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A Note on the Composition of Regular Functions

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Abstract. In this note we give a condition on a function $h : \mathbb{R} \to \mathbb{R}$, both necessary and sufficient, under which the nonlinear composition operator $Hf = h \circ f$ maps the space of all regular functions $f : [a, b] \to \mathbb{R}$ into itself. Moreover, we show that in this case the operator H is automatically bounded in the supremum norm.

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1. Introduction

To the best of our knowledge, regular functions (also called regulated or regularized functions) were introduced in 1954 by Aumann [2]. In Aumann's terminology, a bounded function $f : [a, b] \to \mathbb{R}$ is called regular if it has only removable discontinuities or discontinuities of first kind (jumps). A prominent example is of course monotone functions or, slightly more general, functions of bounded variation.

Regular functions play an important role, for instance, in applications to differential equations with singular right-hand sides or with distributional coefficients [7], or to the Skorokhod problem in stochastics [4].

Composition operators appear in a natural way in virtually every field of nonlinear analysis and its applications. Recall that, given a function $h : \mathbb{R} \to \mathbb{R}$ and a space $X = X[a, b]$ of functions $f : [a, b] \to \mathbb{R}$, the (autononomous) composition operator H generated by h is defined by

$$Hf(t) = (h \circ f)(t) = h(f(t)) \quad (a \le t \le b). \tag{1}$$

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More generally, given a function $h\colon [a,b]\times\mathbb{R}\to\mathbb{R}$, the operator H defined by

$$Hf(t) = h(t, f(t)) \quad (a \le t \le b) \tag{2}$$

is called the (nonautononomous) composition operator generated by h. We point out that the operator H is much more complicated in the nonautonomous case (2) than in the autonomous case (1), and many unexpected features and open problems occur in its study. A detailed account of results and examples may be found in the monograph [1], for a more recent treatment with a special emphasis on spaces of functions of bounded variation (classical or generalized) we refer to Chapter 2 of [10].

In what follows, we will work in the (Banach) space $R[a,b]$ of all regular functions $f : [a,b] \to \mathbb{R}$, equipped with the natural norm

$$||f||_\infty = \sup\{|f(t)| : a \le t \le b\}. \tag{3}$$

As far as we know, the operators (1) and (2) have not been studied yet in this space, and to fill this gap is the purpose of the present note. A basic problem here is to find conditions on the function h, possibly both necessary and sufficient, under which the corresponding operator H maps a function space $X = X[a,b]$ into itself. In the paper [3] the authors analyze the set

$$COP(X) = \{h : H(X) \subseteq X\} \tag{4}$$

for various choices of X and call the description of this set the composition operator problem for the space X. The solution to this problem is in some cases completely trivial, in some cases surprisingly difficult, in some cases simply unknown. For example, a famous result due to Josephy [6] states that

$$COP(BV) = Lip_{loc}(\mathbb{R})$$

in the autonomous case (1), which means that the function $h \circ f$ has bounded variation for all functions f of bounded variation if and only if the outer function h is Lipschitz continuous on each compact interval in $\mathbb{R}$. (The sufficiency of this condition for $H(BV[a,b]) \subseteq BV[a,b]$ is of course trivial, it is the necessity proof which requires the construction of a rather sophisticated counterexample).

Josephy's result apparently led Viloria in the paper [12] to conjecture that also

$$COP(R) = Lip_{loc}(\mathbb{R})$$

in both the autonomous case (1) and the nonautonomous case (2). As a matter of fact, Viloria assumes in advance that $h(t,\cdot) \in Lip_{loc}(\mathbb{R})$ for $a \le t \le b$, and then gives an explicit description of the set $COP(R)$ in the nonautonomous case. However, this a priori assumption is far too strong, as we will see in a moment.

Let us start with the autonomous composition operator (1) generated by some function $h : \mathbb{R} \to \mathbb{R}$. Here the following interesting decomposition theorem for regular functions due to Sierpiński [11] which seems to be rather unknown, may be useful.

Theorem 1.1. *A function f belongs to $R[a, b]$ if and only if it may be represented in the form $f = g \circ \tau$, where $\tau : [a, b] \to [c, d]$ is strictly monotone and $g : [c, d] \to \mathbb{R}$ is continuous.*

Here $[c, d] = [\tau(a), \tau(b)]$ if τ is increasing, and $[c, d] = [\tau(b), \tau(a)]$ if τ is decreasing. An essential part of the proof of Theorem 1.1 consists in defining g on $E := \tau([a, b])$ by means of the obvious formula $g(\tau(t)) = f(t)$, extending g to the closure $\bar{E}$ of E by a limiting process, and then using the classical Tietze extension lemma for continuous functions on closed sets (see, e.g., [5]).

We point out that Theorem 1 has an interesting counterpart for functions of bounded variations which builds on the McShane extension lemma for Lipschitz continuous functions [9] and reads as follows: A function f belongs to $BV[a, b]$ if and only if it may be represented in the form $f = g \circ \tau$, where $\tau : [a, b] \to [c, d]$ is monotone and $g \in Lip[c, d]$, i.e. $g : [c, d] \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L \leq 1$.

From Theorem 1.1 it follows immediately that the condition $h \in C(\mathbb{R})$ is sufficient to guarantee that $H(R[a, b]) \subseteq R[a, b]$ for H as in (1). Of course, a necessary condition on $h : \mathbb{R} \to \mathbb{R}$ under which $H(R[a; b]) \subseteq R[a, b]$ is $h \in R(\mathbb{R})$, as may be seen by considering affine functions f between compact intervals in $\mathbb{R}$. In the notation (4) we may state this as inclusion

$$C(\mathbb{R}) \subseteq COP(R) \subseteq R(\mathbb{R}). \tag{5}$$

2. Mean result

The following example shows that regularity of h on $\mathbb{R}$ is not sufficient for $H(R[a, b]) \subseteq R[a, b]$, and so the second inclusion in (5) is strict.

Example 2.1. Let $A \subset [0, 1]$ be a Cantor set of positive measure, and let $f : [0, 1] \to \mathbb{R}$ be defined by $f(t) := \text{dist}(t, A)$. Clearly, f is continuous, and therefore regular. Moreover, the function $h(u) := \chi_{\{0\}}$ trivially belongs to $R(\mathbb{R})$. However, the function $h \circ f = \chi_A$ is not regular, because it is discontinuous on the uncountable set $\partial A = A$.

This example shows even more than we claimed: continuity of h on the whole real axis, except for one point, does not even imply the very weak condition $H(Lip[a; b]) \subseteq R[a, b]$! Interestingly, one may adapt Example 2.1 to prove that the first inclusion in (5) is actually an equality; this is our main result.

Theorem 2.2. *The composition operator* (1) *maps* $R[a,b]$ *into itself if and only if* $h \in C(\mathbb{R})$. *In this case the operator* (1) *is automatically bounded in the norm* (3).

Proof. The "if" part follows from Theorem 1.1. To prove the "only if" part, observe that $H(R[a,b]) \subseteq R[a,b]$ implies that h must be regular on $\mathbb{R}$. So assume that h is discontinuous at some point $u_0 \in \mathbb{R}$, where without loss of generality

$$h(u_0-) = \lim_{u \to u_0-} h(u) < \lim_{u \to u_0+} h(u) = h(u_0+).$$

Choose $\epsilon > 0$ and $\delta > 0$ such that $h(u) > h(u_0-) + \epsilon$ for $u_0 < u \leq u_0 + \delta$, and define $f : [a,b] \to \mathbb{R}$ by

$$f(t) := h(u_0) + \min\{\mathrm{dist}(t, A), \delta\},$$

where A is as in Example 2.1. Then $Hf = h \circ f$ is discontinuous at each point $t \in A$, and so cannot belong to $R[a,b]$, a contradiction.

It remains to prove that the operator (1) is bounded under the hypotheses of Theorem 2.2. But this is an immediate consequence of the fact that a continuous function $h \colon \mathbb{R} \to \mathbb{R}$ is bounded on the compact interval $[-r, r]$ for each $r > 0$. □

Now let us briefly discuss the case of the nonautonomous operator (2). In [8] it is shown that the operator (2) maps $BV[a,b]$ into itself (and is automatically bounded in the familiar BV-norm) if the function $h(\cdot, u)$ has bounded variation on $[a,b]$, uniformly with respect to $u \in R$, and the function $h(t, \cdot)$ is locally Lipschitz on $\mathbb{R}$, uniformly with respect to $t \in [a,b]$. A parallel sufficient condition may be given for the space $R[a,b]$.

Theorem 2.3. *Suppose that the function* $h(\cdot, u)$ *is regular on* $[a,b]$ *for all* $u \in \mathbb{R}$, *and the function* $h(t, \cdot)$ *is continuous on* $\mathbb{R}$, *uniformly with respect to* $t \in [a,b]$. *Then the composition operator* (2) *maps* $R[a,b]$ *into itself and is bounded in the norm* (3).

The proof of Theorem 2.3 is straightforward, therefore we omit it. Instead, we present another example which shows that the hypotheses given in Theorem 2.3 are sufficient, but not necessary.

Example 2.4. Let $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by $h(0,0) = 1$ and $h(t,u) = 0$ elsewhere on $[0,1] \times \mathbb{R}$. Clearly, the function $h(\cdot, u)$ is regular on $[0,1]$ for each $u \in \mathbb{R}$, but the function $h(0, \cdot) = \chi_{\{0\}}$ is discontinuous at zero. On the other hand, for any $f \in R[0,1]$ the function $g(t) = Hf(t)$ satisfies $g(t) \equiv 0$ on $(0,1]$, and so the right hand limit at zero satisfies

$$\lim_{t \to 0+} g(t) = 0.$$

Consequently, the operator (2) generated by h maps the space $R[0,1]$ into itself and is bounded in the norm (3).

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