Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 33 (2014), 101–117
DOI: 10.4171/ZAA/1501

Asymptotic Behavior of Inexact Infinite Products of Nonexpansive Mappings in Metric Spaces

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Abstract. We study the influence of errors on the convergence of infinite products of nonexpansive mappings in metric spaces. Previously, certain convergence results were proved under the assumption that all exact orbits converge uniformly on the whole space. In the present paper, we improve upon these results by proving the convergence of inexact orbits only assuming uniform convergence of exact orbits on bounded subsets of the metric space. We also provide applications to the convex feasibility problem in Hilbert space.

Keywords. Attracting set, complete metric space, fixed point, inexact orbit, infinite product, nonexpansive mapping, uniform convergence

Mathematics Subject Classification (2010). 47H09, 47H10, 54E50

1. Introduction and preliminaries

Convergence analysis of iterations of nonexpansive mappings [12] is a central topic in Nonlinear Functional Analysis and its applications. Therefore it is natural to ask if convergence of the iterates of nonexpansive mappings is preserved in the presence of computational errors. Affirmative answers to this question are provided in [5]. Related results can be found, for instance, in [4, 6, 7, 16–18]. More precisely, in [5] it is shown that if all exact iterates of a given nonexpansive mapping converge (to fixed points), then this convergence continues to hold for inexact orbits with summable errors. The authors of [17] study the influence of computational errors on the convergence of iterates of nonexpansive mappings in both Banach and metric spaces. It is shown there that if all the orbits of a nonexpansive self-mapping of a metric space X converge to some closed subset F of X, then all inexact orbits with summable errors also converge to F. On the other hand, the authors of [17] also construct examples which show that the convergence of inexact orbits no longer holds when the errors are not summable.

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The convergence of infinite products of nonexpansive mappings is also of major importance because of their many applications in, for example, the investigation of feasibility and optimization problems. See, for instance, [1–3, 8–11, 13–15, 18–25] and references therein. Several aspects of the convergence of (random) infinite products on bounded, closed and convex subsets of a Banach space were thoroughly studied in [22]. In that paper we consider spaces of sequences of nonexpansive mappings on a bounded, closed and convex subset K of a Banach space, equipped with a suitable complete metric, and show that for a generic sequence in these spaces, the corresponding infinite products converge uniformly.

Recall that a property of elements of a complete metric space Z is said to be generic (typical) in Z if the set of all elements of Z which possess this property contains an everywhere dense G_δ subset of Z. In this case we also say that the property holds for a generic (typical) element of Z or that a generic (typical) element of Z has this property [22, 23].

In [22, Theorem 3.1] it is shown that for a generic element $\{B_t\}_{t=1}^{\infty}$ in a certain space of sequences of nonexpansive operators, there exists a nonexpansive retraction P_* onto the common fixed point set F of the operators B_t, $t = 1, 2, \ldots$, such that

$$B_t \cdot \cdots \cdot B_1 x \to P_* x$$

as $t \to \infty$, uniformly for all $x \in K$. It is also shown ([22, Theorem 3.2]) that for a generic sequence of operators $\{B_t\}_{t=1}^{\infty}$ in the same space, all its random products $B_{r(t)} \cdot \cdots \cdot B_{r(1)} x$ also converge to a nonexpansive retraction $P_r : K \to F$, uniformly for all $x \in K$, where $r : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$.

In view of the above discussion, it is natural to ask if the convergence of infinite products is preserved in the presence of computational errors. Affirmative answers to this question are provided in [7, 18]. These answers extend several results which were obtained in [5] for powers of a single operator. More precisely, the results of [5] were developed in [7] by replacing the iterates of a single operator with infinite products taken from a possibly infinite pool. Sections 2 and 4 of [7] are devoted to weak ergodic theorems in metric and Banach spaces, respectively, while Sections 3 and 5 of [7] deal with convergence to fixed points. Note that in [7] all the convergence results were established under the assumptions that the exact infinite products converge and that the computational errors are summable. In [18] uniform convergence of the exact infinite products was, once again, required, but the computational errors were only assumed to converge to zero. Under these assumptions, it still turned out to be possible to establish uniform convergence of the corresponding inexact infinite products. We now quote three results which were proved in [18]. In order to formulate them, we first recall the following notations and assumptions.

Let (X, ρ) be a complete metric space. For each $x \in X$ and each nonempty set $A \subset X$, we denote

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

Let $T_i : X \to X$, $i = 0, 1, \ldots$ satisfy

$$\rho(T_i x, T_i y) \le \rho(x, y), \quad x, y \in X, \ i = 0, 1, \ldots.$$

For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

Theorem 1.1. *Let F be a nonempty and closed subset of X such that*

$$T_i(F) \subset F \quad \textit{for all integers } i \ge 0. \tag{1.1}$$

Let $\mathcal{R}$ be a nonempty set of mappings $r : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ with the following property:

(P1) *If $r \in \mathcal{R}$ and q is a natural number, then $r_q \in \mathcal{R}$, where*

$$r_q(i) := r(i + q) \quad \textit{for all integers } i \ge 0.$$

Assume that the following property holds:

(P2) *For each $\epsilon > 0$, there exists a natural number $n(\epsilon)$ such that for each $r \in \mathcal{R}$ and each $x \in X$,*

$$\rho(T_{r(n(\epsilon))} \cdots T_{r(1)} T_{r(0)} x, F) < \epsilon.$$

Then for each $\epsilon > 0$, there exist $\delta > 0$ and a natural number $\bar{n}$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$\rho(x_{i+1}, T_{r(i)} x_i) \le \delta \quad \textit{for all integers } i \ge 0,$$

the following inequality holds:

$$\rho(x_i, F) < \epsilon \quad \textit{for all integers } i \ge \bar{n}.$$

Theorem 1.2. *Let F be a nonempty and closed subset of X, assume that* (1.1) *holds and let $\mathcal{R}$ be a nonempty set of mappings $r : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ which has property* (P1). *Assume that property* (P2) *holds too.*

Let $\{\delta_i\}_{i=0}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{i \to \infty} \delta_i = 0.$$

Let $\epsilon > 0$ be given. Then there exists a natural number n_0 such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$\rho(x_{i+1}, T_{r(i)} x_i) \le \delta_i, \quad i = 0, 1, \ldots, \tag{1.2}$$

we have $\rho(x_n, F) < \epsilon$ for all integers $n \ge n_0$.

The following corollary is the special case of Theorem 1.2 where the attracting set F is a singleton.

Corollary 1.3. *Assume that the assumptions of Theorem* 1.2 *hold and that* F *is a singleton* $\{\bar{x}\}$. *Let* $\{\delta_i\}_{i=0}^{\infty}$ *be a sequence of positive numbers such that* $\lim_{i\to\infty}\delta_i = 0$. *Then for each* $\epsilon > 0$, *there exists a natural number* n_ϵ *such that for each* $r \in \mathcal{R}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies* (1.2), *we have* $\rho(x_n, \bar{x}) < \epsilon$ *for all integers* $n \ge n_\epsilon$.

The most restrictive assumption in these results is the uniform convergence of exact orbits on the whole space X, which usually holds when the space X is bounded and which does not hold in many important cases such as, for example, the convex feasibility problem in Hilbert spaces. In the present paper, we improve upon these results by establishing the convergence of inexact orbits only assuming uniform convergence of exact orbits on bounded subsets of the metric space.

Our paper is organized as follows. Our two main results are stated in the next section (see Theorems 2.1 and 2.2 below). Theorem 2.1 is proved in Section 3, while Theorem 2.2 is proved in Section 4. In Section 5 we state two extensions of our main results, Theorems 5.1 and 5.2, which are proved in Section 6 and 7, respectively. Finally, in Section 8 we apply our results to the convex feasibility problem in Hilbert spaces.

2. Main results

Let (Z, ρ) be a complete metric space.

For each $x \in Z$ and each nonempty set $A \subset Z$, put

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

Let $T_i : Z \to Z$, $i = 0, 1, \ldots$ satisfy

$$\rho(T_i x, T_i y) \le \rho(x, y), \quad x, y \in Z, \ i = 0, 1, \ldots. \tag{2.1}$$

Fix $\theta \in Z$. For each $x \in Z$ and each $r > 0$, set

$$B(x, r) = \{y \in Z : \ \rho(x, y) \le r\}.$$

Theorem 2.1. *Let* F *be a nonempty, bounded and closed subset of* Z *such that*

$$T_i(F) \subset F \quad \textit{for all integers } i \ge 0. \tag{2.2}$$

Let $\mathcal{R}$ *be a nonempty set of mappings* $r : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ *with property* (P1) *and assume that the following property holds:*

(P3) *For each* $\epsilon, M > 0$, *there exists a natural number* $n(\epsilon, M)$ *such that for each* $r \in \mathcal{R}$ *and each* $x \in B(\theta, M)$,

$$\rho(T_{r(n(\epsilon,M))} \cdots T_{r(1)} T_{r(0)} x, F) < \epsilon.$$

Then for each $\epsilon, M > 0$, *there exist a real number* $\delta > 0$ *and a natural number* $\bar{n}$ *such that for each* $r \in \mathcal{R}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset Z$ *which satisfies*

$$x_0 \in B(\theta, M) \tag{2.3}$$

and

$$\rho(x_{i+1}, T_{r(i)} x_i) \le \delta \quad \textit{for all integers } i \ge 0, \tag{2.4}$$

the following inequality holds:

$$\rho(x_i, F) < \epsilon \quad \textit{for all integers } i \ge \bar{n}.$$

Theorem 2.2. *Let* F *be a nonempty, bounded and closed subset of* Z *such that* (2.2) *holds. Let* $\mathcal{R}$ *be a nonempty set of mappings* $r : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ *with properties* (P1) *and* (P3).

Let $M > 0$ *be given. Then there is* $\bar{\delta} > 0$ *such that for each* $\epsilon > 0$ *and each sequence*

$$\{\delta_i\}_{i=0}^{\infty} \subset (0, \bar{\delta}] \quad \textit{such that } \lim_{i\to\infty} \delta_i = 0,$$

there exists a natural number n_0 *such that for each* $r \in \mathcal{R}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset Z$ *satisfying*

$$x_0 \in B(\theta, M)$$

and

$$\rho(x_{i+1}, T_{r(i)} x_i) \le \delta_i, \quad i = 0, 1, \ldots,$$

we have $\rho(x_n, F) < \epsilon$ *for all integers* $n \ge n_0$.

3. Proof of Theorem 2.1

Let $M, \epsilon > 0$ be given. We show that there exist a real number $\delta > 0$ and a natural number $\bar{n}$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfying (2.3) and (2.4), the following inequality holds:

$$\rho(x_i, F) < \epsilon \quad \text{for all integers } i \ge \bar{n}.$$

Without loss of generality we may assume that

$$\epsilon < \frac{1}{8}, \quad M > 4, \quad F \subset B(\theta, M - 4). \tag{3.1}$$

Set

$$X = \{y \in Z : \ \rho(y, F) \le 2M - 1\}. \tag{3.2}$$

Clearly, X is a closed subset of (Z, ρ), (X, ρ) is a complete metric space and for all integers $i \ge 0$,

$$T_i(X) \subset X. \tag{3.3}$$

It is easy now to see that all the assumptions of Theorem 1.1 hold for the space (X, ρ) and the restrictions of T_i to X, $i = 0, 1, \dots$. Therefore by Theorem 1,1 there exist $\delta > 0$ and a natural number $\bar{n}$ such that the following property holds:

(P4) For each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying $\rho(x_{i+1}, T_{r(i)}(x_i)) \le \delta$ for all integers $i \ge 0$, the inequality $\rho(x_i, F) < \epsilon$ holds for all integers $i \ge \bar{n}$.

We may assume without loss of generality that

$$\delta < (2\bar{n})^{-1}. \tag{3.4}$$

We show that the following property holds:

(P5) If $r : \{0, 1, \dots\} \to \{0, 1, \dots\}$, a sequence $\{x_i\}_{i=0}^{\bar{n}} \subset Z$ satisfies

$$\rho(x_0, F) < 2M - 2, \tag{3.5}$$

and for all integers $i = 0, \dots, \bar{n} - 1$,

$$\rho(x_{i+1}, T_{r(i)}x_i) \le \delta, \tag{3.6}$$

then $\{x_i\}_{i=0}^{\bar{n}} \subset X$.

Assume that $r : \{0, 1, \dots\} \to \{0, 1, \dots\}$ and $\{x_i\}_{i=0}^{\bar{n}} \subset Z$ satisfies (3.5) and (3.6).

By (2.1), (2.2) and (3.6) for each integer $i \in [0, \bar{n} - 1]$,

$$\rho(x_{i+1}, F) \le \rho(x_{i+1}, T_{r(i)}x_i) + \rho(T_{r(i)}x_i, F) \le \delta + \rho(x_i, F).$$

When combined with (3.5), (3.4) and (3.2), this implies that for all $i = 0, \dots, \bar{n}$,

$$\rho(x_i, F) \le \rho(x_0, F) + i\delta < 2M - 2 + \delta\bar{n} < 2M - 1,$$

and $\{x_i\}_{i=0}^{\bar{n}} \subset X$. Thus (P5) holds.

Assume that $r \in \mathcal{R}$ and the sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfies (2.3) and (2.4). By (2.3) and (3.1) inequality (3.5) holds.

Assume that $p \ge 0$ is an integer and that

$$\rho(x_p, F) < 2M - 2.$$

By (P5), (P1), the above inequality, (2.4), (P4) and (3.3),

$$\{x_i\}_{i=p}^{p+\bar{n}} \subset X, \ \rho(x_{p+\bar{n}}, F) < \epsilon < 2M - 2.$$

Together with (3.5), this implies that $\{x_i\}_{i=0}^{\infty} \subset X$. When combined with (2.4) and (P4), this implies that $\rho(x_i, F) < \epsilon$ for all integers $i \ge \bar{n}$. Theorem 2.1 is proved.

4. Proof of Theorem 2.2

We may assume without any loss of generality that

$$M > 4 \quad \text{and} \quad F \subset B(\theta, M-4). \tag{4.1}$$

By Theorem 2.1, there exist a number $\bar{\delta} > 0$ and a natural number n_1 such that the following property holds:

(P6) For each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ which satisfies

$$x_0 \in B(\theta, 2M+4) \quad \text{and} \quad \rho(x_{i+1}, T_{r(i)}x_i) \leq \bar{\delta} \text{ for all integers } i \geq 0,$$

we have $\rho(x_i, F) < 1$ for all integers $i \geq n_1$.

Let $\epsilon > 0$ be given and assume that

$$\{\delta_i\}_{i=0}^{\infty} \subset (0, \bar{\delta}] \quad \text{and} \quad \lim_{i\to\infty} \delta_i = 0. \tag{4.2}$$

By Theorem 2.1, there are a natural number n_2 and a number $\delta \in (0, \bar{\delta})$ such that the following property holds:

(P7) For each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfying

$$x_0 \in B(\theta, 2M+4) \quad \text{and} \quad \rho(x_{i+1}, T_{r(i)}x_i) \leq \delta,\ i = 0, 1, \dots,$$

we have $\rho(x_i, F) < \epsilon$ for all integers $i \geq n_2$.

Choose natural numbers

$$n_3 \geq n_1 + n_2 \quad \text{and} \quad n_0 \geq n_1 + n_2 + n_3 \tag{4.3}$$

such that

$$\delta_i < \delta \quad \text{for all integers } i \geq n_3. \tag{4.4}$$

Assume that

$$r \in \mathcal{R}, \quad \{x_i\}_{i=0}^{\infty} \subset Z, \quad x_0 \in B(\theta, M) \tag{4.5}$$

and

$$\rho(x_{i+1}, T_{r(i)}x_i) \leq \delta_i, \quad i = 0, 1, \dots. \tag{4.6}$$

By (P6), (4.5), (4.6), (4.2) and (4.3),

$$\rho(x_i, F) < 1 \text{ for all integers } i \geq n_1 \quad \text{and} \quad \rho(x_{n_3}, F) < 1.$$

When combined with (4.1), this inequality implies that

$$x_{n_3} \in B(\theta, M-3). \tag{4.7}$$

For each integer $i \geq 0$, set

$$y_i = x_{i+n_3}, \quad \tilde{r}_i = r(i+n_3). \tag{4.8}$$

By (P1), $\tilde{r} \in \mathcal{R}$. By (4.8), (4.6) and (4.4), for each integer $i \geq 0$, we have

$$\rho(y_{i+1}, T_{\tilde{r}(i)} y_i) = \rho(x_{i+n_3+1}, T_{r(i+n_3)} x_{i+n_3}) \leq \delta_{i+n_3} < \delta.$$

When combined with (4.7), (4.8), (P7) and (4.9), this implies (when applied to $\tilde{r}$ and $\{y_i\}_{i=0}^{\infty}$) that for all integers $i \geq n_2$,

$$\epsilon > \rho(y_i, F) = \rho(x_{i+n_3}, F) \quad \text{and} \quad \rho(x_i, F) < \epsilon$$

for all integers $i \geq n_0$. Theorem 2.2 is proved.

5. Extensions of the main results

We use the notations, definitions and assumptions from Section 2.

Theorem 5.1. *Let F be a nonempty and closed subset of Z such that*

$$T_i(F) \subset F \quad \text{for all integers } i \geq 0. \tag{5.1}$$

Let $\mathcal{R}$ be a nonempty set of mappings $r : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ which has properties (P1) *and* (P3).

Assume that $s \geq 0$ is an integer, q is a natural number such that

$$T_s(Z) \quad \text{is bounded}$$

and that the following property holds:

(P8) *for any $r \in \mathcal{R}$, there is an integer $j \in [0, q]$ such that $r(j) = s$.*

Let $\{T_i(\theta) : \ i = 0, 1, \ldots\}$ be bounded and let $M_0 > 0$ be such that

$$T_s(Z) \subset B(\theta, M_0), \tag{5.2}$$

$$\{T_i(\theta) : \quad i = 0, 1, \ldots\} \subset B(\theta, M_0), \tag{5.3}$$

$$\tilde{F} = F \cap B(\theta, M_0 + 2 + q(1 + M_0)). \tag{5.4}$$

Then for each $\epsilon > 0$, there exist a number $\delta \in (0, 1)$ and a natural number $\bar{n} > q$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ which satisfies

$$\rho(x_{i+1}, T_{r(i)} x_i) \leq \delta \quad \text{for all integers } i \geq 0,$$

the following inequality holds:

$$\rho(x_i, \tilde{F}) < \epsilon \quad \text{for all integers } i \geq \bar{n}.$$

Theorem 5.2. *Let F be a nonempty and closed subset of Z such that* (5.1) *holds. Let $\mathcal{R}$ be a nonempty set of mappings $r : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ which has properties* (P1) *and* (P3). *Assume that $s \geq 0$ is an integer and q is a natural number such that $T_s(Z)$ is bounded and* (P8) *holds. Let $M_0 > 0$ be such that* (5.2) *and* (5.3) *hold, and let $\tilde{F}$ be defined by* (5.4).

Let $\epsilon > 0$ be given and assume that

$$\{\delta_i\}_{i=0}^{\infty} \subset (0, \infty) \quad \text{and} \quad \lim_{i \to \infty} \delta_i = 0. \tag{5.5}$$

Then there exist a natural number n_0 such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfying $\rho(x_{+1}, T_{r(i)} x_i) \leq \delta_i$ for all integers $i \geq 0$, the following inequality holds: $\rho(x_i, \tilde{F}) < \epsilon$ for all integers $i \geq n_0$.

6. Proof of Theorem 5.1

Assume that $r \in \mathcal{R}$ and that $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfies

$$\rho(x_{i+1}, T_{r(i)} x_i) \leq 1 \quad \text{for all integers } i \geq 0. \tag{6.1}$$

Let $j > q$ be an integer. By the properties of q and s, and property (P8), there is an integer $p \geq 0$ such that

$$p < j, \quad j - p - 1 \leq q, \quad \text{and} \quad r(p) = s. \tag{6.2}$$

By (6.1), (6.2) and (5.2),

$$\rho(x_{p+1}, \theta) \leq \rho(x_{p+1}, T_{r(p)} x_p) + \rho(T_{r(p)}(x_p), \theta) \leq 1 + M_0. \tag{6.3}$$

We show by induction that for all integers $i \geq 0$,

$$\rho(x_{p+i+1}, \theta) \leq (M_0 + 1)(i + 1). \tag{6.4}$$

Clearly, (6.4) holds for $i = 0$. Assume that $i \geq 0$ is an integer and that (6.4) holds.

Then by (6.1), (2.1), (5.3) and (6.4),

$$\begin{aligned}
&\rho(x_{p+i+2}, \theta) \\
&\leq \rho(x_{p+i+2}, T_{r(p+i+1)}, x_{p+i+1}) + \rho(T_{r(p+i+1)} x_{p+i+1}, T_{r(p+i+1)} \theta) + \rho(T_{r(p+i+1)} \theta, \theta) \\
&\leq 1 + \rho(x_{p+i+1}, \theta) + M_0 \\
&\leq (M_0 + 1)(i + 2).
\end{aligned}$$

Thus (6.4) holds for all integers $i \geq 0$ and, in particular, by (6.2),

$$\rho(x_j, \theta) \leq M_0 + 1 + (j - p - 1)(M_0 + 1) \leq (M_0 + 1)(q + 1).$$

Thus we have shown that the following property holds:

(P9) For each $r \in \mathcal{R}$ and each $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfying (6.1),

$$\rho(x_j, \theta) \le (M_0 + 1)(q + 1) \quad \text{for all integers } j > q. \tag{6.5}$$

Let $\epsilon \in (0, 1)$ be given. By (P3), there exists a natural number n_0 such that the following property holds:
(P10) For each $r \in \mathcal{R}$ and each $x \in B(\theta, (q + 1)(M_0 + 1))$,

$$\rho(T_{r(n_0)} \cdots T_{r(1)} T_{r(0)} x, F) < \epsilon.$$

Put

$$\bar{n} = n_0 + q + 1. \tag{6.6}$$

Assume that $r \in \mathcal{R}$, $\{x_i\}_{i=0}^{\infty} \subset Z$ and

$$x_{i+1} = T_{r(i)} x_i \quad \text{for all integers } i \ge 0. \tag{6.7}$$

By (6.7) and (P9),

$$x_j \in B(\theta, (q + 1)(M_0 + 1)) \quad \text{for all integers } j > q. \tag{6.8}$$

For all integers $i \ge 0$, set

$$\tilde{r}(i) = r(i + q + 1), \quad \tilde{x}_i = x_{i+q+1}. \tag{6.9}$$

By (6.9), (P1) and (6.7), $\tilde{r} \in \mathcal{R}$ and for all integers $i \ge 0$,

$$\tilde{x}_{i+1} = x_{i+q+2} = T_{r(i+q+1)} x_{i+q+1} = T_{\tilde{r}(i)} \tilde{x}_i. \tag{6.10}$$

By (6.8) and (6.9), for all integers $i \ge 0$,

$$\tilde{x}_i \in B(\theta, (q + 1)(M_0 + 1)). \tag{6.11}$$

By (6.7), (6.9), (6.6), the inclusion $\tilde{r} \in \mathcal{R}$, (6.11) and (P10),

$$\begin{aligned} \rho(T_{r(\bar{n})} \cdots T_{r(1)} T_{r(0)} x_0, F) &= \rho(T_{r(\bar{n})} \cdots T_{r(q+1)} x_{q+1}, F) \\ &= \rho(T_{\tilde{r}(n_0)} \cdots T_{\tilde{r}(0)} \tilde{x}_0, F) \\ &< \epsilon. \end{aligned} \tag{6.12}$$

Thus we have shown that the following property holds:
For each $r \in \mathcal{R}$ and each $x \in Z$,

$$\rho(T_{r(\bar{n})} \cdots , T_{r(1)} T_{r(0)} x, F) < \epsilon.$$

Since ϵ is an arbitrary element of $(0, 1)$, we conclude that (P2) holds with $X = Z$ (see Theorem 1.1). Thus the assertion of Theorem 1.1 holds with $X = Z$.

Let $\epsilon \in (0, 1)$ be given. By Theorem 1.1, there exist a natural number n_ϵ and a number $\delta \in (0, \epsilon)$ such that the following property holds:

(P11) for each $r \in \mathcal{R}$ and each $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfying

$$\rho(x_{i+1}, T_{r(i)}x_i) \le \delta \quad \text{for all integers } i \ge 0, \tag{6.13}$$

we have

$$\rho(x_i, F) < \epsilon \quad \text{for all integers } i \ge n_\epsilon. \tag{6.14}$$

We may assume without loss of generality that $n_\epsilon > q$.

Assume that $r \in \mathcal{R}$ and $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfies (6.13). By (P11), (6.14) holds. By (6.13), the inequality $\delta < 1$ and (P9),

$$\rho(x_i, \theta) \le (M_0 + 1)(q + 1) \quad \text{for all integers } i > q. \tag{6.15}$$

It follows from (5.4), (6.14), (6.15) and the inequality $n_\epsilon > q$, $\epsilon \in (0, 1)$, that for all integers $i \ge n_\epsilon$,

$$\rho(x_i, \tilde{F}) < \epsilon.$$

Theorem 5.1 is proved.

7. Proof of Theorem 5.2

By Theorem 5.1, there are $\delta \in (0, 1)$ and a natural number $n_1 > q$ such that the following property holds:

(P12) for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfying

$$\rho(x_{i+1}, T_{r(i)}(x_i)) \le \delta \quad \text{for all integers } i \ge 0,$$

the inequality $\rho(x_i, \tilde{F}) < \epsilon$ holds for all integers $i \ge n_1$.

By (5.5), there is a natural number n_2 such that

$$\delta_i < \delta \quad \text{for all integers } i \ge n_2. \tag{7.1}$$

Put

$$n_0 = n_1 + n_2. \tag{7.2}$$

Assume that $r \in \mathcal{R}$, $\{x_i\}_{i=0}^{\infty} \subset Z$ and

$$\rho(x_{i+1}, T_{r(i)}x_i) \le \delta_i \quad \text{for all integers } i \ge 0. \tag{7.3}$$

For all integers $i \ge 0$, set

$$\tilde{x}_i = x_{i+n_2} \quad \text{and} \quad \tilde{r}(i) = r(i + n_2). \tag{7.4}$$

By (7.4) and (P1), $\tilde{r} \in \mathcal{R}$. By (7.4), (7.3) and (7.1), we have for all integers $i \ge 0$

$$\rho(\tilde{x}_{i+1}, T_{\tilde{r}(i)}\tilde{x}_i) = \rho(x_{i+1+n_2}, T_{r(i+n_2)}x_{i+n_2}) \le \delta_{i+n_2} \le \delta.$$

When combined with (P12) and (7.4), this implies that for all integers $i \ge n_1$,

$$\rho(x_{i+n_2}, \tilde{F}) = \rho(\tilde{x}_i, \tilde{F}) < \epsilon. \tag{7.5}$$

By (7.5) and (7.2), for all integers $i \ge n_1 + n_2 = n_0$, we have $\rho(x_i, \tilde{F}) < \epsilon$. This completes the proof of Theorem 5.2.

8. Applications to the convex feasibility problem

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $|| \cdot ||$.

For each $x \in X$ and each nonempty set $A \subset X$, put

$$\rho(x, A) = \inf\{||x - y|| : \ y \in A\}.$$

It is well known that the following proposition holds.

Proposition 8.1. *Let D be a nonempty and closed convex subset of X. Then for each $x \in X$, there is a unique point $P_D(x) \in D$ satisfying*

$$||x - P_D(x)|| = \inf\{||x - y|| : \ y \in D\}.$$

Moreover,

$$||P_D(x) - P_D(y)|| \le ||x - y|| \quad \textit{for all } x, y \in X$$

and for each $x \in X$ and each $z \in D$,

$$||z - P_D(x)||^2 + ||x - P_D(x)||^2 \le ||z - x||^2.$$

Let m be a natural number and suppose that $C_1, \dots, C_m$ are nonempty, closed and convex subsets of X. Set

$$C = \bigcap_{i=1}^{m} C_i. \tag{8.1}$$

We assume that $C \neq \emptyset$. We are also going to use the following assumption.

(A) For each $\epsilon > 0$ and each $M > 0$, there exists a number $\delta = \delta(\epsilon, M) > 0$ such that for each $x \in B(0, M)$ satisfying $\rho(x, C_i) \le \delta$, $i = 1, \dots, m$, the inequality $\rho(x, C) \le \epsilon$ holds.

It is well known that the following proposition holds.

Proposition 8.2. *If the space X is finite-dimensional, then assumption* (A) *holds.*

For each integer $p \ge 0$ and each $i \in \{0, \dots, m-1\}$, set

$$T_{pm+i} = P_{C_{i+1}}. \tag{8.2}$$

Let $l \ge m$ be a natural number. Denote by $\mathcal{R}$ the set of all mappings $r : \{0, 1, \dots\} \to \{1, \dots, m\}$ such that for each integer $p \ge 0$ and each $s \in \{1, \dots .m\}$, there is

$$i \in \{p, \dots, p + l - 1\} \quad \text{such that} \quad T_{r(i)} = P_{C_s}. \tag{8.3}$$

It is easy to see that property (P1) holds. Let $M_0 > 0$ be such that

$$B(0, M_0) \cap C \neq \emptyset. \tag{8.4}$$

Theorem 8.3. *Let $\epsilon > 0$, $M > 0$ and $\delta \in (0,1)$ be such that*

$$\text{if } x \in B(0, 2M_0 + M) \text{ and } \rho(x, C_i) \le \delta,\ i = 1, \dots, m, \quad \text{then } \rho(x, C) \le \frac{\epsilon}{4}, \tag{8.5}$$

and suppose that the natural number k_0 satisfies

$$k_0 > \left(\delta^{-1} l (M_0 + M)\right)^2. \tag{8.6}$$

Assume that $r \in \mathcal{R}$ and that $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$||x_0|| \le M, \quad x_{i+1} = T_{r(i)}(x_i), \quad i = 0, 1, \dots. \tag{8.7}$$

Then the sequence $\{x_i\}_{i=0}^{\infty}$ converges in the norm topology of X, $\lim_{i\to\infty} x_i \in C$ and

$$||x_j - \lim_{i\to\infty} x_i|| \le \epsilon \quad \text{for all integers } j \ge k_0 l.$$

Proof of Theorem 8.3. Fix

$$\theta \in B(0, M_0) \cap C \tag{8.8}$$

(see (8.4)). By (8.7), (8.8), (8.2) and Proposition 8.1, for all integer $i \ge 0$,

$$||x_{i+1} - \theta|| = ||T_{r(i)} x_i - T_{r(i)} \theta|| \le ||x_i - \theta|| \le ||x_0 - \theta|| \le M_0 + M. \tag{8.9}$$

By (8.9) (8.8), (8.7), (8.2) and Proposition 8.1,

$$\begin{aligned}
(M_0 + M)^2 &\ge ||x_0 - \theta||^2 \\
&\ge ||x_0 - \theta||^2 - ||x_{k_0 l} - \theta||^2 \\
&= \sum_{i=0}^{k_0 l - 1} [||x_i - \theta||^2 - ||x_{i+1} - \theta||^2] \\
&\ge \sum_{i=0}^{k_0 l - 1} ||x_i - x_{i+1}||^2 \\
&= \sum_{j=0}^{k_0 - 1} \sum_{i=jl}^{(j+1)l-1} ||x_i - x_{i+1}||^2.
\end{aligned}$$

This implies that there is an integer $j \in \{0, \dots, k_0 - 1\}$ such that

$$\sum_{i=jl}^{(j+1)l-1} ||x_i - x_{i+1}||^2 \le (M_0 + M)^2 k_0^{-1}.$$

This inequality implies in its turn that for all $i = jl, \dots, (j+1)l - 1$,

$$||x_i - x_{i+1}||^2 \le (M_0 + M)^2 k_0^{-1}$$

and for each $i = jl, \dots, (j+1)l - 1$,

$$||x_i - x_{i+1}|| \le (M_0 + M)k_0^{-\frac{1}{2}}.$$

Therefore we have for each $i = jl + 1, \dots, (j+1)l$,

$$||x_i - x_{jl}|| \le (M_0 + M)lk_0^{-\frac{1}{2}}.$$

When combined with (8.7), (8.2), (8.3) and (8.6), this inequality implies that for each $s \in \{1, \dots, m\}$,

$$\rho(x_{jl}, C_s) \le l(M_0 + M)k_0^{-\frac{1}{2}} < \delta. \tag{8.10}$$

By (8.9) and (8.8),

$$||x_{jl}|| \le ||x_{jl} - \theta|| + ||\theta|| \le 2M_0 + M. \tag{8.11}$$

By (8.10), (8.11) and (8.5), $\rho(x_{jl}, C) \le \frac{\epsilon}{4}$ and there is

$$y \in C \quad \text{such that} \quad ||x_{jl} - y|| < \frac{\epsilon}{2}. \tag{8.12}$$

By (8.12), (8.7), (8.2), Proposition 8.1 and the inequality $j > k_0$,

$$||x_i - y|| < \frac{\epsilon}{2} \quad \text{for all integers } i \ge k_0 l \ge jl. \tag{8.13}$$

Since ϵ is any positive number, we conclude that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence, there exists $\lim_{i\to\infty} x_i$ in the norm topology and

$$\lim_{i\to\infty} ||x_i - y|| \le \frac{\epsilon}{2}. \tag{8.14}$$

Since ϵ is any positive number, we have by (8.12), $\lim_{i\to\infty} x_i \in C$. By (8.14) and (8.13), $||x_i - \lim_{j\to\infty} x_j|| < \epsilon$ for all integers $i \ge k_0 l$. Theorem 8.3 is proved. □

Theorems 8.3 and 2.1 now imply our next result.

Theorem 8.4. *Assume that the set C is bounded. Then for each $M, \epsilon > 0$, there exist a number $\delta > 0$ and a natural number $\bar{n}$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies $||x_0|| \le M$ and $||x_{i+1} - T_{r(i)}(x_i)|| \le \delta$ for all integers $i \ge 0$, the following inequality holds:*

$$\rho(x_i, C) < \epsilon \quad \textit{for all integers } i \ge \bar{n}.$$

Next, we note the following consequence of Theorems 8.3 and 2.2.

Theorem 8.5. *Assume that the set C is bounded and let $M > 0$ be given. Then there exists a number $\bar{\delta} > 0$ such that for each $\epsilon > 0$ and each sequence $\{\delta_i\}_{i=0}^{\infty} \subset (0, \bar{\delta}]$ satisfying $\lim_{i\to\infty} \delta_i = 0$, there is a natural number n_0 such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies $||x_0|| \le M$ and $||x_{i+1} - T_{r(i)}(x_i)| \le \delta_i$ for all integers $i \ge 0$, the following inequality holds:*

$$\rho(x_i, C) < \epsilon \quad \textit{for all integers } i \ge \bar{n}_0.$$

Combining Theorems 8.3 and 5.1, we arrive at our next result.

Theorem 8.6. *Assume that there is a natural number $s \in \{1, \dots, m\}$ such that the set C_s is bounded. Then for each $\epsilon > 0$, there exist a number $\delta \in (0, 1)$ and a natural number $\bar{n}$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies*

$$||x_{i+1} - T_{r(i)}(x_i)|| \le \delta$$

for all integers $i \ge 0$, the following inequality holds:

$$\rho(x_i, C) < \epsilon \quad \textit{for all integers } i \ge \bar{n}.$$

Finally, Theorems 8.3 and 5.2 yield our last result.

Theorem 8.7. *Assume that there is a natural number $s \in \{1, \dots, m\}$ such that the set C_s is bounded. Let $\epsilon > 0$ be given and let a sequence $\{\delta_i\}_{i=0}^{\infty} \subset (0, \infty)$ satisfy $\lim_{i\to\infty} \delta_i = 0$. Then there is a natural number n_0 such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies $||x_{i+1} - T_{r(i)}(x_i)|| \le \delta_i$ for all integers $i \ge 0$, the following inequality holds:*

$$\rho(x_i, C) < \epsilon \quad \textit{for all integers } i \ge n_0.$$

Acknowledgement. This research was supported by the Israel Science Foundation (Grant No. 647/07), the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

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Received October 17, 2012; revised June 21, 2013