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Atomic Decomposition for Morrey Spaces

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Abstract. The Hardy space $H^p(\mathbb{R}^n)$ substitutes for the Lebesgue space $L^p(\mathbb{R}^n)$. When $p > 1$, then the Hardy space $H^p(\mathbb{R}^n)$ coincides with the Lebesgue spaces $L^p(\mathbb{R}^n)$. This is shown by using the reflexivity of the function spaces. The atomic decomposition is readily available for $H^p(\mathbb{R}^n)$ with $0 < p < \infty$. This idea can be applied to many function spaces. As example of such an attempt, we now propose here a non-smooth decomposition of Morrey spaces. As applications, we consider the Olsen inequality. In the end of this article, we compare our results with existing ones and propose some possibility of extensions, which are left as future works.

Keywords. Morrey spaces, fractional integral operators, atoms

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1. Introduction

Morrey spaces are tools for PDE. For $0 < q \le p < \infty$, recall that Morrey spaces are defined by the norm given by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, dy\right)^{\frac{1}{q}}$$

for measurable functions $f : \mathbb{R}^n \to \mathbb{C}$, where $\mathcal{D}(\mathbb{R}^n)$ denotes the set of all dyadic cubes. We denote by $\mathcal{Q}(\mathbb{R}^n)$ the set of all cubes whose edges are parallel to the coordinate axes.

We aim here to prove the following decomposition result about the functions in Morrey spaces.

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Theorem 1.1. *Suppose that the parameters p, q, s, t satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad q < t, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathcal{M}_t^s(\mathbb{R}^n)$ *and* $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ *fulfill*

$$\|a_j\|_{\mathcal{M}_t^s} \le |Q_j|^{\frac{1}{s}}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p} < \infty.$$

Then $f \equiv \sum_{j=1}^\infty \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{M}_q^p} \le C_{p,q,s,t} \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p}. \tag{1}$$

The next assertion concerns the decomposition of functions in $\mathcal{M}_q^p(\mathbb{R}^n)$.

Theorem 1.2. *Suppose that the real parameters p, q, L satisfy*

$$1 < q \le p < \infty, \quad L \in \mathbb{N} \cup \{0\}.$$

Let $f \in \mathcal{M}_q^p(\mathbb{R}^n)$. *Then there exists a triplet* $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ *and* $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ *such that* $f = \sum_{j=1}^\infty \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and that, for all* $v > 0$,

$$|a_j| \le \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^\alpha a_j(x)\,dx = 0, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_q^p} \le C_v \|f\|_{\mathcal{M}_q^p}$$

for all multi-indices α with $|\alpha| \le L$. Here the constant $C_v > 0$ is independent of f.

Theorem 1.2 is a special case of Theorem 1.3 to follow, which concerns the decomposition of Hardy-Morrey spaces. Recall that, for $0 < q \le p < \infty$, the Hardy-Morrey space $H\mathcal{M}_q^p(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{H\mathcal{M}_q^p} = \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{\mathcal{M}_q^p}$ is finite, where $e^{t\Delta} f$ stands for the heat extension of f for $t > 0$;

$$e^{t\Delta} f(x) = \left\langle \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x - \cdot|^2}{4t} \right), f \right\rangle \quad (x \in \mathbb{R}^n).$$

Theorem 1.3. *Let $0 < q \le p < \infty$ and $L \ge 0$ be a fixed integer. Let* $f \in H\mathcal{M}_q^p(\mathbb{R}^n)$. *Then there exist* $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ *and* $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ *such that* $f = \sum_{j=1}^\infty \lambda_j a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and that, for all* $v > 0$,

$$|a_j| \le \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^\alpha a_j(x)\,dx = 0, \quad \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_q^p} \le C \|f\|_{H\mathcal{M}_q^p} \tag{2}$$

for all α with $|\alpha| \le L$. Here the constant C_v does not depend upon f.

Theorem 1.1 has the following counterpart.

Theorem 1.4. *Let* $d_q = \max([n(\frac{1}{q}-1)], 0)$. *Suppose that the parameters* p, q, s, t *satisfy*

$$0 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad q < t, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}_t^s(\mathbb{R}^n)$ *and* $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ *fulfill*

$$\|a_j\|_{\mathcal{M}_t^s} \le |Q_j|^{\frac{1}{s}}$$

and

$$\operatorname{supp}(a_j) \subset Q_j, \quad \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{\min(1,q)} \right)^{\frac{1}{\min(1,q)}} \right\|_{\mathcal{M}_q^p} < \infty, \quad \int_{\mathbb{R}^n} x^{\alpha} a_j(x)\, dx = 0$$

for all α *with* $|\alpha| \le d_q$. *Then* $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{H\mathcal{M}_q^p} \le C_{p,q,s,t} \left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^{\min(1,q)} \right)^{\frac{1}{\min(1,q)}} \right\|_{\mathcal{M}_q^p}.$$

Remark that in [13] Jia and Wang considered the case when $q \le 1$. Theorem 1.1 seems new and even in Theorem 1.2–1.4 we do not have to postulate $q \le 1$. About the relation between $\mathcal{M}_q^p(\mathbb{R}^n)$ and $H\mathcal{M}_q^p(\mathbb{R}^n)$ when $q > 1$, we have the following assertion:

Proposition 1.5. *Let* $1 < q \le p < \infty$.

1. *If* $f \in \mathcal{M}_q^p(\mathbb{R}^n)$, *then* $f \in H\mathcal{M}_q^p(\mathbb{R}^n)$.
2. *If* $f \in H\mathcal{M}_q^p(\mathbb{R}^n)$, *then* f *is represented by a locally integrable function and the representative belongs to* $\mathcal{M}_q^p(\mathbb{R}^n)$.

Proposition 1.5 was investigated by Zorko [42]; see [14] as well. We refer to [1, 9] for more recent characterizations. We supply a detailed proof of Proposition 1.5 in Section 2.

As an application of Theorem 1.1, we can reprove the following Olsen inequality about the fractional integral operator I_α, where $I_\alpha\,(0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

The following result is known:

Proposition 1.6. *Assume that the parameters* p, q, s, t *and* α *satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad 0 < \alpha < n, \quad \frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then I_α *is bounded from* $\mathcal{M}_q^p(\mathbb{R}^n)$ *to* $\mathcal{M}_t^s(\mathbb{R}^n)$.

Based upon Proposition 1.6, we can prove the following result.

Theorem 1.7. *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $1 < q \le q_0 < \infty$ and $1 < r \le r_0 < \infty$. Suppose that*

$$q > r, \quad \frac{1}{p_0} > \frac{\alpha}{n}, \quad \frac{1}{q_0} \le \frac{\alpha}{n},$$

and that

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{r}{r_0} = \frac{p}{p_0}.$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\|g\|_{\mathcal{M}^{q_0}_q} \cdot \|f\|_{\mathcal{M}^{p_0}_p},$$

where the constant C is independent of f and g.

This result recaptures [30, Proposition 1.8]. Note that a detailed calculation shows that Theorem 1.7 is not a mere combination of Proposition 1.6 and Lemma 1.8.

Lemma 1.8. *Let $1 \le q_1 \le p_1 < \infty$ and $1 \le q_2 \le p_2 < \infty$. Define*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|f \cdot g\|_{\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^{p_1}_{q_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2}}.$$

We write $\infty' = 1$ and $s' = \frac{s}{s-1}$ for $1 < s < \infty$. We have the following corollary:

Proposition 1.9. *Let $1 < s \le \infty$, $0 < \alpha < n$, $1 \le s' < p \le p_0 < \infty$, $q > r$, $\frac{1}{p_0} > \frac{\alpha}{n}$, $\frac{1}{q_0} \le \frac{\alpha}{n}$, $\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{r}{r_0} = \frac{p}{p_0}$. Let $\Omega \in L^s(\mathbb{S}^{n-1})$ satisfy, for any $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. Then,*

$$\|g \cdot I_{\Omega,\alpha}(f)\|_{\mathcal{M}^{r_0}_r} \le C \, \|g\|_{\mathcal{M}^{r_0}_r} \, \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \, \|f\|_{\mathcal{M}^{p_0}_p} \, ,$$

where

$$I_{\Omega,\alpha} f(x) \equiv \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy.$$

The definition of $I_{\Omega,\alpha}$ dates back to [5]. Proposition 1.9 is a direct consequence of Theorem 1.7, the next lemma and the boundedness of the Hardy-Littlewood maximal operator M.

Lemma 1.10 ([12, Lemma 1])**.** *If $1 < s \le \infty$, then we have*

$$|I_{\Omega,\alpha} f(x)| \le C \, \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \, |I_\alpha F(x)|,$$

where $F(x) \equiv M\left(|f|^{s'}\right)(x)^{\frac{1}{s'}}$.

In [27], decompositions of Morrey spaces are considered. The second author applied them to pseudo-differential operators in [24]. As we did in [19], by using Theorem 1.1 we can prove the boundedness of pseudo-differential operators, which we omit the detail.

Hardy-Morrey spaces have a characterization by using the grand maximal operator. To formulate the result, we recall the following two fundamental notions.

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.

2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| \,:\, t>0,\, \psi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n), \tag{3}$$

where we choose and fix a large integer N.

The following proposition can be proved.

Proposition 1.11. *Let* $0 < q \le p < \infty$. *Then*

$$\|\mathcal{M}f\|_{\mathcal{M}^p_q} \sim \|f\|_{H\mathcal{M}^p_q} \quad \textit{for all } f \in \mathcal{S}'(\mathbb{R}^n).$$

When $p \le 1$, this proposition is contained in [13]. Here for the sake of convenience, we outline the proof of Proposition 1.11 in Section 2. We also remark that Theorems 1.3 with $p = q \le 1$ and 1.4 with $p = q \le 1$ are included in [11, Theorems 2.1 and 2.2].

We plan to prove Theorems 1.1–1.4 in the following manner. First of all, we concentrate on Theorem 1.1 in Subsection 3.1. Next, by mimicking the proof of Theorem 1.1, we prove Theorem 1.4. Finally, we prove Theorem 1.3, which includes Theorem 1.2. Note that Theorem 1.2 is included in Theorem 1.3 as we prove just in the beginning of Subsection 3.3. Necessary lemmas for the proofs are stated in each subsection. Section 4 is devoted to the proof of Theorem 1.7.

2. Proofs of Propositions 1.5 and 1.11

2.1. Proof of Proposition 1.5. To prove Proposition 1.5, we need duality. Recall that when $1 < q \le p < \infty$, then the predual space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ of the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is given by

$$\mathcal{H}^{p'}_{q'}(\mathbb{R}^n) = \left\{ g = \sum_{j=1}^\infty \mu_j b_j \,:\, \{\mu_j\}_{j=1}^\infty \in \ell^1(\mathbb{N}),\ \text{each } b_j \text{ is a } (p',q')\text{-block} \right\}.$$

Here "by a (p', q')-block" we mean an $L^{q'}(\mathbb{R}^n)$-function supported on a cube Q with $L^{q'}(\mathbb{R}^n)$-norm less than $|Q|^{\frac{1}{q'}-\frac{1}{p'}}$. The norm of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is defined by

$$\|g\|_{\mathcal{H}^{p'}_{q'}} = \inf \sum_{j=1}^{\infty} |\mu_j|,$$

where inf is over all admissible expressions above. A fundamental fact about this space is that

$$\|f\|_{\mathcal{M}^p_q} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)|\, dx \,:\, \|g\|_{\mathcal{H}^{p'}_{q'}} = 1 \right\}.$$

With this in mind, we prove Proposition 1.5.

1. Denote by $B(R) = \{x \in \mathbb{R}^n \,:\, |x| < R\}$ for $R > 0$. Since

$$\|f\|_{L^1(B(R))} \le C R^{-\frac{n}{p+n}} \|f\|_{\mathcal{M}^p_q},$$

   we have $f \in \mathcal{S}'(\mathbb{R}^n)$. As is described in [6, Section 2], we have a pointwise estimate $|e^{t\Delta} f| \le Mf$, where M denotes the Hardy-Littlewood maximal operator. Since M is shown to be bounded in [3], we have $f \in H\mathcal{M}^p_q(\mathbb{R}^n)$.
2. Let $f \in H\mathcal{M}^p_q(\mathbb{R}^n)$. Then $\{e^{t\Delta} f\}_{t>0}$ is a bounded set of $\mathcal{M}^p_q(\mathbb{R}^n)$, which admits a predual as we have seen. Therefore, there exists a sequence $\{t_j\}_{j=1}^{\infty}$ decreasing to 0 such that $\{e^{t_j\Delta} f\}_{j=1}^{\infty}$ converges to a function g in the weak-* topology of $\mathcal{M}^p_q(\mathbb{R}^n)$. Meanwhile, it can be shown that $\lim_{t\downarrow 0} e^{t\Delta} f = f$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$. Since the weak-* topology of $\mathcal{M}^p_q(\mathbb{R}^n)$, is stronger than the topology of $\mathcal{S}'(\mathbb{R}^n)$, it follows that $f = g \in \mathcal{M}^p_q(\mathbb{R}^n)$.

2.2. Proof of Proposition 1.11. The proof is similar to Hardy spaces with variable exponents [19]. We content ourselves with stating two fundamental estimates (4) and (5).

We define the (discrete) maximal function with respect to φ by

$$M_{\mathrm{heat}} f(x) \equiv \sup_{j \in \mathbb{Z}} |e^{2^j \Delta} f(x)| \quad (x \in \mathbb{R}^n).$$

Recall that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function is defined by

$$\mathcal{M} f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| \,:\, t > 0,\, \psi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n),$$

where $\mathcal{F}_N$ is given by

$$\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}.$$

Suppose that we are given an integer $L \gg 1$. We write

$$M^*_{\rm heat}f(x) \equiv \sup_{j\in\mathbb{Z}} \left(\sup_{y\in\mathbb{R}^n} \frac{|e^{2^j\Delta}f(y)|}{(1+4^j|x-y|^2)^L} \right) \quad (x \in \mathbb{R}^n).$$

The next lemma connects $M^*_{\rm heat}$ with $M_{\rm heat}$ in terms of the usual Hardy-Littlewood maximal function M.

Lemma 2.1 ([19, Lemma 3.2], [23, §4]). *For $0<\theta<1$, there exists L_θ so that for all $L \geq L_\theta$, we have*

$$M^*_{\rm heat}f(x) \leq CM\left[\sup_{k\in\mathbb{Z}}|e^{2^k\Delta}f|^\theta\right](x)^{\frac{1}{\theta}} = M^{(\theta)}[M_{\rm heat}f](x) \quad (x\in\mathbb{R}^n) \tag{4}$$

all $f \in \mathcal{S}'(\mathbb{R}^n)$, where $M^{(\theta)}$ is the powered maximal operator given by

$$M^{(\theta)}g(x) \equiv M[|g|^\theta](x)^{\frac{1}{\theta}} \quad (x\in\mathbb{R}^n)$$

for measurable functions g.

In the course of the proof of [19, Theorem 3.3], we have shown

$$\mathcal{M}f(x) \sim \sup_{\tau\in\mathcal{F}_N,\, j\in\mathbb{Z}} |\tau^j * f(x)| \lesssim M^*_{\rm heat}f(x) \tag{5}$$

once we fix an integer $L \gg 1$ and $N \gg 1$.

With the fundamental pointwise estimates (4) and (5), Proposition 1.11 can be proven.

3. Proofs of Theorems 1.1 – 1.4

3.1. Proof of Theorem 1.1. By decompositing Q_j suitably, we may assume that each Q_j is dyadic. To prove this, we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \geq N$. Let us assume in addition that a_j are non-negative. Fix a positive (p',q')-block $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ with the associated cube Q.

Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that $Q_j = 2^jQ$ for each $j \in \mathbb{N}$. Then we have

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \sum_{j=1}^\infty \lambda_j \int_Q a_j(x)g(x)\,dx \leq \sum_{j=1}^\infty \lambda_j \|a_j\|_{L^q(Q)}\|g\|_{L^{q'}(Q)}$$

from $f=\sum_{j=1}^\infty \lambda_j a_j$. By the size condition of a_j and g, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx \le \sum_{j=1}^\infty \lambda_j |Q|^{\frac{1}{q}-\frac{1}{s}}|Q_j|^{\frac{1}{s}}|Q|^{\frac{1}{q'}-\frac{1}{p'}} \le \sum_{j=1}^\infty \lambda_j |Q|^{\frac{1}{p}-\frac{1}{s}}|Q_j|^{\frac{1}{s}}.$$

Note that $\left\|\sum_{j=1}^\infty \lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_q} \ge \|\lambda_{j_0}\chi_{Q_{j_0}}\|_{\mathcal{M}^p_q} = |Q_{j_0}|^{\frac{1}{p}}\lambda_{j_0}$ for each j_0. Consequently, it follows from the condition $p<s$ that

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx \le \sum_{j=1}^\infty |Q|^{\frac{1}{p}-\frac{1}{s}}|Q_j|^{\frac{1}{s}-\frac{1}{p}}\left\|\sum_{j=1}^\infty \lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_q} \le C\left\|\sum_{j=1}^\infty \lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_q}.$$

Conversely assume that Q contains each Q_j. Then we have

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \sum_{j=1}^\infty \lambda_j\int_{Q_j} a_j(x)g(x)\,dx \le \sum_{j=1}^\infty \lambda_j\|a_j\|_{L^t(Q_j)}\|g\|_{L^{t'}(Q_j)}.$$

By the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \sum_{j=1}^\infty \lambda_j\int_{Q_j} a_j(x)g(x)\,dx \le \sum_{j=1}^\infty \lambda_j|Q_j|^{\frac{1}{t}-\frac{1}{s}}|Q_j|^{\frac{1}{s}}\|g\|_{L^{t'}(Q_j)}.$$

Thus, in terms of the Hardy-Littlewood maximal operator M, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,dx &\le \sum_{j=1}^\infty \lambda_j|Q_j| \times \inf_{y\in Q_j} M[|g|^{t'}](y)^{\frac{1}{t'}}\\ &\le \int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty \lambda_j\chi_{Q_j}(y)\right) M[|g|^{t'}](y)^{\frac{1}{t'}}\,dy\\ &\le \int_{\mathbb{R}^n}\left(\sum_{j=1}^\infty \lambda_j\chi_{Q_j}(y)\right)\chi_Q(y) M[|g|^{t'}](y)^{\frac{1}{t'}}\,dy.\end{aligned}$$

If we let κ be the operator norm of the maximal operator M on $L^{\frac{q'}{t'}}(\mathbb{R}^n)$, then we obtain $\kappa^{-\frac{1}{t'}}\chi_Q M[|g|^{t'}]^{\frac{1}{t'}}$ is a (p',q')-block. Indeed, it is supported on a cube Q and it satisfies

$$\begin{aligned}\|\kappa^{-\frac{1}{t'}}\chi_Q M[|g|^{t'}]^{\frac{1}{t'}}\|_{L^{q'}} &\le (\|\kappa^{-1}\chi_Q M[|g|^{t'}]\|_{L^{\frac{q'}{t'}}})^{\frac{1}{t'}}\\ &\le (\|\chi_Q|g|^{t'}\|_{L^{\frac{q'}{t'}}})^{\frac{1}{t'}}\\ &= \|g\|_{L^{q'}}\\ &\le |Q|^{\frac{1}{q'}-\frac{1}{p'}}.\end{aligned}$$

Hence, we obtain $\int_{\mathbb{R}^n} f(x)g(x)\,dx \le \kappa^{\frac{1}{t'}}\left\|\sum_{j=1}^\infty \lambda_j\chi_{Q_j}\right\|_{\mathcal{M}^p_q}$.This is the desired result.

3.2. Proof of Theorem 1.4. Recall again that the grand maximal operator $\mathcal{M}$ was given by

$$\mathcal{M}f(x) = \sup\{|\varphi_t * f(x)| \,:\, \varphi \in \mathcal{F}_N,\, t > 0\} \quad (x \in \mathbb{R}^n).$$

Then we know that

$$\mathcal{M}a_j(x) \le C\left(\chi_{3Q_j}(x)Ma_j(x) + (M\chi_{Q_j}(x))^{\frac{n+d_q+1}{n}}\right),$$

where $d_q = [\max(\frac{n}{q} - n, 0)]$. See [19, (5.2)] for more details. The first term can be controlled by an argument similar to Theorem 1.1. The second term can be handled by using the Fefferman-Stein maximal inequality for Morrey spaces.

Proposition 3.1 ([26, Theorem 2.2], [31, Lemma 2.5])**.** *Let $1 < q \le p < \infty$ and $1 < r \le \infty$. Then*

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}^p_q} \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}^p_q}$$

for all sequences of measurable functions $\{f_j\}_{j=1}^{\infty}$.

3.3. Proof of Theorem 1.3. We invoke the following lemma. We refer to [29].

Lemma 3.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $d \in \{0, 1, 2, \ldots\}$ and $j \in \mathbb{Z}$. Then there exist collections of cubes $\{Q^*_{j,k}\}_{k\in K_j}$ and functions $\{\eta_{j,k}\}_{k\in K_j} \subset C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, which are all indexed by a set K_j for every j, and a decomposition*

$$f = g_j + b_j, \quad b_j = \sum_{k\in K_j} b_{j,k},$$

such that

(i) *Define $\mathcal{O}_j \equiv \{y \in \mathbb{R}^n : \mathcal{M}f(y) > 2^j\}$ and consider its Whitney decomposition. Then the cubes $\{200Q^*_{j,k}\}_{k\in K_j}$ have the bounded intersection property, and*

$$\mathcal{O}_j = \bigcup_{k\in K_j} Q^*_{j,k} = \bigcup_{k\in K_j} 200Q^*_{j,k}. \tag{6}$$

(ii) *Consider the partition of unity with respect to $\{Q^*_{j,k}\}_{k\in K_j}$. Denote it by $\{\eta_{j,k}\}_{k\in K_j}$. Then each function $\eta_{j,k}$ is supported in $Q^*_{j,k}$ and*

$$\sum_{k\in K_j} \eta_{j,k} = \chi_{\{y\in\mathbb{R}^n : \mathcal{M}f(y)>2^j\}}, \quad 0 \le \eta_{j,k} \le 1.$$

(iii) *The distribution g_j satisfies the inequality:*

$$\mathcal{M}g_j(x) \le C \left(\mathcal{M}f(x)\chi_{\mathcal{O}_j{}^c}(x) + 2^j \sum_{k \in K_j} \frac{\ell_{j,k}{}^{n+d+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+d+1}} \right)$$

for all $x \in \mathbb{R}^n$.

(iv) *Each distribution $b_{j,k}$ is given by $b_{j,k} = (f - c_{j,k})\eta_{j,k}$ with a certain polynomial $c_{j,k} \in \mathcal{P}_d(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} b_{j,k}(x) q(x)\, dx = 0 \text{ for all } q \in \mathcal{P}_d(\mathbb{R}^n),$$

and

$$\mathcal{M}b_{j,k}(x) \le C \left(\mathcal{M}f(x)\chi_{Q^*_{j,k}}(x) + 2^j \cdot \frac{\ell_{j,k}{}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n \setminus Q^*_{j,k}}(x) \right) \quad (7)$$

for all $x \in \mathbb{R}^n$.

In the above, $x_{j,k}$ and $\ell_{j,k}$ denote the center and the side-length of $Q^*_{j,k}$, respectively, and the implicit constants are dependent only on n.

For the proof of Theorem 1.3, we need an auxiliary norm. Define

$$\|f\|_{L^q((1+|\cdot|)^\alpha)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^q (1 + |x|)^\alpha \, dx \right)^{\frac{1}{q}}$$

for a measurable function f on $\mathbb{R}^n$. Recall that a non-negative measurable function w is an A_1-weight if w satisfies

$$Mw(x) \le Cw(x) \quad (x \in \mathbb{R}^n).$$

Lemma 3.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Keep to the same notation as Lemma 3.2. Then we have*

$$|\langle b_j, \varphi \rangle| \le C \|\chi_{\mathcal{O}_j} \mathcal{M}f\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)} \quad (8)$$

and

$$|\langle g_j, \varphi \rangle| \le C \|\chi_{\mathcal{O}_j{}^c} \cdot \mathcal{M}f + 2^j \chi_{\mathcal{O}_j}\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}, \quad (9)$$

where the constants C in (8) and (9) depend on φ but not on j or k.

Proof. Denote by $B(x,r)$ the ball centered at x and radius $r > 0$. We shall write $B(r) = B(0,r)$ as before.

By the subadditivity of $\mathcal{M}$ given by (3), we have

$$|\langle b_j, \varphi \rangle| \le C \inf_{x \in B(1)} \mathcal{M}b_j(x) \le C \inf_{x \in B(1)} \sum_{k \in K_j} \mathcal{M}b_{j,k}(x).$$

Indeed, for some large constant $M = M_\varphi$, we have $M^{-1}\varphi \in \mathcal{F}_N$, so that

$$|\langle b_j, \varphi\rangle| \le M \inf_{x\in B(1)} \mathcal{M}b_j(x).$$

Observe also that $CM\chi_Q(x) \ge \frac{|Q|}{|Q|+|x-x_Q|^n}$, if Q is a cube centered at x_Q. It follows from (7) that

$$\begin{aligned}\sum_{k\in K_j} \mathcal{M}b_{j,k}(x) &\le C \sum_{k\in K_j} \left(\mathcal{M}f(x)\chi_{Q^*_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x-x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n\setminus Q^*_{j,k}}(x)\right)\\ &\le C\left(\mathcal{M}f(x)\chi_{\mathcal{O}_j}(x) + 2^j \sum_{k\in K_j} M\chi_{Q^*_{j,k}}(x)^{\frac{n+d+1}{n}}\right).\end{aligned}$$

Thus, from this pointwise estimate, we deduce

$$\begin{aligned}&\|\mathcal{M}b_j\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\ &\le C \left\| \mathcal{M}f\cdot\chi_{\mathcal{O}_j} + 2^j \sum_{k\in K_j} (M\chi_{Q^*_{j,k}})^{\frac{n+d+1}{n}} \right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\ &\le C \left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)} + C\left\| 2^j \sum_{k\in K_j} (M\chi_{Q^*_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\ &= C \left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\ &\quad + C\left(\left\| 2^j \left(\sum_{k\in K_j} (M\chi_{Q^*_{j,k}})^{\frac{n+d+1}{n}}\right)^{\frac{n}{n+d+1}} \right\|_{L^{\frac{n+d+1}{n}q}\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\right)^{\frac{n+d+1}{n}}.\end{aligned}$$

Now by the Fefferman-Stein inequality for A_1-weighted Lebesgue spaces [2],

$$\begin{aligned}&\|\mathcal{M}b_j\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\ &\le C\left\|\chi_{\mathcal{O}_j}\mathcal{M}f\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\ &\quad + C\left(\left\| 2^j \left(\sum_{k\in K_j} \chi_{Q^*_{j,k}}{}^{\frac{n+d+1}{n}}\right)^{\frac{n}{n+d+1}} \right\|_{L^{\frac{n+d+1}{n}q}\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\right)^{\frac{n+d+1}{n}}.\end{aligned}$$

By the definition of $\mathcal{O}_j$, we have

$$
\begin{aligned}
&\|\mathcal{M}b_j\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}+C\left\|2^j\sum_{k\in K_j}\chi_{Q^*_{j,k}}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}+C\left\|2^j\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}.
\end{aligned}
$$

Thus, (8) is proved.

In the same way we can prove (9). Indeed, by using the Fefferman-Stein inequality for A_1-weighted Lebesgue spaces [2], we obtain

$$
\begin{aligned}
&\|\mathcal{M}g_j\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j{}^c}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}+C\left\|\sum_{k\in K_j}\frac{2^j\cdot{\ell_{j,k}}^{n+d+1}}{(\ell_{j,k}+|\cdot-x_{j,k}|)^{n+d+1}}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j{}^c}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}+C\left\|\sum_{k\in K_j}2^j(M\chi_{Q^*_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}\\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j{}^c}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}+C\left\|2^j\chi_{\mathcal{O}_j}\right\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)}.
\end{aligned}
$$

Thus, (9) is proven. □

The key observation is the following.

Lemma 3.4. *In the notation of Lemma 3.2, in the topology of $\mathcal{S}'(\mathbb{R}^n)$, we have $g_j\to 0$ as $j\to-\infty$ and $b_j\to 0$ as $j\to\infty$. In particular,*

$$f=\sum_{j=-\infty}^{\infty}(g_{j+1}-g_j)$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$.

Proof. We claim first that

$$(1+|\cdot|)^{-\frac{n}{q}+\frac{n}{2p}}\mathcal{M}f\in L^q(\mathbb{R}^n). \tag{10}$$

Consider first the case when $q \ge 1$. Then we have

$$\begin{aligned}
\|(1+|\cdot|)^{-\frac{n}{q}+\frac{n}{2p}}\mathcal{M}f\|_{L^q} &\le C\sum_{j=1}^{\infty} 2^{-j\frac{n}{q}+\frac{jn}{2p}}\|\mathcal{M}f\|_{L^q(B(2^j))}\\
&= C\sum_{j=1}^{\infty} 2^{-j\frac{n}{q}+\frac{jn}{2p}+j\frac{n}{q}-\frac{jn}{p}}|B(2^j)|^{\frac{1}{p}-\frac{1}{q}}\|\mathcal{M}f\|_{L^q(B(2^j))}\\
&\le C\sum_{j=1}^{\infty} 2^{-\frac{jn}{q}+\frac{jn}{q}-\frac{jn}{2p}}\|f\|_{H\mathcal{M}^p_q}\\
&\le C\|f\|_{H\mathcal{M}^p_q}\\
&< \infty,
\end{aligned}$$

proving (10).

Assume instead that $0 < q \le 1$. Then we have

$$\begin{aligned}
\left(\|(1+|\cdot|)^{-\frac{n}{q}+\frac{n}{2p}}\mathcal{M}f\|_{L^q}\right)^q &\le C\sum_{j=1}^{\infty}(2^{-\frac{jn}{q}+\frac{jn}{2p}}\|\mathcal{M}f\|_{L^q(B(2^j))})^q\\
&= C\sum_{j=1}^{\infty}\left(2^{-\frac{jn}{q}+\frac{jn}{2p}+\frac{jn}{q}-\frac{jn}{p}}|B(2^j)|^{\frac{1}{p}-\frac{1}{q}}\|\mathcal{M}f\|_{L^q(B(2^j))}\right)^q\\
&\le C\sum_{j=1}^{\infty}\left(2^{-\frac{jn}{q}+\frac{jn}{q}-\frac{jn}{2p}}\|f\|_{H\mathcal{M}^p_q}\right)^q\\
&\le C(\|f\|_{H\mathcal{M}^p_q})^q\\
&< \infty,
\end{aligned}$$

proving (10) again.

Let us next show that $b_j \to 0$ as $j \to \infty$ in $\mathcal{S}'(\mathbb{R}^n)$. Once this is proved, then we have $f = \lim_{j\to\infty} g_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Let us choose a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$|\langle b_j, \varphi\rangle| \le C \inf_{x\in B(1)} \mathcal{M}b_j(x) \le C\|\mathcal{M}b_j\|_{L^q(B(1))},$$

where C does depend on φ. Hence it follows from (8) that $\langle b_j, \varphi\rangle \to 0$ as $j \to \infty$. Likewise by using (9), we obtain

$$|\langle g_j, \varphi\rangle| \le \|2^j \cdot \chi_{\mathcal{O}_j} + \chi_{(\mathcal{O}_j)^c}\cdot \mathcal{M}f\|_{L^q\left((1+|\cdot|)^{-n+\frac{qn}{2p}}\right)},$$

Hence, $g_j \to 0$ as $j \to -\infty$. Consequently, it follows that $f = \lim_{j\to\infty} g_j = \lim_{j,k\to\infty}\sum_{l=-k}^{j}(g_{l+1} - g_l)$ in $\mathcal{S}'(\mathbb{R}^n)$. □

Now let us prove Theorem 1.3.

Proof of Theorem 1.3. For each $j \in \mathbb{Z}$, consider the level set

$$\mathcal{O}_j \equiv \{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2^j\}.$$

Then it follows immediately from the definition that

$$\mathcal{O}_{j+1} \subset \mathcal{O}_j.$$

If we invoke Lemma 3.2, then f can be decomposed;

$$f = g_j + b_j, \quad b_j = \sum_k b_{j,k}, \quad b_{j,k} = (f - c_{j,k})\eta_{j,k}$$

where each $b_{j,k}$ is supported in a cube $Q^*_{j,k}$ as is described in Lemma 3.2.

We know that

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j),$$

with the sum converging in the sense of distributions. Here, going through the same argument as the one in [29, pp. 108–109], we have an expression

$$f = \sum_{j,k} A_{j,k}, \quad g_{j+1} - g_j = \sum_k A_{j,k} \quad (j \in \mathbb{Z})$$

in the sense of distributions, where each $A_{j,k}$, supported in $Q^*_{j,k}$, satisfies the pointwise estimate $|A_{j,k}(x)| \le C_0 2^j$ for some universal constant C_0 and the moment condition $\int_{\mathbb{R}^n} A_{j,k}(x)q(x)\,dx = 0$ for every $q \in \mathcal{P}_d(\mathbb{R}^n)$. With these observations in mind, let us set

$$a_{j,k} \equiv \frac{A_{j,k}}{C_0 2^j}, \quad \kappa_{j,k} \equiv C_0 2^j.$$

Then we automatically obtain that each $a_{j,k}$ satisfies

$$|a_{j,k}| \le \chi_{Q^*_{j,k}}, \quad \int_{\mathbb{R}^n} x^\alpha a_{j,k}(x)\,dx = 0 \quad (|\alpha| \le L)$$

and that $f = \sum_{j,k} \kappa_{j,k} a_{j,k}$ in the topology of $H\mathcal{M}^p_q(\mathbb{R}^n)$, once we prove the estimate of coefficients. Rearrange $\{a_{j,k}\}$ and so on to obtain $\{a_j\}$ and so on.

To establish (2) we need to estimate

$$\alpha \equiv \left\| \left(\sum_{j=-\infty}^{\infty} |\lambda_j \chi_{Q_j}|^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}^p_q}.$$

Since $\{(\kappa_{j,k},Q^*_{j,k})\}_{j,k}=\{(\lambda_j,Q_j)\}_j$ we have $\alpha=\left\|\left(\sum_{j=-\infty}^{\infty}\sum_{k\in K_j}|\kappa_{j,k}\chi_{Q^*_{j,k}}|^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_q}$. If we insert the definition of κ_j, then we have

$$\alpha = C_0\left\|\left(\sum_{j=-\infty}^{\infty}\sum_{k\in K_j}|2^j\chi_{Q^*_{j,k}}|^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_q} = C_0\left\|\left(\sum_{j=-\infty}^{\infty}2^{jv}\sum_{k\in K_j}\chi_{Q^*_{j,k}}\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_q}.$$

Observe that (6) together with the bounded overlapping property yields

$$\chi_{\mathcal{O}_j}(x) \le \sum_{k\in K_j}\chi_{Q^*_{j,k}}(x) \le \chi_{200Q^*_{j,k}}(x) \le M\chi_{\mathcal{O}_j}(x) \quad (x\in\mathbb{R}^n).$$

Thus, we have $\alpha \le C\left\|\left(\sum_{j=-\infty}^{\infty}\left(2^j\chi_{\mathcal{O}_j}\right)^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_q}$. Recall that $\mathcal{O}_j \supset \mathcal{O}_{j+1}$ for each $j\in\mathbb{Z}$. Consequently we have

$$\sum_{j=-\infty}^{\infty}\left(2^j\chi_{\mathcal{O}_j}(x)\right)^v \sim \left(\sum_{j=-\infty}^{\infty}2^j\chi_{\mathcal{O}_j}(x)\right)^v \sim \left(\sum_{j=-\infty}^{\infty}2^j\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}(x)\right)^v.$$

Thus, we obtain

$$\alpha \le C\left\|\left(\sum_{j=-\infty}^{\infty}\left(2^j\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}\right)^v\right)^{\frac{1}{v}}\right\|_{\mathcal{M}^p_q} = \left\|\sum_{j=-\infty}^{\infty}2^j\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}\right\|_{\mathcal{M}^p_q}.$$

It follows from the definition of $\mathcal{O}_j$ that we have $2^j < \mathcal{M}f(x)$ for all $x\in\mathcal{O}_j$. Hence, we have $\alpha \le C\left\|\sum_{j=-\infty}^{\infty}\chi_{\mathcal{O}_j\setminus\mathcal{O}_{j+1}}\mathcal{M}f\right\|_{\mathcal{M}^p_q} \le C\|\mathcal{M}f\|_{\mathcal{M}^p_q}$. This is the desired result. $\square$

4. Proof of Theorem 1.7

First, we prove two lemmas. The first one is recorded as [7, Lemma 2.2] or [8, Lemma 2.1]. Here for the sake of convenience, we supply the whole proof.

Lemma 4.1. *There exists a constant depending only on n and α such that, for every cube Q, we have $I_\alpha\chi_Q(x) \ge C\ell(Q)^\alpha\chi_Q(x)$ for all $x\in Q$.*

Proof. Let us set

$$Q = \{x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n : \max(|x_1-z_1|,|x_2-z_2|,\ldots,|x_n-z_n|) < r\},$$

where $z = (z_1, z_2, \ldots, z_n)$ denotes the center of Q and $r > 0$ denotes half of the side-length. Let $x \in Q$. Then we have

$$\begin{aligned} I_\alpha \chi_Q(x) &= \int_{|y_1-z_1|,|y_2-z_2|,\ldots,|y_n-z_n|<r} \frac{1}{|x-y|^{n-\alpha}}\,dy \\ &= r^n \int_{|ry_1-z_1|,|ry_2-z_2|,\ldots,|ry_n-z_n|<r} \frac{1}{|x-ry|^{n-\alpha}}\,dy \\ &= r^\alpha \int_{|y_1-z_{\frac{1}{r}}|,|y_2-z_{\frac{2}{r}}|,\ldots,|y_n-z_{\frac{n}{r}}|<1} \frac{1}{|\frac{x}{r}-y|^{n-\alpha}}\,dy \\ &= r^\alpha \int_{|y_1|,|y_2|,\ldots,|y_n|<1} \frac{1}{|\frac{x}{r}-\frac{z}{r}-y|^{n-\alpha}}\,dy \end{aligned}$$

by using the dilation and translation. Set

$$C_0 \equiv \min_{\substack{w=(w_1,w_2,\ldots,w_n) \\ |w_1|,|w_2|,\ldots,|w_n|<1}} \int_{|y_1|,|y_2|,\ldots,|y_n|<1} \frac{dy}{|w-y|^{n-\alpha}}.$$

Then we have $I_\alpha \chi_Q(x) \geq C_0 r^\alpha \chi_Q(x)$. Since $2r = \ell(Q)$, we obtain the desired result. □

To prove the next estimate, we need the Adams inequality asserting that I_α is bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ whenever p, q, s, t satisfies $1 < q \leq p < \infty$, $1 < t \leq s < \infty$, $\frac{t}{s} = \frac{q}{p}$ and $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$.

Lemma 4.2. *Let $L = 0, 1, 2, \ldots$. Suppose that A is an $L^\infty(\mathbb{R}^n)$-function supported on a cube Q. Assume in addition that $\int_{\mathbb{R}^n} x^\beta a(x)\,dx = 0$ for all multi-indices β with $|\beta| \leq L$. Then,*

$$|I_\alpha A(x)| \leq C_{\alpha,L} \|A\|_{L^\infty} \ell(Q)^\alpha \sum_{k=1}^\infty \frac{1}{2^{k(n+L+1-\alpha)}} \chi_{2^k Q}(x) \quad (x \in \mathbb{R}^n). \tag{11}$$

Proof. Suppose first that $x \in 2Q$. Then

$$|I_\alpha A(x)| \leq \|A\|_{L^\infty} \int_Q \frac{1}{|x-y|^{n-\alpha}}\,dy \leq C_\alpha \|A\|_{L^\infty} \ell(Q)^\alpha. \tag{12}$$

Next, suppose that $x \in 2^{k+1}Q \setminus 2^k Q$ for some $k \in \mathbb{N}$. We write out $I_\alpha A$ in full:

$$I_\alpha A(x) = \int_Q \frac{A(y)}{|x-y|^{n-\alpha}}\,dy.$$

We freeze x and denote by $P_{x,L}(y)$ the Taylor polynomial of order L at $y = c_Q$ of the function $y \mapsto |x-y|^{-n+\alpha}$. Then we have

$$I_\alpha A(x) = \int_Q A(y) \left(\frac{1}{|x-y|^{n-\alpha}} - P_{x,L}(y) \right) dy.$$

Since $x \in 2^{k+1}Q \setminus 2^kQ$, thus $C^{-1}|x-y| \le |x-c_Q| \le C|x-y|$ for all $y \in Q$. Thus, for all $y \in Q$, we observe

$$\begin{aligned}\left|\frac{1}{|x-y|^{n-\alpha}} - P_{x,L}(y)\right| &\le C_L \ell(Q)^{L+1} \sup_{z\in Q}\sup_{|\beta|=L+1}\left|\left(\frac{\partial}{\partial z}\right)^{\beta}\frac{1}{|x-z|^{n-\alpha}}\right| \\ &\le C_{\alpha,L}\frac{\ell(Q)^{L+1}}{|x-c_Q|^{n+L+1-\alpha}}.\end{aligned}$$

If we insert this pointwise estimate, we obtain

$$|I_\alpha A(x)| \le C_{\alpha,L}\|A\|_{L^\infty}\int_Q \frac{\ell(Q)^{L+1}}{|x-c_Q|^{n+L+1-\alpha}}\,dy = C_{\alpha,L}\|A\|_{L^\infty}\frac{\ell(Q)^{n+L+1}}{|x-c_Q|^{n+L+1-\alpha}}. \quad (13)$$

With (12) and (13) in mind, let us prove (11). If $x \in 2Q$, then from (12) we conclude

$$\begin{aligned}|I_\alpha A(x)| &\le C_\alpha \|A\|_{L^\infty}\ell(Q)^\alpha \\ &= C_\alpha \|A\|_{L^\infty}\ell(Q)^\alpha \chi_{2Q}(x) \\ &= C_\alpha 2^{n+L+1-\alpha}\|A\|_{L^\infty}\ell(Q)^\alpha \frac{\chi_{2^1Q}(x)}{2^{1\cdot(n+L+1-\alpha)}} \\ &\le C_{\alpha,L}\|A\|_{L^\infty}\ell(Q)^\alpha \sum_{k=1}^\infty \frac{1}{2^{k(n+L+1-\alpha)}}\chi_{2^kQ}(x).\end{aligned}$$

Hence (11) holds. If we assume $x \in 2^{k_0+1}Q \setminus 2^{k_0}Q$ for some $k_0 \in \mathbb{N}$, then we use (13) to obtain

$$\begin{aligned}|I_\alpha A(x)| &\le C_{\alpha,L}\|A\|_{L^\infty}\frac{\ell(Q)^{n+L+1}}{|x-c_Q|^{n+L+1-\alpha}} \\ &= C_{\alpha,L}\|A\|_{L^\infty}(2^{k_0}\ell(Q))^{-n-L-1+\alpha}\ell(Q)^{n+L+1} \\ &= C_{\alpha,L}\|A\|_{L^\infty}2^{-(k_0+1)(n+L+1-\alpha)}\ell(Q)^\alpha \\ &= C_{\alpha,L}\|A\|_{L^\infty}\ell(Q)^\alpha \frac{\chi_{2^{k_0+1}Q}(x)}{2^{(k_0+1)(n+L+1-\alpha)}} \\ &\le C_{\alpha,L}\|A\|_{L^\infty}\ell(Q)^\alpha \sum_{k=1}^\infty \frac{1}{2^{k(n+L+1-\alpha)}}\chi_{2^kQ}(x).\end{aligned}$$

Hence (11) holds in this case as well. □

Now we prove Theorem 1.7. We decompose f according to Theorem 1.2 with $L > \alpha - \frac{n}{q_0} - 1$; $f = \sum_{j=1}^\infty \lambda_j a_j$, where $\{Q_j\}_{j=1}^\infty \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill (2). Then by Lemma 4.2, we obtain

$$|g(x)I_\alpha f(x)| \le \sum_{j=1}^\infty\sum_{k=1}^\infty \frac{\lambda_j}{2^{k(n+L+1-\alpha)}}\left(\ell(Q_j)^\alpha |g(x)|\chi_{2^kQ_j}(x)\right).$$

Therefore, we conclude

$$\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\|g\|_{\mathcal{M}^{q_0}_q}\left\|\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{\lambda_j \ell(2^kQ_j)^{\alpha-\frac{n}{q_0}}}{2^{k(n+L+1)}}\cdot\frac{\ell(2^kQ_j)^{\frac{n}{q_0}}}{\|g\|_{\mathcal{M}^{q_0}_q}}|g|\chi_{2^kQ_j}\right\|_{\mathcal{M}^{r_0}_r}.$$

For each $(j,k)\in\mathbb{N}\times\mathbb{N}$, write

$$\kappa_{jk}\equiv\frac{\lambda_j \ell(2^kQ_j)^{\alpha-\frac{n}{q_0}}}{2^{k(n+L+1)}},\quad b_{jk}\equiv\frac{\ell(2^kQ_j)^{\frac{n}{q_0}}}{\|g\|_{\mathcal{M}^{q_0}_q}}|g|\chi_{2^kQ_j}.$$

Then, $\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{\lambda_j \ell(2^kQ_j)^{\alpha-\frac{n}{q_0}}}{2^{k(n+L+1)}}\cdot\frac{\ell(2^kQ_j)^{\frac{n}{q_0}}}{\|g\|_{\mathcal{M}^{q_0}_q}}|g|\chi_{2^kQ_j}=\sum_{j,k=1}^\infty \kappa_{jk}b_{jk}$, each b_{jk} is supported on a cube 2^kQ_j and $\|b_{jk}\|_{\mathcal{M}^{q_0}_q}\le \ell(2^kQ_j)^{\frac{n}{q_0}}$. Observe also that $q_0>r_0$ and that $q>r$. Thus, by Theorem 1.1, it follows that

$$\begin{aligned}\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} &\le C\|g\|_{\mathcal{M}^{q_0}_q}\left\|\sum_{j=1}^\infty\sum_{k=1}^\infty \kappa_{jk}\chi_{2^kQ_j}\right\|_{\mathcal{M}^{r_0}_r}\\ &\le C\|g\|_{\mathcal{M}^{q_0}_q}\left\|\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{\lambda_j \ell(2^kQ_j)^{\alpha-\frac{n}{q_0}}}{2^{k(n+L+1)}}\chi_{2^kQ_j}\right\|_{\mathcal{M}^{r_0}_r}.\end{aligned}$$

Observe that $\chi_{2^kQ_j}\le 2^{kn}M\chi_{Q_j}$. Hence, if we choose $0<\theta<r$ so that $L>\alpha-\frac{n}{q_0}-1+\theta n-n$, then we have

$$\begin{aligned}\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} &\le C\|g\|_{\mathcal{M}^{q_0}_q}\left\|\sum_{j=1}^\infty \lambda_j \ell(Q_j)^{\alpha-\frac{n}{q_0}}(M\chi_{Q_j})^\theta\right\|_{\mathcal{M}^{r_0}_r}\\ &\le C\|g\|_{\mathcal{M}^{q_0}_q}\left\|\sum_{j=1}^\infty (M[\lambda_j^{\frac{1}{\theta}}\ell(Q_j)^{(\alpha-\frac{n}{q_0})/\theta}\chi_{Q_j}])^\theta\right\|_{\mathcal{M}^{r_0}_r}\\ &\le C\|g\|_{\mathcal{M}^{q_0}_q}\left(\left\|\left\{\sum_{j=1}^\infty (M[\lambda_j^{\frac{1}{\theta}}\ell(Q_j)^{(\alpha-\frac{n}{q_0})/\theta}\chi_{Q_j}])^\theta\right\}^{\frac{1}{\theta}}\right\|_{\mathcal{M}^{r_0\theta}_{r\theta}}\right)^{\frac{1}{\theta}}.\end{aligned}$$

By virtue of Proposition 3.1, the Fefferman-Stein inequality for Morrey spaces, with $f_j=\lambda_j^{\frac{1}{\theta}}\ell(Q_j)^{\frac{\alpha-\frac{n}{q_0}}{\theta}}\chi_{Q_j}$, we can remove the maximal operator and we obtain

$$\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r}\le C\|g\|_{\mathcal{M}^{q_0}_q}\left\|\sum_{j=1}^\infty \lambda_j\ell(Q_j)^{\alpha-\frac{n}{q_0}}\chi_{Q_j}\right\|_{\mathcal{M}^{r_0}_r}.$$

We distinguish two cases here.

1. If $\alpha = \frac{n}{q_0}$, then $p_0 = r_0$ and $p = r$. Thus, we can use (1).
2. If $\alpha > \frac{n}{q_0}$, then, by the Adams inequality and Lemma 4.1, we obtain

$$\left\| \sum_{j=1}^{\infty} \lambda_j \ell(Q_j)^{\alpha - \frac{n}{q_0}} \chi_{Q_j} \right\|_{\mathcal{M}_r^{r_0}} \le C \left\| I_{\alpha - \frac{n}{q_0}} \left[\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right] \right\|_{\mathcal{M}_r^{r_0}} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_p^{p_0}}.$$

Thus, we are still in the position of using (1).

5. Concluding remarks

In this section, we compare our results with earlier ones.

5.1. Decompositions and characterizations of Morrey spaces. Morrey spaces can be characterized by means of decompositions. See [15] for the Calderón-Zygmund decomposition, for example. Roughly speaking, there are two types of decompositions other than this decomposition. One is the non-smooth decomposition and the other is the smooth decomposition. Theorems 1.1–1.4 fall under the scope of the non-smooth decomposition, while the decompositions obtained in [16, 17, 22, 27, 28, 35, 36, 40] are smooth decompositions. The smooth decompositions hinges upon the theory of spaces of Triebel-Lizorkin type spaces developed in [10, 16, 17, 21, 34]. If we restrict ourselves to Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^n)$ with $1 < q \le p < \infty$, the result in [18, Theorem 4.2] connects $\mathcal{M}_q^p(\mathbb{R}^n)$ and the framework of Triebel-Lizorkin-Morrey spaces above. The idea is generalized widely to other function spaces; see [11] and [17, Theorem 9.8]. In the framework of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, more and more are investigated; see [35–38] and the textbook [41]. For the definitions of these function spaces, we refer to [10, 17, 41].

5.2. Other function spaces in $\mathbb{R}^n$. Our results promise extension to many other function spaces such as Herz spaces, generalized Morrey spaces and B_σ spaces. Hardy spaces with variable exponents are investigated from this aspect; we already have a counterpart of Theorem 1.1–1.4; see [4, 19, 25]. For Orlicz spaces, we refer to [20].

5.3. Sharpness of t in Thoerem 1.1. In Theorem 1.1, we postulated $t > q$. But this condition is absolutely necessary. Indeed, if we had Theorem 1.1 for $t \le q$, then we would have Theorem 1.7 for $r \ge q$. This contradicts a counterexample in [30, Section 4].

5.4. A passage to non-doubling measures. A passage to the setting of $\mathbb{R}^n$ equipped with the Radon measure μ satisfying $\mu(B(x,r)) \le Cr^a$, where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Tolsa defined Hardy spaces for such measures μ in [32] and obtained a Littlewood-Paley characterization in [33]. Hardy spaces by means of the grand maximal operator and Hardy spaces by means of the atoms are shown to be equivalent in [32]. By using this grand maximal operator, it seems possible to extend Theorem 1.1 and 1.2 to the space defined in [26]. The extension to this direction is left as our future work.

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