2 are close with respect to $\mathcal{H}^\star$ if each member of K_1 is close to some member of K_2 with respect to $\mathcal{H}$, and vice-versa. This metric space also is compact [8].

Here are some of our results. Let

$$\mathcal{R}(\omega) = \{\omega(x, f) : (x, f) \in I \times C(I, I)\}$$

be the range of ω. We show that

(1) both $\mathcal{R}(\omega)$ and $\mathcal{K} \setminus \mathcal{R}(\omega)$ are dense in $\mathcal{K}$,
(2) $\mathcal{R}(\omega)$ is path-connected,
(3) $\mathcal{R}(\omega)$ is the disjoint union of a dense G_δ subset of $\mathcal{K}$ and a first category F_σ subset of $\mathcal{K}$.

The set $\mathcal{R}(\omega)$ also is self-similar. Let $h_{[a,b],I} : [a, b] \to I$ be the linear homeomorphism such that $h_{[a,b],I}(a) = 0$ and $h_{[a,b],I}(b) = 1$. If $\mathcal{S}_{[a,b]} = \{F \in \mathcal{R}(\omega) : F \subseteq [a, b]\}$, then $\mathcal{R}(\omega) = h_{[a,b],I}(\mathcal{S}_{[a,b]})$.

Similarly, let

$$\mathcal{R}(\mathcal{L}) = \{\mathcal{L}(f) : f \in C(I, I)\}$$

be the range of $\mathcal{L}$ in $\mathcal{K}^\star$. We show that

(1) $\mathcal{R}(\mathcal{L})$ is a porous subset of $\mathcal{K}^\star$,
(2) $\mathcal{R}(\mathcal{L})$ is path-connected,
(3) $\mathcal{K} \in \overline{\mathcal{R}(\mathcal{L})}$,
(4) $\mathcal{R}(\mathcal{L}) \subsetneq \overline{\mathcal{R}(\mathcal{L})}$.

If $F \in \mathcal{R}(\mathcal{L})$, by $F \subseteq (a, b)$ we mean that $\omega \subseteq (a, b)$ for every $\omega \in F$. Now, let $(a, b) \subseteq [0, 1]$ and set $\mathcal{T}_{(a,b)} = \{F \in \mathcal{L}(f) : F \subseteq (a, b)\}$, and $\mathcal{T}_{(0,1)} = \{F \in \mathcal{L}(f) : F \subseteq (0, 1)\}$. Then, $\mathcal{R}(\mathcal{L})$ also demonstrates a self-similar structure, as $h_{[a,b],I}(\mathcal{T}_{(a,b)}) = \mathcal{T}_{(0,1)}$.

We proceed through several sections. After establishing notation and definitions in Section 2, we study the map $\mathcal{L} : C(I, I) \to \mathcal{K}^\star$ in Section 3. Section 4 focuses on the results concerning $\omega : I \times C(I, I) \to \mathcal{K}$.

2. Definitions and background material

Let $X = (X, d)$ be a compact metric space and let $\mathcal{C}(X, X)$ be the class of continuous self-maps of X.

Definition 2.1. A topological dynamical system (X, f) is a compact metric space X and a map $f \in \mathcal{C}(X, X)$.

In the following X will always denote a compact metric space. For $f \in \mathcal{C}(X, X)$ and any integer n, f^n denotes the n-th iterate of f. Let $P(f)$

be the set of periodic points of f. For each $x \in X$, we denote by $\omega(x, f)$ the *ω-limit set* of f; that is, the set of limit points of the sequence $\{f^k(x)\}_{k\geq 0}$. Let $\Lambda(f) = \cup_{x\in X}\omega(x, f)$ and $\mathcal{L}(f) = \{\omega(x, f) : x \in X\}$. We let $I = [0, 1]$, the closed unit interval of the real line. If $f \in \mathcal{C}(I, I)$, then $\Lambda(f)$ is closed [18].

Porosity. Let $P = (P, d)$ be a metric space, with $B_r(x) = \{y \in P : d(x, y) < r\}$ for $x \in P$ and $r > 0$. Take $M \subseteq P$, $x \in P$ and $R > 0$. Let

$$r(x, R, M) = \sup\{r > 0 : \text{ there exists } z \text{ in } P \text{ such that } B_r(z) \subset B_R(x) \setminus M\}.$$

The number

$$P(M, x) = 2 \limsup_{R\to 0^+} \frac{r(x, R, M)}{R},$$

is called the *porosity of M at x* [20].

Solenoidal sets. [5] An interval J is called periodic (of period k) or k-periodic if $J, \dots, f^{k-1}(J)$ are pairwise disjoint and $f^k(J) = J$. The set $\cup_{i=0}^{k-1} f^i(J)$ is the orbit of J and is denoted by $\operatorname{orb} J$.

Let $J_0 \supset J_1 \supset \cdots$ be periodic intervals with periods $m_0, m_1, \dots$. Obviously m_{i+1} is a multiple of m_i for all i. If $m_i \to \infty$ then the intervals $\{J_i\}_{i=0}^\infty$ are said to be generating and any invariant closed set $S \subseteq Q = \cap_{i=0}^\infty \operatorname{orb} J_i$ is called a solenoidal set; if Q is nowhere dense then we call Q a solenoid.

Basic sets. [5] Let J be an n-periodic interval, and let $M = \operatorname{orb} J$ be the orbit of J . Consider a set

$$\{x \in M : \text{ for any relative neighborhood } U \text{ of } x \text{ in } M \text{ we have } \overline{\operatorname{orb} U} = M\};$$

it is easy to see that this is a closed invariant set. It is called a basic set and denoted by $B(M, f)$ provided it is infinite. The set $B(M, f)$ is perfect [5, Theorem 4.1].

Path-connected topological spaces. A topological space X is *path-connected* (or *pathwise connected*) if for every two points x, y in X, there is a continuous function f from $[0, 1]$ to X such that $f(0) = x$ and $f(1) = y$.

3. The map $\mathcal{L} : C(I, I) \to \mathcal{K}^\star$

In this section we focus our attention on the map $\mathcal{L} : C(I, I) \to \mathcal{K}^\star$ which takes f in $C(I, I)$ to its collection of ω-limit sets $\mathcal{L}(f) = \{\omega(x, f) : x \in I\}$. As our first result shows, the elements of $\mathcal{R}(\mathcal{L})$ can be extremely complicated, as $\mathcal{K}$ is contained in the closure of $\mathcal{R}(\mathcal{L})$.

Proposition 3.1. *The set $\mathcal{K} = \{F \subseteq [0, 1] : F \text{ is closed}\}$ is contained in the closure of $\mathcal{R}(\mathcal{L})$.*

Proof. As $\mathcal{K}$ is a compact metric space, it is totally bounded. Hence, fixed $\epsilon > 0$, there exists an ϵ-net; that is, there exists a finite collection $\mathcal{T}_\epsilon = \{T_1, T_2, \ldots, T_m\}$ of elements of $\mathcal{K}$ compact such that $\mathcal{K} \subseteq \cup_{i=1}^m B_\epsilon(T_i)$, $T_i \cap T_j = \emptyset$ whenever $i \neq j$, and $|T_i|$ is finite for all i. Let $f \in C(I,I)$ so that $T_i \in \mathcal{L}(f)$ for all $1 \leq i \leq m$. Then, $\mathcal{H}^\star(\mathcal{K}, \mathcal{L}(f)) < \epsilon$. □

As mentioned in the introduction, $\Lambda(f)$ and $\mathcal{L}(f)$ are closed for any f in $C(I,I)$, as is each of the ω-limit sets $\omega(x,f)$. The next two examples show, however, that $\mathcal{R}(\mathcal{L})$ is not a closed subset of $\mathcal{K}^\star$.

Example 3.2. Recall that $\mathcal{K}$ is contained in the closure of $\mathcal{R}(\mathcal{L})$, and $\{\{x\} : x \in [0,1]\} \subsetneq \mathcal{K}$. If $f \in C(I,I)$ such that $\{\{x\} : x \in [0,1]\} \subseteq \mathcal{L}(f)$, then $\omega(x,f) = \{x\}$ for any x in $[0,1]$, and this precludes $T \in \mathcal{L}(f)$ for any T in $\mathcal{K} \setminus \{\{x\} : x \in [0,1]\}$.

Example 3.3. Let $f(x) = x$ on I, and for any $\epsilon > 0$, choose $\frac{1}{n} < \epsilon$. An appropriate polygonal function (*the Bruckner sawtooth function*) that possesses the orbit

$$0 \to \frac{1}{n} \to \frac{2}{n} \to \cdots \to \frac{n-1}{n} \to 1 \to \frac{n-\frac{1}{2}}{n} \to \frac{n-\frac{3}{2}}{n} \to \cdots \to \frac{\frac{1}{2}}{n} \to 0$$

has a periodic orbit that spans I, and has the property that $\|f - f_n\| \leq \frac{1}{n}$. Then f_n uniformly converges to f, and since $\mathcal{K}^\star$ is compact there exists $\{f_{n_k}\}_{k\in\mathbb{N}} \subseteq \{f_n\}_{n\in\mathbb{N}}$ so that $\lim_{k\to\infty} \mathcal{L}(f_{n_k}) = K^\star$ exists. Then $\{\{x\} : x \in [0,1]\} \cup [0,1] \subseteq K^\star$. If there exists $g \in C(I,I)$ so that $\{\{x\} : x \in [0,1]\} \subseteq \mathcal{L}(g)$, then $\omega(x,g) = \{x\}$ for any x in $[0,1]$, so that $[0,1] \notin \mathcal{L}(g)$.

Two of the principal results of this section are found with Theorems 3.4 and 3.5. The first result shows just how particular the elements $\mathcal{L}(f)$ are in $\mathcal{K}^\star$, as $\mathcal{R}(\mathcal{L})$ is porous in $\mathcal{K}^\star$. Interestingly enough, while nowhere dense, $\mathcal{R}(\mathcal{L})$ has no isolated point, and it is, in fact, path-connected.

Theorem 3.4. *$\mathcal{R}(\mathcal{L})$ is porous in $\mathcal{K}^\star$. In particular, $P(\mathcal{R}(\mathcal{L}), \mathcal{L}(f)) = 1$ for all f in $C(I,I)$.*

Proof. Fix f in $C(I,I)$, $n \in \mathbb{N}$, and $0 < \epsilon < \frac{1}{2}$. Since $\mathcal{L}(f)$ is compact in $\mathcal{K}^\star$, we can take $K^\star = \{K_1, K_2, \ldots, K_m\}$ in $\mathcal{K}^\star$ so that

(1) $K^\star$ is an ϵ-net of $\mathcal{L}(f)$;
(2) $|K_i| \geq 2$ for any i;
(3) $\operatorname{diam}(K_i) \geq \frac{n-1}{n} 2\epsilon$ for any i.

Now, if $E^\star \in \mathcal{K}^\star$ such that $\mathcal{H}^\star(E^\star, K^\star) < \frac{n-1}{n}\epsilon$, then $|K| \geq 2$ for any $K \in E^\star$. In particular,

$$\mathcal{R}(\mathcal{L}) \cap B_{\frac{n-1}{n}\epsilon}(K^\star) = \emptyset, \quad \text{and} \quad P(\mathcal{R}(\mathcal{L}), \mathcal{L}(f)) \geq 2\left[\frac{(\frac{n-1}{n})\epsilon}{2\epsilon}\right] = \frac{n-1}{n}.$$

Now, let $n \to +\infty$. □

Theorem 3.5. *$\mathcal{R}(\mathcal{L})$ is path-connected in $\mathcal{K}^\star$.*

Proof. Let $f_{\frac{1}{2}} : I \to I$ be the constant map given by $f_{\frac{1}{2}}(x) = \frac{1}{2}$, for all $x \in I$. Since, by [19, Theorem 1.4], $\mathcal{L} : C(I, I) \to \mathcal{K}^\star$ is continuous at every constant map, it is, in particular, continuous at $f_{\frac{1}{2}}$. Therefore, for any $\epsilon > 0$ there is $\delta > 0$ so that $\mathcal{L}(g) \subseteq B_\epsilon(\{\frac{1}{2}\})$, whenever $\|g - f_{\frac{1}{2}}\| < \delta$. Now, fix some g in $C(I, I)$ and take $0 < \epsilon \leq 1$. Consider the homeomorphism $h_\epsilon : I \to J = [\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}]$ so that $h_\epsilon(0) = \frac{1}{2} - \frac{\epsilon}{2}$, $h_\epsilon(1) = \frac{1}{2} + \frac{\epsilon}{2}$, and h_ϵ is linear on $[0, 1]$. Let $f_g^\epsilon = h_\epsilon \circ g \circ h_\epsilon^{-1} : J \to J$, and define $\tilde{f}_g^\epsilon : I \to I$ so that

1. $\tilde{f}_g^\epsilon | J = f_g^\epsilon$,
2. $\tilde{f}_g^\epsilon | J = \begin{cases} f_g^\epsilon(\frac{1}{2} - \frac{\epsilon}{2}) & \text{if } x \in [0, \frac{1}{2} - \frac{\epsilon}{2}] \\ f_g^\epsilon(\frac{1}{2} + \frac{\epsilon}{2}) & \text{if } x \in [\frac{1}{2} + \frac{\epsilon}{2}, 1]. \end{cases}$

It follows that $\mathcal{L}(\tilde{f}_g^\epsilon) = \mathcal{L}({f_g}^\epsilon)$ since $\|\tilde{f}_g^\epsilon - f_{\frac{1}{2}}\| \leq \frac{\epsilon}{2}$, so that $\tilde{f}_g^\epsilon([0, 1]) \subseteq J$ and $f_g^\epsilon(J) \subseteq J$. Since $\mathcal{L}$ is continuous at $f_{\frac{1}{2}}$, it follows that $\mathcal{L}(\tilde{f}_g^\epsilon) \to \{\{\frac{1}{2}\}\}$ as $\epsilon \to 0$, for any $g \in C(I, I)$. Since g and f_g^ϵ are topologically conjugate, it follows that $h_\epsilon(\mathcal{L}(g)) = \mathcal{L}({f_g}^\epsilon) = \mathcal{L}(\tilde{f}_g^\epsilon)$, and since as $\epsilon \to 1$ $h_\epsilon \to i_d$, where i_d denotes the identity map on $[0, 1]$, we conclude that $\mathcal{L}(\tilde{f}_g^\epsilon) \to \mathcal{L}(g)$ as $\epsilon \to 1$. In particular, $\mathcal{L}(g)$ and $\mathcal{L}(f_{\frac{1}{2}}) = \{\{\frac{1}{2}\}\}$ are in the same path-connected component of $\mathcal{R}(\mathcal{L})$, for any g in $C(I, I)$. □

Remark 3.6. While $\mathcal{R}(\mathcal{L})$ is path-connected, the paths established in the theorem are not the only ones found between elements of $\mathcal{R}(\mathcal{L})$. If $f \neq g$ in $C(I, I)$ are topologically conjugate, then $\mathcal{L}(f)$ and $\mathcal{L}(g)$ are path-wise connected. Suppose $f \neq g$ in $C(I, I)$ with $h : I \to I$ a homeomorphism such that $g = h \circ f \circ h^{-1}$. Let h_α, $\alpha \in [0, 1]$, be a continuous injective deformation such that $h_0 = i_d$, $h_1 = h$. Then $\{\mathcal{L}(h_\alpha \circ f \circ {h_\alpha}^{-1}) : 0 \leq \alpha \leq 1\}$ is a path in $\mathcal{R}(\mathcal{L})$ with $\mathcal{L}(f) = \mathcal{L}(h_0 \circ f \circ h_0^{-1})$ and $\mathcal{L}(g) = \mathcal{L}(h_1 \circ f \circ h_1^{-1})$.

With the next theorem, we begin a series of results which establish self-similarity in the range of $\mathcal{L}$. Theorem 3.7 shows that for any interval $[a, b]$ in $[0, 1]$, we can find a function g in $C(I, I)$ so that $C(I, I)$ is replicated in miniature by $\overline{B_\epsilon(g)}$ on $[a, b]$, where $\epsilon = \frac{b-a}{2}$.

Theorem 3.7. *Let $[a, b] \subseteq [0, 1]$ and set $\epsilon = \frac{b-a}{2}$. There exists g in $C(I, I)$ so that $\overline{B_\epsilon(g)}|[a, b] = h_{I,[a,b]} \circ C(I, I) \circ h_{[a,b],I}$.*

Proof. Let $g \in C(I, I)$ so that $g(x) = \frac{a+b}{2}$ for any x in $[a, b]$. If $f \in C(I, I)$ so that $\|f - g\| \leq \epsilon = \frac{b-a}{2}$, then

$$\frac{a+b}{2} - \frac{b-a}{2} = a \leq f(x) \leq b = \frac{a+b}{2} + \frac{b-a}{2},$$

for any x in $[a,b]$, so that $f([a,b]) \subseteq [a,b]$. Therefore, $\overline{B_\epsilon(g)}|[a,b] \subseteq C([a,b],[a,b])$. Clearly,

$$C([a,b],[a,b]) \subseteq B_\epsilon(g)|[a,b] \subseteq \overline{B_\epsilon(g)}|[a,b].$$

Since $h_{I,[a,b]}$ and $h_{[a,b],I}$ are continuous, $h_{I,[a,b]} \circ f \circ h_{[a,b],I} : [a,b] \to [a,b]$ is continuous for every f in $C(I,I)$, so that $h_{I,[a,b]} \circ C(I,I) \circ h_{[a,b],I} \subseteq C([a,b],[a,b])$.

If $l \in C([a,b],[a,b])$ then $f = h_{[a,b],I} \circ l \circ h_{I,[a,b]} \in C(I,I)$ so that $l = h_{I,[a,b]} \circ f \circ h_{[a,b],I}$. Thus, $h_{I,[a,b]} \circ C(I,I) \circ h_{[a,b],I} = C([a,b],[a,b])$. Finally, of course, if $f \neq l$ in $C(I,I)$, then $h_{I,[a,b]} \circ f \circ h_{[a,b],I} \neq h_{I,[a,b]} \circ l \circ h_{[a,b],I}$ in $C([a,b],[a,b])$ so that the transformation from $C(I,I)$ to $C([a,b],[a,b])$ is also one to one. □

Corollary 3.8. *Let* $[a,b] \subseteq [0,1]$ *and* $\epsilon = \frac{b-a}{2}$. *There exists* g *in* $C(I,I)$ *so that the range of* $\mathcal{L}$ *restricted to* $\overline{B_\epsilon(g)}|[a,b]$ *satisfies*

$$\mathcal{R}(\mathcal{L} : \overline{B_\epsilon(g)}|[a,b] \to \mathcal{K}^\star) = h_{I,[a,b]}(\mathcal{R}(\mathcal{L})).$$

In Theorem 3.7 we begin with an interval $[a,b]$ in I, and find an appropriate function g in $C(I,I)$. Our next two results show that we can just as well begin with a function f in $C(I,I)$ and an $\epsilon > 0$, and find an appropriate g in $B_\epsilon(f)$ with a corresponding interval $[a,b]$.

Theorem 3.9. *For every* f *in* $C(I,I)$ *and any* $0 < \epsilon \leq 1$ *there exist* g *in* $B_\epsilon(f)$, $0 < \delta < \epsilon$ *and* $[a,b] \subseteq I$ *so that* $b - a = 2\delta$, $\overline{B_\delta(g)} \subset B_\epsilon(f)$, *and* $\overline{B_\delta(g)}|[a,b] = h_{I,[a,b]} \circ C(I,I) \circ h_{[a,b],I}$.

Proof. Let $f \in C(I,I)$ with $\epsilon > 0$. Since f is uniformly continuous, there exist $0 < \delta' < \frac{\epsilon}{2}$ so that $|x-y| < \delta'$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$, and z in $[0,1]$ so that $f(z) = z$. Now, suppose that $z \in (0,1)$, and choose δ so that $\delta < \min\{\delta', z, 1-z\}$. Let $g \in C(I,I)$ so that $g \in B_{\frac{\epsilon}{2}}(f)$ and $g(y) = z$ for any $y \in [z-\delta, z+\delta] = [a,b]$, and consider $\overline{B_\delta(g)}$.

Now, suppose that $z \in \{0,1\}$. We assume that $z = 0$; the case that $z = 1$ is similar. Set $\delta = \frac{\delta'}{2}$, and take g in $C(I,I)$ so that $g \in B_{\frac{\epsilon}{2}}(f)$ and $g(y) = \delta$ for any y in $[0,\delta'] = [a,b]$. Consider $\overline{B_\delta(g)}$.

In either case, we show that $\overline{B_\delta(g)}$ is the set which satisfies our theorem's conclusion. This is done as in our previous theorem's proof. □

Corollary 3.10. *For every* f *in* $C(I,I)$ *and any* $\epsilon > 0$ *there exist* g *in* $B_\epsilon(f)$, $0 < \delta < \epsilon$ *and* $[a,b] \subseteq I$ *so that* $b - a = 2\delta$ *and*

$$\overline{B_\delta(g)} \subset B_\epsilon(f),$$

and

$$\mathcal{R}(\mathcal{L} : \overline{B_\delta(g)}|[a,b] \to \mathcal{K}^\star) = h_{I,[a,b]} \circ \mathcal{R}(\mathcal{L}).$$

We conclude our study of $\mathcal{R}(\mathcal{L})$ with the next theorem. Stronger than Corollaries 3.8 and 3.10, Theorem 3.11 shows that all of the elements of $\mathcal{R}(\mathcal{L})$ which lie in any interval (a, b) found in I are a scaled copy of those found in $(0, 1)$.

Theorem 3.11. *Let* $(a, b) \subseteq [0, 1]$, *and set* $\mathcal{T}_{(0,1)} = \{\mathcal{L}(f) : \mathcal{L}(f) \subseteq (0, 1), f \in C(I, I)\}$, $\mathcal{T}_{(a,b)} = \{\mathcal{L}(f) : \mathcal{L}(f) \subseteq (a, b), f \in C(I, I)\}$. *Then*

$$\mathcal{T}_{(0,1)} = h_{[a,b],I}(\mathcal{T}_{(a,b)}),$$

so that $\mathcal{T}_{(0,1)}$ *and* $\mathcal{T}_{(a,b)}$ *are homeomorphic.*

Proof. For each f in $C(I, I)$, $\mathcal{L}(f)$ is closed in $(\mathcal{K}, \mathcal{H})$ by [4]. Let $f \in C(I, I)$. By $\mathcal{L}(f) \subseteq (a, b)$ we intend $\omega(x, f) \subseteq (a, b)$ for all x in I.

We first show that $h_{I,[a,b]}(\mathcal{T}_{(0,1)}) \subseteq \mathcal{T}_{(a,b)}$. Let $g \in C(I, I)$, so that $\mathcal{L}(g) \in \mathcal{T}_{(0,1)}$, and set $f_g = h_{I,[a,b]} \circ g \circ h_{[a,b],I}$. Since $g \in C(I, I)$, it follows that $f_g \in C([a, b], [a, b])$ and $\mathcal{L}(f_g) = h_{I,[a,b]}(\mathcal{L}(g)) \subseteq (a, b)$. We extend $f_g : [a, b] \to [a, b]$ to $f : I \to I$ so that

$$f(x) = \begin{cases} f_g(a) & \text{if } x \in [0, a] \\ f_g(x) & \text{if } x \in [a, b] \\ f_g(b) & \text{if } x \in [b, 1]. \end{cases}$$

Since $\mathcal{L}(f) = \mathcal{L}(f_g)$, we have $h_{I,[a,b]}(\mathcal{L}(g)) = \mathcal{L}(f)$ contained in $\mathcal{T}_{(a,b)}$, so that $h_{I,[a,b]}(\mathcal{T}_{(0,1)}) \subseteq \mathcal{T}_{(a,b)}$.

It remains to show that $h_{[a,b],I}(\mathcal{T}_{(0,1)}) \supseteq \mathcal{T}_{(a,b)}$. Let $F \in \mathcal{T}_{(a,b)}$, so that $F = \mathcal{L}(f)$ for some f in $C(I, I)$, and $\mathcal{L}(f) \subseteq (a, b)$. It suffices to show that $F = \mathcal{L}(g)$, where $g \in C([a, b], [a, b])$. Now, since $\mathcal{L}(f) \subseteq (a, b)$, there exists $\epsilon > 0$ so that $\overline{B_\epsilon(\Lambda(f))} \subset (a, b)$ and since $f(\Lambda(f)) = \Lambda(f)$, $f(\overline{B_\epsilon(\Lambda(f))}) \subset (a, b)$, too. Now, let $x \in I$. There exists $N = N(x) \in \mathbb{N}$ so that $f^n(x) \in \overline{B_\epsilon(\Lambda(f))}$ whenever $n > N(x)$. If we let $y = f^{N+1}(x)$, then $\omega(y, g) = \omega(x, f)$, where g is any extension of $f|\overline{B_\epsilon(\Lambda(f))}$ to all of $[a, b]$. In particular, $\mathcal{L}(f) \subseteq \mathcal{L}(g)$. It remains to show that we can take g so that $\mathcal{L}(f) = \mathcal{L}(g)$.

Set $[\alpha, \beta] = \overline{\text{conv}}(\Lambda(f))$, the closed convex hull of $\Lambda(f)$, and let $y_1 \in (\alpha - \epsilon, \alpha)$ so that $\gamma(y_1, f) \subseteq B_\epsilon(\Lambda(f))$. By [3, Corollary IV.10] this would actually imply that $\gamma(y_1, f) \subseteq B_\epsilon(\Lambda(f)) \cap (y_1, \beta + \epsilon)$. If no such y_1 exists, set $y_1 = \alpha$. Let $M = \max \overline{\gamma(y_1, f)}$. If $M > \beta$, take $y_2 \in \gamma(y_1, f)$ so that $\beta < y_2 < M$. If $M \leq \beta$, take $y_2 \in (\beta, \beta + \epsilon)$ so that $\gamma(y_2, f) \subseteq B_\epsilon(\Lambda(f)) \cap (y_1, \beta + \epsilon)$. Again, by [3, Corollary IV.10], this would imply that $\gamma(y_2, f) \subseteq (y_1, y_2)$. If no such y_2 exists, set $y_2 = \beta$. We note that, by our choice of y_1 and y_2, $\{\omega(x, f) : x \in \overline{B_\epsilon(\Lambda(f))} \cap [y_1, y_2]\} = \mathcal{L}(f)$.

We now define $g : [a, b] \to [a, b]$

$$g(x) = \begin{cases} f(y_1) & \text{if } x \in [a, y_1] \\ f(x) & \text{if } x \in [y_1, \alpha] \cup [\beta, y_2] \\ f(y_2) & \text{if } x \in [y_2, b] \end{cases}$$

and if $x \in [\alpha, \beta]$, then

$$g(x) = \begin{cases} y_1 & \text{whenever } f(x) \leq y_1 \\ f(x) & \text{whenever } y_1 \leq f(x) \leq y_2 \\ y_2 & \text{whenever } f(x) \geq y_2. \end{cases}$$

It remains to verify that $\mathcal{L}(g) \subseteq \mathcal{L}(f)$.

If $x \in [a, y_1]$, then $\omega(x, g) = \omega(y_1, f) \in \mathcal{L}(f)$, and if $x \in [y_2, b]$, then $\omega(x, g) = \omega(y_2, f) \in \mathcal{L}(f)$. Now, let $x \in (y_1, y_2)$, and consider $\gamma(x, g)$. If $\gamma(x, g) \subseteq (y_1, y_2)$, then $\gamma(x, g) = \gamma(x, f)$, so that $\omega(x, g) = \omega(x, f) \in \mathcal{L}(f)$. If $\gamma(x, g) \cap \{y_1, y_2\} \neq \emptyset$, we can take $n \in \mathbb{N}$ minimal so that $g^n(x) = y_i \in \{y_1, y_2\}$. Then $\omega(x, g) = \omega(y_i, f) \in \mathcal{L}(f)$. □

As the next proposition shows, we cannot extend the conclusion of Theorem 3.11 to those elements of $\mathcal{R}(\mathcal{L})$ found in the closed interval $[a, b]$.

Proposition 3.12. *Let* $[a, b] \subseteq [0, 1]$ *so that* $0 < a < b < 1$. *There exists* $F \in \mathcal{R}(\mathcal{L})$ *such that* $\omega \subseteq [a, b]$ *for any* $\omega \in F$, *but* $F \notin \{\mathcal{L}(f) : f \in C([a, b], [a, b])\}$.

Proof. Let $f \in C(I, I)$ so that f has a unique infinite and maximal ω-limit set ω_0, which is generated by a solenoidal system. It follows that $\omega_0 = Q \dot{\cup} C$, where Q is a Cantor set and C is a dense and countable set of points isolated in ω_0. Now, suppose that $a = \min \omega_0 \in C$, $b = \max \omega_0 \in C$, and $\omega \subseteq [a, b]$ for all ω in $\mathcal{L}(f)$. (The reader is referred to [10] for the construction of such a function f.) If $x \in I$ for which $\omega(x, f) = \omega_0$, then $\gamma(x, f) \cap [0, a[\neq \emptyset$ and $\gamma(x, f) \cap]b, 1] \neq \emptyset$, since each of the intervals $[a, \min Q]$ and $[\max Q, b]$ is wandering. Moreover, a and b are isolated in $\Lambda(f)$. We conclude that if $F = \mathcal{L}(f)$, then $F \notin \{\mathcal{L}(g) : g \in C([a, b], [a, b])\}$. In particular, if $\omega_0 \in \mathcal{L}(g)$ for some $g \in C([a, b], [a, b])$, then as we saw earlier, the maximal ω-limit set which contains ω_0 cannot be solenoidal. But if ω_0 is contained in the basic set $B(M, g)$, then a and b are no longer isolated points of $\Lambda(g)$, since $B(M, g)$ is perfect. □

4. The map $\omega : I \times C(I, I)$

This section concerns the map $\omega : I \times C(I, I) \to \mathcal{K}$ which takes (x, f) in $I \times C(I, I)$ to the ω-limit set $\omega(x, f)$. Fundamental to much of the analysis is the characterization of ω-limit sets for continuous self-maps of the interval developed in [1]. The first result is an elementary lemma which takes advantage of the density of the finite sets in $\mathcal{K}$.

Lemma 4.1. $\mathcal{K}$ *is contained in the closure of* $\mathcal{R}(\omega)$.

Proof. Let $\epsilon > 0$. Any element $F \in \mathcal{K}$ can be ϵ-approximated by a finite set S in $\mathcal{K}$ so that $\mathcal{H}(F, S) < \epsilon$. Moreover, there exists f in $C(I, I)$ so that $\omega(x, f) = S$. □

We record several immediate corollaries of Lemma 4.1.

Corollary 4.2. *$\mathcal{R}(\omega)$ and $\mathcal{K} \setminus \mathcal{R}(\omega)$ are dense in $\mathcal{K}$, so that $\mathcal{R}(\omega) \subsetneq \overline{\mathcal{R}(\omega)}$ and $\text{int}(\mathcal{R}(\omega)) = \emptyset$.*

Proof. From [1] it follows that

$$\mathcal{K} \setminus \{\omega(x, f) : x \in I, f \in C(I, I)\}$$

is dense in $(\mathcal{K}, \mathcal{H})$. Precisely, every $K \in \mathcal{K}$ is approximable by finite sets. Fix $K \in \mathcal{K}$ and $\epsilon > 0$. There exists F finite such that $\mathcal{H}(K, F) < \frac{\epsilon}{2}$. Replace one point of F with a suitably small interval around it and obtain a new compact set F' such that $F' \notin \mathcal{R}(\omega)$ and $\mathcal{H}(F, F') < \frac{\epsilon}{2}$. Hence, $\mathcal{H}(K, F') < \epsilon$. □

Theorem 3.5 establishes that $\mathcal{R}(\mathcal{L})$ is path-connected in $\mathcal{K}^\star$, as any element of $\mathcal{L}(f)$ in $\mathcal{R}(\mathcal{L})$ is path-connected to $\{\{\frac{1}{2}\}\}$, also in $\mathcal{R}(\mathcal{L})$. If $F \in \mathcal{R}(\omega)$, then $F \in \mathcal{L}(f)$ for some f in $C(I, I)$, so that F is, in fact, path connected to $\{\frac{1}{2}\} \in \mathcal{R}(\omega)$.

Corollary 4.3. *$\mathcal{R}(\omega)$ is path-connected in $\mathcal{K}$.*

A main result of this section is the following theorem.

Theorem 4.4. *$\mathcal{R}(\omega)$ is the disjoint union of a dense G_δ subset of $\mathcal{K}$ and a first category F_σ subset of $\mathcal{K}$.*

Theorem 4.4 is an immediate consequence of Propositions 4.5 and 4.6. Obviously, this result relies on the work found in [1]. The lack of a "clean" characterization of the elements of $\mathcal{R}(\mathcal{L})$ seems a serious obstacle to the development of an analogous result for the range of $\mathcal{L} : C(I, I) \to \mathcal{K}^\star$.

Proposition 4.5. $\{F \in \mathcal{K} : F \text{ is nowhere dense}\}$ *is a dense G_δ subset of $\mathcal{K}$.*

Proof. Let I_j be an enumeration of the open intervals in $(0, 1)$ with rational end-points, and set $\mathcal{K}_j = \{F \in \mathcal{K} : I_j \subset F\}$. One verifies easily that $\mathcal{K}_j$ is closed in $\mathcal{K}$.

Suppose now that $F \in \mathcal{K}$ is not nowhere dense, so that for some $a \lneq b$ in $[0, 1]$ we have $[a, b] \subseteq F$. There exists some $I_j \subset [a, b]$, so that $I_j \subset F$ and $F \in \mathcal{K}_j$. In particular, $\{F \in \mathcal{K} : F \text{ is nowhere dense}\} = \mathcal{K} \setminus \cup_{j=1}^{\infty} I_j$ is a G_δ subset of $\mathcal{K}$. That $\{F \in \mathcal{K} : F \text{ is nowhere dense}\}$ is dense in $\mathcal{K}$ follows from the observation that the finite sets are dense in $\mathcal{K}$. □

Proposition 4.6. *The set*

$$\{F \in \mathcal{K} : F \text{ is a finite union of nondegenerate closed intervals}\}$$

is a first category F_σ subset of $\mathcal{K}$.

Proof. We show that

$$F_{n,m}=\{F\in\mathcal{K}\colon F=\dot\cup_{i=1}^{k}I_i \text{ s.t. } k\le n \text{ and } I_i=[a_i,b_i] \text{ with } b_i-a_i\ge\frac{1}{m} \text{ for all } i\}$$

is closed in $\mathcal{K}$. Let $\{F_i\}_{i\in\mathbb{N}}\subseteq F_{n,m}$ so that $F_i\to F$ in $\mathcal{K}$. We show that $F\in F_{n,m}$. Suppose, to the contrary, that there exist disjoint open intervals $\{(a_i,b_i)\}_{i=1}^{l}$ with $l\gneq n$, such that $F\subset\cup_{i=1}^{l}(a_i,b_i)$ and $F\cap(a_i,b_i)\neq\emptyset$ for all i. Since $F_i\to F$ in $\mathcal{K}$, there is N in $\mathbb{N}$ so that, whenever $j>N$, $F_j\subset\cup_{i=1}^{l}(a_i,b_i)$ and $F_j\cap(a_i,b_i)\neq\emptyset$, $1\le i\le l$. But this contradicts F_j having at most n components. Now, suppose that there exists an open interval (a,b) so that $(a,b)\cap F$ is closed, non-empty, and $b-a<\frac{1}{m}$. Since $F_i\to F$, for sufficiently large j, $F_j\cap(a,b)$ also is both closed and non-empty. But this contradicts F_j having components of length at least $\frac{1}{m}$. Our conclusion follows from the observation that $\cup_{n=1}^{\infty}\cup_{m=1}^{\infty}F_{n,m}=\{F\in\mathcal{K}: F$ is a finite union of nondegenerate closed intervals$\}$. □

Theorem 4.7. *$\mathcal{R}(\omega)$ is a dense $G_{\delta\sigma}$ subset of $\mathcal{K}$.*

Proof. The theorem follows from Propositions 4.5 and 4.6, and Theorem 1.1, the characterization of ω-limit sets found in [1]. □

The following two corollaries are consequences of Theorem 3.7 and Theorem 3.9, respectively.

Corollary 4.8. *Let $[a,b]\subseteq[0,1]$ and set $\epsilon=\frac{b-a}{2}$. There exists g in $C(I,I)$ so that*

$$\mathcal{R}(\omega:[a,b]\times\overline{B_\epsilon(g)}\to\mathcal{K}))=h_{I,[a,b]}(\mathcal{R}(\omega)).$$

Corollary 4.9. *For every f in $C(I,I)$ and any $\epsilon>0$ there exist g in $B_\epsilon(f)$, $0<\delta<\epsilon$ and $[a,b]\subseteq I$ so that $\overline{B_\delta(g)}\subset B_\epsilon(f)$, $b-a=2\delta$, and $\mathcal{R}(\omega:[a,b]\times\overline{B_\delta(g)}\to\mathcal{K})=h_{I,[a,b]}(\mathcal{R}(\omega))$.*

Another main result of the section is Theorem 4.10. While similar to Theorem 3.11, it is a bit stronger as we may consider all of the ω-limit sets found in $[a,b]$, and not just those found in (a,b).

Theorem 4.10. *Let $[a,b]\subseteq[0,1]$, and set*

$$S_{[a,b]}=\{\omega(x,f):\omega(x,f)\subseteq[a,b],x\in I,f\in C(I,I)\}.$$

Then $\mathcal{R}(\omega)=h_{[a,b],I}(S_{[a,b]})$.

Proof. We first show that $\mathcal{R}(\omega))\subseteq h_{[a,b],I}(S_{[a,b]})$. This is equivalent to showing that $h_{I,[a,b]}(\mathcal{R}(\omega))\subseteq S_{[a,b]}$. Let $\tilde\omega\in\mathcal{R}(\omega)$; say $\tilde\omega=\omega(\tilde x,\tilde f)$, where $(\tilde x,\tilde f)\in I\times C(I,I)$. Then $h_{I,[a,b]}(\tilde\omega)=\omega=\omega(h_{I,[a,b]}(\tilde x),\, h_{I,[a,b]}\circ\tilde f\circ h_{[a,b],I})$, where $h_{I,[a,b]}(\tilde\omega)=\omega\subseteq[a,b]$, $h_{I,[a,b]}(\tilde x)\in[a,b]\subseteq I$, and $h_{I,[a,b]}\circ\tilde f\circ h_{[a,b],I}\in C([a,b],[a,b])$. We now extend $h_{I,[a,b]}\circ\tilde f\circ h_{[a,b],I}$ to $f\in C(I,I)$ via Tietze extension theorem, so that

(1) $h_{I,[a,b]}(\tilde{\omega}) = \omega = \omega(h_{I,[a,b]}(\tilde{x}), f) \subseteq [a,b]$,
(2) $f \in C(I,I)$.

It follows that $h_{I,[a,b]}(\tilde{\omega}) \in S_{[a,b]}$, so that

$$h_{I,[a,b]}(\mathcal{R}(\omega)) \subseteq S_{[a,b]}.$$

We now show that $h_{[a,b],I}(S_{[a,b]}) \subseteq \mathcal{R}(\omega)$.

Let $\omega \in S_{[a,b]}$. Then, by [1], ω is either a finite union of nondegenerate closed intervals or a closed and nowhere dense set in $[a,b]$. Let $h_{[a,b],I}(\omega) = \tilde{\omega}$. Clearly, $\tilde{\omega}$ is either a finite union of nondegenerate closed intervals or a closed and nowhere dense set in $[0,1]$. Thus, again, by [1], there exists $(\tilde{x},\tilde{f}) \in I \times C(I,I)$ so that $\tilde{\omega} = \omega(\tilde{x},\tilde{f})$, so that $\tilde{\omega} \in \mathcal{R}(\omega)$. Thus, $h_{[a,b],I}(S_{[a,b]}) \subseteq \mathcal{R}(\omega)$. Our conclusion follows. □

The main result of [4] shows that, for a fixed f in $C(I,I)$, the set $\{\omega(x,f) : x \in I\}$ is closed in $\mathcal{K}$. The next result fixes, instead, x in $[0,1]$ and considers the set $\{\omega(x,f) : f \in C(I,I)\}$.

Proposition 4.11. *Let $x \in [0,1]$ with $\mathcal{S}_x = \{\omega(x,f) : f \in C(I,I)\}$. Then $\mathcal{S}_x$ contains a dense and open subset of $\mathcal{R}(\omega)$.*

Proof. Let $F \in \mathcal{R}(\omega)$ so that $x \notin F$, with $f \in C(I,I)$ and $y \in I$ such that $\omega(y,f) = F$, and $\epsilon > 0$ such that $x \notin \overline{B_\epsilon(\omega(y,f))}$. Take f_1 so that

(1) $f_1|\overline{B_\epsilon(\omega(y,f))} = f|\overline{B_\epsilon(\omega(y,f))}$,
(2) $f_1(x) = z \in \gamma(y,f) \cap B_\epsilon(\omega(y,f))$,

such that $\gamma(z,f) \subseteq B_\epsilon(\omega(y,f))$, and extend f_1 defined on $\overline{B_\epsilon(\omega(y,f)} \cup \{x\}$ to all of I via the Tietz extension theorem.

By $\omega(x,f_1) = \omega(z,f_1)$, $\gamma(z,f_1) = \gamma(z,f)$, it follows that $\omega(x,f_1) = \omega(z,f)$. Since $z \in \gamma(y,f)$, it follows that $\omega(z,f) = \omega(y,f)$. Thus, $\omega(x,f_1) = F \in \mathcal{S}_x$. Let $\mathcal{D}_x = \{F \in \mathcal{R}(\omega) : F \text{ is finite and } x \notin F\}$. Then $\mathcal{D}_x \subseteq \mathcal{S}_x$, and since $\mathcal{D}_x$ is dense in $\mathcal{R}(\omega)$, $\mathcal{S}_x$ is dense in $\mathcal{R}(\omega)$, too. Moreover, if $F \in \mathcal{D}_x$, then there is $\epsilon > 0$ so that $x \notin B_\epsilon(F)$. Should $F_1 \in \mathcal{R}(\omega)$ such that $\mathcal{H}(F,F_1) < \epsilon$, then $x \notin F_1$, and $F_1 \in \mathcal{S}_x$. □

For a fixed x, $\{\omega(x,f) : f \in C(I,I)\}$ is dense and open in $\mathcal{K}$. The next lemma decribes the origin of the "holes" found in $\mathcal{R}(\omega) \setminus \{\omega(x,f) : f \in C(I,I)\}$.

Lemma 4.12. *Suppose $F \in \mathcal{K}$ is nowhere dense, infinite and contains an isolated point. If $x_0 \in F$, then $F \notin \{\omega(x_0,f) : f \in C(I,I)\}$.*

Proof. Suppose, to the contrary, that there exists f in $C(I,I)$ such that $\omega(x,f) = F$, with $y \in F$ and $\epsilon > 0$ so that $F \cap B_\epsilon(y) = y$. Since F is strongly invariant with respect to f, it follows that $\gamma(x,f) \subseteq F$, and since $y \in F$, there exists $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $f^{n_k}(x) \to y$. Moreover, y is isolated in F, so we can just as well take $\{n_k\} \subseteq \mathbb{N}$ such that $f^{n_k}(x) = y$. Now, if $f^{n_1}(x) = y$ and $f^{n_2}(x) = y$, with $0 < k = n_2 - n_1$, then $f^k(y) = y$. We conclude that $\omega(x, f) = \omega(y, f)$ is periodic, and this contradicts $\omega(x, f) = F$. □

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References

[1] Agronsky, S. J., Bruckner, A. M., Ceder, J. C. and Pearson, T. L., The structure of ω-limit sets for continuous functions. *Real Anal. Exchange* 15 (1989/1990), 483 – 510.

[2] Agronsky, S. J., Bruckner, A. M. and Laczkovich, M., Dynamics of typical continuous functions. *J. Lond. Math. Soc. (2)* 40 (1989), 227 – 243.

[3] Block, L. S. and Coppel, W. A., *One-Dimensional Dynamics.* Lect. Notes Math. 1513. Berlin: Springer 1992.

[4] Blokh, A., Bruckner, A. M., Humke, P. D. and Smital, J., The space of ω-limit sets of a continuous map of the interval. *Trans. Amer. Math. Soc.* 348 (1996), 1357 – 1372.

[5] Blokh, A., The "spectral" decomposition for one-dimensional maps. In: *Dynamics Reported* (eds.: C. K. R. T. Jones et al.). Dynam. Report. Expositions Dynam. Systems (N.S.) 4. Berlin: Springer 1995, 1 – 59.

[6] Bowen, R., Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* 153 (1971), 401 – 414.

[7] Brin, M. and Stuck, G., *An Introduction to Dynamical Systems.* Cambridge: Cambridge Univ. Press 2004.

[8] Bruckner, A. M., Stability in the family of ω-limit sets of continuous self maps of the interval. *Real Anal. Exchange* 22 (1996/1997), 52 – 57.

[9] Bruckner, A. M., Bruckner, J. B. and Thomson, B. S., *Real Analysis.* New Jersey: Prentice-Hall 1997.

[10] Bruckner, A. M. and Ceder, J., Chaos in terms of the map $x \mapsto \omega(x, f)$. *Pacific J. Math.* 156 (1992), 63 – 96.

[11] Bruckner, A. M. and Smital, J., The structure of ω-limit sets for continuous maps of the interval. *Math. Bohem.* 117 (1992), 42 – 47.

[12] Bruckner, A. M. and Smital, J., A characterization of ω-limit sets of maps of the interval with zero topological entropy. *Ergodic Theory Dynam. Systems* 13 (1993)(3), 7 – 19.

[13] D'Aniello, E., Darji, U. B. and Steele, T. H., Ubiquity of odometers in topological dynamical systems. *Topology Appl.* 156 (2008)(2), 240 – 245.

[14] D'Aniello, E., Humke, P. D. and Steele, T. H., The space of adding machines generated by continuous self maps of manifolds. *Topology Appl.* 157 (2010), 954 – 960.

[15] D'Aniello, E. and Steele, T. H., Approximating ω-limit sets with periodic orbits. *Aequationes Math.* 75 (2008)(1-2), 93 – 102.

[16] Dinaburg, E. I., The relation between topological entropy and metric entropy (in Russian). *Dokl. Akad. Nauk SSSR* 190 (1970), 19 – 22; Engl. transl.: *Soviet Math. Dokl.* 11 (1969), 13 – 16.

[17] Lehning, H., Dynamics of typical continuous functions. *Proc. Amer. Math. Soc.* 123 (1995), 1703 – 1707.

[18] Šharkovskii, A. N., On a theorem of G. D. Birkhoff (in Ukranian). *Dopodovidi Akad. Nauk Ukrain. RSR, Ser. A* 1967 (1967), 429 – 432.

[19] Smital, J. and Steele, T. H., Stability of dynamical structures under perturbation of the generating function. *J. Difference Equ. Appl.* 15 (2009), 77 – 86.

[20] Zajíček, L., Porosity and σ-porosity. *Real Anal. Exchange* 13 (1987/1988), 314 – 350.

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