

# Existence and Stability of Periodic Planar Standing Waves in Phase-Transitional Elasticity with Strain-Gradient Effects II: Examples

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**Abstract.** In a companion paper, we investigated phase-transitional elasticity models with strain-gradient effect, and established the existence of non-constant planar periodic standing waves in these models by variational methods. The variational methods enable us to deal with existence of periodic waves no matter the unknowns are scalar or not. Here, we list specific phase-transitional models with strain-gradient effects and list conditions that guarantee the existence of non-constant periodic waves. Also, when the unknowns are scalars, we do a phase-plane analysis, and compare the results obtained by phase-plane analysis with those obtained by our general variational methods. Finally, we briefly discuss relations between spectral and nonlinear stabilities by using a change of variables introduced by Kotschote to transform our system to a strictly parabolic system to which general results of Johnson-Zumbrun and Howard-Zumbrun apply.

**Keywords.** Elasticity, strain-gradient effect, periodic wave, Hamiltonian system.

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## 1. Introduction

In the companion paper [23], extending investigations of Antman & Malek-Madani, Schechter & Shearer, Slemrod, Barker & Lewicka & Zumbrun ([1, 2, 8, 19]), and others, we investigated phase-transitional elasticity models with strain-gradient effect. We obtained general existence results for non-constant planar periodic standing waves in these models by variational methods. The use of variational methods enables us to deal with cases where the unknowns in our systems are vector-valued. In this paper, we list some specific models following the line of research in [8] and specify the conditions which guarantee the

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existence of non-constant periodic waves in these models. For the 1D models, we make use of the Hamiltonian structure of the corresponding ODE system and do a phase-plane analysis to get information about the wave structures. It is interesting to compare these results obtained by phase-plane analysis with those obtained by variational methods. Through practical computations for these models, we see that the results obtained by the above two different lines match very well. for deformations of arbitrary dimension and general, physical, viscosity and strain-gradient terms. Previous investigations considered one-dimensional phenomenological models with artificial viscosity/strain gradient effect, for which the existence reduces to the study of a standard (scalar) non-linear oscillator. For our variational analysis, we require that the mean vector of the unknowns over one period be in the elliptic region with respect to the corresponding pure inviscid elastic model. Previous such results were confined to one-dimensional deformations in models with artificial viscosity–strain-gradient coefficients.

The paper is organized as follows. In Section 2, we recall the related equations. In Section 3, we list the models following the line of research in [8]. The specific conditions which guarantee the existence of non-constant periodic standing waves are listed in Section 4 for the corresponding models in Section 3. In Section 5, we carry out a phase-plane analysis for the 1D models and compare conditions led by variational methods. In Section 6, we briefly discuss stability.

## 2. Equations and specific models

Adopting the notations in [1, 3, 8, 17, 23], we recall the linear momentum equations of isothermal elasticity with strain-gradient effect:

$$\xi_{tt} - \nabla_X \cdot \left( DW(\nabla\xi) + \mathcal{Z}(\nabla\xi, \nabla\xi_t) - \mathcal{E}(\nabla^2\xi) \right) = 0. \quad (2.1)$$

For the physics and restrictions on the Piola-Kirchhoff stress  $\nabla W$ , deformation gradient  $\nabla_X\xi$ , viscous stress tensor  $\mathcal{Z}$  see [23] and the above references.

In this paper, we focus on the interesting subclass of planar solutions, which are solutions in the full 3D space that depend only on a single coordinate direction. That is, we investigate deformations  $\xi$  given by

$$\xi(X) = X + U(z), \quad X = (x, y, z), \quad U = (U_1, U_2, U_3) \in \mathbb{R}^3.$$

Corresponding to the above deformation or displacement  $\xi$ , the deformation gradient with respect to  $X$

$$F = \begin{pmatrix} 1 & 0 & U_{1,z} \\ 0 & 1 & U_{2,z} \\ 0 & 0 & 1 + U_{3,z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix}. \quad (2.2)$$

We shall denote  $V = (\tau, u) = (\tau_1, \tau_2, \tau_3, u_1, u_2, u_3)$ , where  $\tau_1 = U_{1,z}$ ,  $\tau_2 = U_{2,z}$ ,  $\tau_3 = 1 + U_{3,z}$  and  $u_1 = U_{1,t}$ ,  $u_2 = U_{2,t}$ ,  $u_3 = U_{3,t}$  with the physical constraint  $\tau_3 > 0$ , corresponding to  $\det F > 0$  in the region of physical feasibility of  $V$ .

Writing  $W(\tau) = W \left( \begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix} \right)$ , we see that for all  $F$  as in (2.1) there holds

$$\nabla_X \cdot (DW(F)) = (D_\tau W(\tau))_z.$$

That is, the planar equations inherit a vector-valued variational structure echoing the matrix valued variational structure (note that the left hand side is the divergence of  $DW(F)$ ).

In this paper, we study local models by specifying the related terms in system (2.1) as follows:

**1. Elastic potential  $W$ .** As described in [FP], we shall study the phase-transitional elastic potential:

$$W(F) = |F^T F - C_-|^2 \cdot |F^T F - C_+|^2,$$

which is a potential for anisotropic material with material frame indifference property. Here

$$C_\pm = (F^T F)_\pm := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm \varepsilon \\ 0 & \pm \varepsilon & 1 + \varepsilon^2 \end{pmatrix}, \quad F_\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm \varepsilon \\ 0 & 0 & 1 \end{pmatrix}.$$

we have then

$$W(F) = \left( 2\tau_1^2 + 2(\tau_2 - \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \right) \left( 2\tau_1^2 + 2(\tau_2 + \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \right),$$

We study models involving this phase-transitional elasticity potential. It is important to notice that the elastic potential here does not satisfy the asymptotic behavior  $W \rightarrow +\infty$  when  $\det F \rightarrow 0^+$  that was assumed in [23]. Especially this is an irrelevant assumption for shear models. Hence the related models are local models for the real physics.

**2. Viscous stress tensor  $\mathcal{Z}$ .** We use the following tress tensor which is compatible with the principles of continuum mechanics (see [8]):

$$\mathcal{Z}(F, Q) = 2(\det F) \operatorname{sym}(QF^{-1})F^{-1,T}.$$

Here we note that the related Cauchy stress tensor  $T_2 = 2(\det F)^{-1} \mathcal{Z}F^T = 2\operatorname{sym}(QF^{-1})$  is the Lagrangian version of the stress tensor  $2\operatorname{sym} \nabla v$  written in the Eulerian coordinates. For incompressible fluids  $2\operatorname{div}(\operatorname{sym} \nabla v)$ , giving the usual parabolic viscous regularization of the fluid dynamics evolutionary system.

**3. The strain-gradient term  $\mathcal{E}$ .** For the strain-gradient effect we will choose  $\Psi(P) = \frac{1}{2}|P|^2$  as a mathematically natural first step, so that  $\mathcal{E}(\nabla^2\xi) = \nabla_X \cdot \nabla^2\xi = \Delta_X F$ , which is an extension of the 1D case of [S1]. We see that it is the strain-gradient term that makes the models have abundant wave phenomena.

**The system.** As in [23], we shall use  $x \in \mathbb{R}^1$  as the space variable instead of  $z$ . Our system is (see also [23]):

$$\begin{cases} \tau_t - u_x = 0 \\ u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(\tau_x)\tau_{xx})_x. \end{cases} \quad (2.3)$$

now with  $\sigma := -D_\tau W(\tau)$ ,  $d(\cdot) := D^2\Psi(\cdot) = Id$  and

$$b(\tau) = \tau_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (2.4)$$

### 3. Hamiltonian structure

From the analysis in [23], we see that all traveling periodic waves are standing. The traveling wave ODE system reduces to the following form with an integral constant  $q$ :

$$\begin{cases} -\tau'' = -D_\tau W(\tau) + q \\ q = \{D_\tau W(\tau) - \tau''\} \Big|_{x=x_0}. \end{cases} \quad (3.1)$$

If we take the Hamiltonian point of view, the corresponding Hamiltonian for the above system is

$$H(\tau, \tau') = \frac{1}{2}|\tau'(x)|^2 + V(\tau, \tau'),$$

where  $V(\tau, \tau') := q \cdot \tau(x) - W(\tau(x))$ . The periodic solutions of the system are confined to the surface  $H(\tau, \tau') \equiv \text{constant}$ .

The form of  $W$  contains complete information for our purpose. In the following, we list the elastic potential and related information for the phase-transitional models we shall deal with in this paper for completeness and future study. To get these models, we fix one or two directions of  $\tau$  as zero, or, in the incompressible case,  $\tau_3 \equiv 1$  (as described in [2, 8], the latter is an imposed constraint, that is compensated for in the  $\tau_3$  equation by a Lagrange multiplier corresponding to pressure). We refer the reader to [8, Section 3] for details of the derivations of these models.

**3.1. 2D incompressible shear model.** This model corresponds to setting  $\tau_3 = 1$ .

$$W(\tau) = \left(2\tau_1^2 + 2(\tau_2 - \varepsilon)^2 + (|\tau|^2 - \varepsilon^2)^2\right) \left(2\tau_1^2 + 2(\tau_2 + \varepsilon)^2 + (|\tau|^2 - \varepsilon^2)^2\right). \quad (3.2)$$

Its gradient components are

$$\begin{aligned} D_{\tau_1} W(\tau) &= 8\tau_1(|\tau|^2 + 1 - \varepsilon^2)\{2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2\}; \\ D_{\tau_2} W(\tau) &= 8\tau_2(|\tau|^2 + 1 - \varepsilon^2)\{2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2\} - 32\tau_2\varepsilon^2. \end{aligned}$$

The Hessian components are

$$\begin{aligned} w_{11} &:= D_{\tau_1\tau_1} W(\tau) \\ &= 8(|\tau|^2 + 1 - \varepsilon^2 + 2\tau_1^2)\{2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2\} + 32\tau_1^2(|\tau|^2 + 1 - \varepsilon^2)^2; \\ w_{12} &= w_{21} \\ &:= D_{\tau_1\tau_2} W(\tau) \\ &= 16\tau_1\tau_2\{2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2\} + 32\tau_1\tau_2(|\tau|^2 + 1 - \varepsilon^2)^2; \\ w_{22} &:= D_{\tau_2\tau_2} W(\tau) \\ &= 8(|\tau|^2 + 1 - \varepsilon^2 + 2\tau_2^2)[2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2] + 32[\tau_2^2(|\tau|^2 + 1 - \varepsilon^2)^2 - \varepsilon^2]. \end{aligned}$$

**3.2. 1D shear model I:**  $\tau_3 \equiv 1$ ;  $\tau_2 \equiv 0$ . The elastic potential becomes

$$W(\tau) = \left(2\tau_1^2 + 2\varepsilon^2 + (\tau_1^2 - \varepsilon^2)^2\right)^2. \quad (3.3)$$

The first order derivative is

$$D_{\tau_1} W(\tau) = 8\tau_1(\tau_1^2 + 1 - \varepsilon^2)\{2(\tau_1^2 + \varepsilon^2) + (\tau_1^2 - \varepsilon^2)^2\}.$$

The second order derivative is

$$D_{\tau_1\tau_1} W(\tau) = 8(3\tau_1^2 + 1 - \varepsilon^2)\{2(\tau_1^2 + \varepsilon^2) + (\tau_1^2 - \varepsilon^2)^2\} + 32\tau_1^2(\tau_1^2 + 1 - \varepsilon^2)^2.$$

**3.3. 1D shear model II:**  $\tau_3 \equiv 1$ ;  $\tau_1 \equiv 0$ . Correspondingly, the elastic potential becomes

$$W(\tau) = \left(2(\tau_2 - \varepsilon)^2 + (\tau_2^2 - \varepsilon^2)^2\right) \times \left(2(\tau_2 + \varepsilon)^2 + (\tau_2^2 - \varepsilon^2)^2\right) \quad (3.4)$$

The first order derivative is

$$D_{\tau_2} W(\tau) = 8\tau_2(\tau_2^2 + 1 - \varepsilon^2)\{2(\tau_2^2 + \varepsilon^2) + (\tau_2^2 - \varepsilon^2)^2\} - 32\tau_2\varepsilon^2;$$

The second order derivative is

$$D_{\tau_2\tau_2} W(\tau) = 8(3\tau_2^2 + 1 - \varepsilon^2)\{2(\tau_2^2 + \varepsilon^2) + (\tau_2^2 - \varepsilon^2)^2\} + 32\{\tau_2^2(\tau_2^2 + 1 - \varepsilon^2)^2 - \varepsilon^2\}.$$

**3.4. 1D compressible model III.** In this case  $\tau_1 = \tau_2 \equiv 0$  and we denote  $\tau = \tau_3$ . The potential and its derivatives are given below. The elastic potential becomes

$$W(\tau) = \left(2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)^2\right)^2 \quad (3.5)$$

The first order derivative is

$$D_{\tau_3}W(\tau) = 8\tau_3(\tau_3^2 - 1 - \varepsilon^2) \{2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)^2\}.$$

The second order derivative is

$$w_{33} := D_{\tau_3\tau_3}W(\tau) = 8(3\tau_3^2 - 1 - \varepsilon^2)\{2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)^2\} + 32\tau_3^2(\tau_3^2 - 1 - \varepsilon^2)^2.$$

**3.5. 2D compressible models.** First, we consider the case  $\tau = (\tau_2, \tau_3)^T \in \mathbb{R}_+^2$ . The elastic potential  $W$  and derivatives are as follows.

$$W(\tau) = \left(2(\tau_2 - \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)\right) \left(2(\tau_2 + \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)\right). \quad (3.6)$$

The gradient components are

$$\begin{aligned} D_{\tau_2}W(\tau) &= 8\tau_2(|\tau|^2 - \varepsilon^2)\{2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} - 32\tau_2\varepsilon^2 \\ D_{\tau_3}W(\tau) &= 8\tau_3(|\tau|^2 - 1 - \varepsilon^2)\{2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\}. \end{aligned}$$

Similarly, we have the Hessian components

$$\begin{aligned} w_{22} &:= D_{\tau_2\tau_2}W(\tau) \\ &= 8(|\tau|^2 - \varepsilon^2 + 2\tau_2^2)[2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2] + 32[\tau_2^2(|\tau|^2 - \varepsilon^2)^2 - \varepsilon^2]; \\ w_{23} &= w_{32} \\ &:= D_{\tau_3\tau_2}W(\tau) \\ &= 16\tau_2\tau_3\{2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_2\tau_3(|\tau|^2 - \varepsilon^2)(|\tau|^2 - 1 - \varepsilon^2); \\ w_{33} &:= D_{\tau_3\tau_3}W(\tau) \\ &= 8(|\tau|^2 - 1 - \varepsilon^2 + 2\tau_3^2)\{2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_3^2(|\tau|^2 - 1 - \varepsilon^2)^2. \end{aligned}$$

Second, if we fix the  $\tau_2$  direction and let  $\tau := (\tau_1, \tau_3)^T$ , we get another 2D compressible model. We omit the details here as the form is obvious.

**3.6. The full 3D model.** In this case  $\tau = (\tau_1, \tau_2, \tau_3)^T \in \mathbb{R}_+^3$ . corresponding to the phase-transitional elastic potential function  $W$ , we list the components of  $D^2W(\tau) := (w_{ij})_{3 \times 3}$ .

$$w_{11} = 8(|\tau|^2 + 2\tau_1^2 - \varepsilon^2) \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_1^2(|\tau|^2 - \varepsilon^2)^2;$$

$$w_{12} = w_{21}$$

$$= 16\tau_1\tau_2 \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_1\tau_2(|\tau|^2 - \varepsilon^2)^2;$$

$$w_{13} = w_{31}$$

$$= 16\tau_1\tau_3 \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_1\tau_3(|\tau|^2 - \varepsilon^2)(|\tau|^2 - 1 - \varepsilon^2);$$

$$w_{23} = w_{32}$$

$$= 16\tau_2\tau_3 \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_2\tau_3(|\tau|^2 - \varepsilon^2)(|\tau|^2 - 1 - \varepsilon^2);$$

$$w_{33} = 8(|\tau|^2 + 2\tau_3^2 - 1 - \varepsilon^2) \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_3^2(|\tau|^2 - 1 - \varepsilon^2)^2;$$

$$w_{22} = 8(|\tau|^2 - \varepsilon^2 + 2\tau_2^2) [2(\tau_1^2 + \tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2] + 32[\tau_2^2(|\tau|^2 - \varepsilon^2)^2 - \varepsilon^2].$$

**3.7. Justification of phenomenological models.** We note that, for the case of 1D shear flow, the coefficients given by (2.4) become  $b, d \equiv \text{constant}$ , and the elastic potential  $W$  is of a generalized double-well form. Thus, we recover from first principles the type of phenomenological model studied in [19–22, 24], though with a slightly modified potential refining the quartic double-well potential assumed in the phenomenological models. The 2D shear flow gives a natural extension to multi-dimensional deformations, which is also interesting from the pure Calculus of Variations point of view (see the following section), as a physically relevant example of a vectorial “real Ginzburg-Landau” problem of the type studied on abstract grounds by many authors. Finally, we note that the various compressible models give a different extension of the phenomenological models, to the case of “real” or nonconstant viscosity.

## 4. Existence of periodic solutions for specific models

In this section, we focus on the existence of periodic waves for the incompressible models (i.e. models with  $\tau_3 \equiv 1$ ), namely the 1D models I, II and the 2D incompressible model. First note that these models do not involve the  $\tau_3$  direction. Hence we have no condition corresponding to (A1) in [23]<sup>1</sup>. However, the specific phase-transitional elastic potential energy function  $W$  has a good growth rate when  $|\tau| \rightarrow +\infty$  for  $\tau = \tau_1, \tau_2$  or  $(\tau_1, \tau_2)$ . This will make our functionals coercive. Hence we have the following:

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<sup>1</sup>The condition that  $W(\tau) \rightarrow +\infty$  as  $\tau_3 \rightarrow 0^+$ .

**Theorem 4.1.** *For the incompressible models, there exist non-constant periodic standing waves if the mean  $m$  (either vector or scalar) satisfies*

$$\sigma\{D^2W(m)\} \cap \mathbb{R}_-^1 \neq \emptyset; \quad \left(\frac{2\pi}{T}\right)^2 < \lambda(m). \quad (4.1)$$

When  $m$  is scalar, assumption “ $\sigma\{D^2W(m)\} \cap \mathbb{R}_-^1 \neq \emptyset$ ” means that  $m$  lies in the elliptic region of the viscoelasticity system (2.1).

*Proof.* It is easy to see the corresponding functionals are coercive, weakly lower semi-continuous functionals on the reflexive Banach spaces  $H_{T,0}^1$ . Hence Corollary 6.11 applies. The verification that the global minimizers respectively are not zero is entirely the same as in [23, Lemma 6.18] by considering the second variation.  $\square$

Next, we specify the corresponding conditions in Theorem 4.1 for these incompressible models.

**4.1. 1D shear model I.** The condition is

$$\left(\frac{2\pi}{T}\right)^2 < -8(3m^2 + 1 - \varepsilon^2)\{2(m^2 + \varepsilon^2) + (m^2 - \varepsilon^2)^2\} - 32m^2(m^2 + 1 - \varepsilon^2)^2. \quad (4.2)$$

In particular, if  $m = 0$ , the condition reads

$$\left(\frac{2\pi}{T}\right)^2 < -8(1 - \varepsilon^2)(2\varepsilon^2 + \varepsilon^4). \quad (4.3)$$

Condition (4.2) illustrate that our assumption is not a void assumption. Also, in the mean zero case, (4.2) holds only if  $\varepsilon > 1$ . Comparing this with the existence result by phase-plane analysis (see in particular Section 5.1), we see that these results match very well.

**4.2. 1D shear model II.** The condition is

$$\begin{aligned} \left(\frac{2\pi}{T}\right)^2 &< -8(3m^2 + 1 - \varepsilon^2)\{2(m^2 + \varepsilon^2) + (m^2 - \varepsilon^2)^2\} \\ &\quad - 32\{m^2(m^2 + 1 - \varepsilon^2)^2 - \varepsilon^2\}. \end{aligned} \quad (4.4)$$

In particular, if  $m = 0$ , the condition reads

$$\left(\frac{2\pi}{T}\right)^2 < -8(1 - \varepsilon^2)(2\varepsilon^2 + \varepsilon^4) + 32\varepsilon^2 = 8(\varepsilon^6 + \varepsilon^4 + 2\varepsilon^2). \quad (4.5)$$

Condition (4.4) implies in particular that for any  $\varepsilon > 0$ , we have long-periodic oscillatory waves. Similarly, for any given  $T > 0$ , we have oscillatory waves as long as  $\varepsilon > 0$  large enough. Comparing with the phase-plane analysis (see Section 5.2), we see that the related obtained wave phenomena match very well.



**4.3. 2D incompressible shear model.** In this case  $D^2W(m)$  is given by its components

$$\begin{aligned} w_{11} &:= D_{\tau_1\tau_1}W(m) \\ &= 8(|m|^2+1-\varepsilon^2+2m_1^2)\{2(|m|^2+\varepsilon^2)+(|m|^2-\varepsilon^2)^2\}+32m_1^2(|m|^2+1-\varepsilon^2)^2; \\ w_{12} &= w_{21} \\ &:= D_{\tau_1\tau_2}W(m) \\ &= 16m_1m_2\{2(|m|^2+\varepsilon^2)+(|m|^2-\varepsilon^2)^2\}+32m_1m_2(|m|^2+1-\varepsilon^2)^2; \\ w_{22} &:= D_{\tau_2\tau_2}W(m) \\ &= 8(|m|^2+1-\varepsilon^2+2m_2^2)[2(|m|^2+\varepsilon^2)+(|m|^2-\varepsilon^2)^2]+32[m_2^2(|m|^2+1-\varepsilon^2)^2-\varepsilon^2]. \end{aligned}$$

The corresponding condition is  $(\frac{2\pi}{T})^2 < \lambda(m)$ . This is obviously a rather mild condition. To see this, we can consider in particular the mean  $m = (m_1, 0)^T$  or  $(0, m_2)^T$ . Then the results on the two 1D incompressible models readily give the conclusion because we have diagonal matrices.

Based on the analysis of these conditions, we have in particular ( $m = 0$  case):

**Theorem 4.2.** *For the 2D shear model, 1D shear model I and II, we have the following existence result of periodic viscous traveling/standing waves:*

- (1) *Given any  $\varepsilon > \varepsilon_0 \geq 0$ , for any  $T$  satisfying  $T > T(\varepsilon) > 0$ , system (3.1) hence (2.1) has a nonconstant periodic solution with some appropriate integral constant  $q$ ; For the 1D model II we have  $\varepsilon_0 = 0$ .*
- (2) *Given any  $T > 0$ , for any  $\varepsilon$  satisfying  $\varepsilon > \varepsilon(T) > 0$ , system (3.1) hence (2.1) has a nonconstant periodic solution with some appropriate integral constant  $q$ .*

**Remark 4.3.** From the above theorem, we see that for the 2D shear model, we have infinitely many nontrivial periodic viscous traveling waves with appropriate corresponding  $q$  values. In particular, we have a sequence of waves with minimum positive period  $T \rightarrow +\infty$ .

## 5. 1D existence by phase-plane analysis

In this section, we discuss how to generate periodic waves for 1D models. In (3.1), the integral constant  $q = D_\tau W(\tau_-) - \tau_-''$ . Here  $(\tau_-, \tau_-'' )$  is the vector evaluated at some specific space value  $x_0$ . If there indeed exist periodic-T waves,  $q = \frac{1}{T} \int_0^T DW(\tau(x)) dx$ .

By the variational formulation and the usual bootstrap argument, we conclude that the periodic waves are classical solutions of the system (3.1). Hence there must be points (say  $x_0, x_1$ ) in a period  $[0, T]$  such that  $\tau'(x_0) = 0$  and

$\tau''(x_1) = 0$ , etc., if such periodic solution did exist for  $\tau$  scalar. The reason is that  $\tau(x)$  cannot be always monotone and convex in view of periodicity (this applies to all derivatives). Hence we can make the integration constant have the form  $q = D_\tau W(\tau(x_1))$  for convenience a priori. Then we can show existence, which in turn guarantees the a priori assumption. Hence, we could assume  $q = D_\tau W(\tau_-)$  to show existence. We adopt this convention in the following analysis.

The guiding principle is that the ODE systems are planar Hamiltonian systems. To get complete and clear pictures of the phase-portraits, we just need to specify the “potential energy” term  $V(\tau, \tau_-)$  in the Hamiltonian  $H(\tau, \tau')$ .

**5.1. 1D shear model I:**  $\tau_2 \equiv 0$ ;  $\tau_3 \equiv 1$ . In this section, we denote  $\tau = \tau_1$ . We use similar notation in other sections. Recall the elastic potential  $W(\tau) = \left(2\tau_1^2 + 2\varepsilon^2 + (\tau_1^2 - \varepsilon^2)^2\right)^2$  and its first and second order derivatives

$$\begin{aligned} D_{\tau_1} W(\tau) &= 8\tau_1(\tau_1^2 + 1 - \varepsilon^2)\{2(\tau_1^2 + \varepsilon^2) + (\tau_1^2 - \varepsilon^2)^2\}; \\ D_{\tau_1\tau_1} W(\tau) &= 8(3\tau_1^2 + 1 - \varepsilon^2)\{2(\tau_1^2 + \varepsilon^2) + (\tau_1^2 - \varepsilon^2)^2\} + 32\tau_1^2(\tau_1^2 + 1 - \varepsilon^2)^2. \end{aligned}$$

The traveling wave ODE and corresponding Hamiltonian system are

$$\tau'' = W'(\tau) - W'(\tau_-). \tag{5.1}$$

$$\begin{cases} \tau' = \tau' \\ \tau'' = W'(\tau) - W'(\tau_-). \end{cases} \tag{5.2}$$

We write the Hamiltonian system as follows:

$$\frac{|\tau'|^2}{2} = H(\tau, \tau') - V(\tau; \tau_-) \equiv E - V(\tau, \tau_-). \tag{5.3}$$

Here  $E$  are constants corresponding to energy level curves of  $H(\tau, \tau')$  and  $V(\tau, \tau_-) := q\tau - W(\tau)$ .

First, we determine the number of equilibria of the Hamiltonian system, hence focus on the solution of  $W'(\tau) = q$ .

Note that  $W'(\tau)$  is an odd function on the real line, hence we just need to study its graph on the interval  $(0, \infty)$ . In view of the expression of  $W'(\tau)$ , we need consider the cases: (1)  $0 < \varepsilon \leq 1$ ; (2)  $\varepsilon > 1$ .

For the case  $0 < \varepsilon < 1$ , we have  $W''(\tau) > 0$  for  $\tau$  real, hence  $W'(\tau)$  is strictly monotone increasing and

$$W'(0) = 0, \quad W'(\tau) > 0 \text{ for } \tau > 0, \quad W'(\tau) < 0 \text{ for } \tau < 0.$$

Hence in this case, for any given  $\tau_-$ , the solution of  $W'(\tau) = W'(\tau_-)$  is  $\tau_-$  and unique.

Similar analysis holds true for  $\varepsilon = 1$ . Considering the definition of  $V(\tau, \tau_-)$ , we have

**Proposition 5.1.** *When  $0 < \varepsilon \leq 1$ ,  $V(\tau; \tau_-)$  has exactly one critical point  $\tau_-$ , which must be a global maximum.*

**Remark 5.2.** In this case, our Hamiltonian system admits no periodic orbit for any  $\tau_-$  (or equivalently, for any  $q$ ).

Next, consider the case  $\varepsilon > 1$ . In this case, we can see from the expression of  $W'(\tau)$  that  $W'(\tau)$  has three distinct zeros:  $-\sqrt{\varepsilon^2 - 1}$ ,  $0$ ,  $\sqrt{\varepsilon^2 - 1}$ . A qualitative graph of  $W'(\tau_1)$  is as follows (see Figure 1):

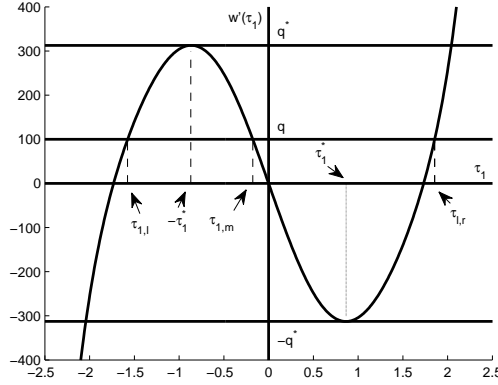


Figure 1: Graph of  $W'(\tau_1)$

The corresponding graph of the potential  $V(\tau_1, \tau_{1,-})$  is shown in Figure 2.

**Proposition 5.3.** *The function  $W'(\tau)$  has exactly two critical points.*

*Proof.* By symmetry, we do the following computations: Denote  $\tau_1^2 := X$  and  $\varepsilon^2 := a > 1$ . We want to show that the function

$$f(X) := (3X + 1 - a)[2(X + a) + (X - a)^2] + 4X(X + 1 - a)^2$$

has exactly one zero when  $X > 0$ .

First, noting that  $f(0) < 0$  and  $f(\frac{a-1}{3}) > 0$ , we know that  $f(X)$  has a root on  $(0, \frac{a-1}{3})$ . Also note that  $f(X) > 0$  on  $[\frac{a-1}{3}, \infty)$ , hence we just need to show that  $f(X)$  admits a unique zero on  $(0, \frac{a-1}{3})$ . Computing the derivative, we have

$$f'(X) = 3[7X^2 + 10(1 - a)X + a^2 + 2a + 2(1 - a)^2].$$

Denote

$$\Delta = 100(1 - a)^2 - 28[a^2 + 2a + 2(1 - a)^2].$$

If  $\Delta \leq 0$ , we know that  $f'(X) \geq 0$ , hence  $f(X)$  is monotone increasing, which implies that  $f(X)$  admits a unique zero;

If  $\Delta > 0$ , we will have two positive roots for  $f'(X) = 0$  and the smaller one is  $\frac{10(a-1)-\sqrt{\Delta}}{14}$ . However, we can show that  $\frac{10(a-1)-\sqrt{\Delta}}{14} \geq \frac{a-1}{3}$ , hence the function  $f(X)$  is monotone increasing on the interval  $(0, \frac{a-1}{3})$ , which also implies the uniqueness of the zero.  $\square$

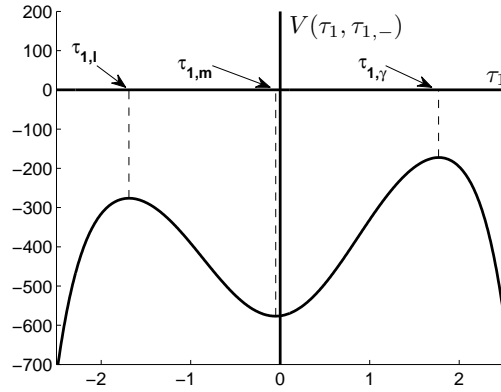


Figure 2: Graph of  $V(\tau_1, \tau_1, -)$

Now we have a clear picture on the potential  $W'(\tau)$  (see the graph of  $W'(\tau_1)$  in Figure 1).

**Proposition 5.4.** *When  $\varepsilon > 1$ , the function  $W'(\tau)$  is an odd function with 3 zeros and 2 critical points and goes to infinity when  $\tau \rightarrow +\infty$ .*

Denote the two critical values of  $W'(\tau)$  as  $q^* > 0$  and  $-q^*$ , for convenience denoting  $Q = q^*$ . Then we have the following property:

**Proposition 5.5.** *Assume  $\varepsilon > 1$ . When  $|q| > Q$ , the equation  $W'(\tau) = q$  has exactly one solution. When  $|q| = Q$ , the equation  $W'(\tau) = q$  has exactly 2 solutions. When  $|q| < Q$ , the equation  $W'(\tau) = q$  has exactly 3 solutions.*

As the solutions of  $W'(\tau) = q$  correspond to the critical points of  $V(\tau; \tau_-)$ , we have:

**Theorem 5.6.** *For  $|q| \geq Q$ , the Hamiltonian system admits no periodic orbit. For  $|q| < Q$ , the Hamiltonian system admits a family of nontrivial periodic orbits. Further if  $q = 0$ , the Hamiltonian system also admits a heteroclinic orbit.*

*Proof.* For  $|q| \geq Q$ , we know that  $V(\tau; \tau_-)$  has a global maximum and hence the Hamiltonian admits no periodic orbit. For the case  $|q| < Q$ , we know that the potential  $V(\tau; \tau_-)$  must have exactly 3 critical points with 2 local maxima

and 1 local minimum. Also, we know that  $V(\tau; \tau_-)$  has strictly lower energy at the local minimum than at the two local maxima. Hence the existence of a family of periodic orbits follows. Further, when  $q = 0$ , the energies at the two local maxima of  $V(\tau; \tau_-)$  are the same, hence we get an heteroclinic orbit.  $\square$

**Remark 5.7.** We may compare the two energy values of  $V(\tau; \tau_-)$  at the two local maxima. If they are equal (when  $q = 0$  in particular), we have a heteroclinic orbit. In general, they are not equal to each other, which yields a homoclinic orbit.

**5.2. 1D shear model II:**  $\tau_1 \equiv 0; \tau_3 \equiv 1$ . Recall the elastic potential

$$W(\tau) = \left(2(\tau_2 - \varepsilon)^2 + (\tau_2^2 - \varepsilon^2)^2\right) \times \left(2(\tau_2 + \varepsilon)^2 + (\tau_2^2 - \varepsilon^2)^2\right)$$

and the relevant derivatives

$$\begin{aligned} D_{\tau_2} W(\tau) &= 8\tau_2(\tau_2^2 + 1 - \varepsilon^2)\{2(\tau_2^2 + \varepsilon^2) + (\tau_2^2 - \varepsilon^2)^2\} - 32\tau_2\varepsilon^2; \\ D_{\tau_2\tau_2} W(\tau) &= 8(3\tau_2^2 + 1 - \varepsilon^2)\{2(\tau_2^2 + \varepsilon^2) + (\tau_2^2 - \varepsilon^2)^2\} + 32\{\tau_2^2(\tau_2^2 + 1 - \varepsilon^2)^2 - \varepsilon^2\}. \end{aligned}$$

As above, we list the Hamiltonian system and the potential  $V(\tau; \tau_-)$ . The system is:

$$\begin{cases} \tau' = \tau' \\ \tau'' = W'(\tau) - W'(\tau_-). \end{cases} \tag{5.4}$$

The potential is  $V(\tau; \tau_-) = q\tau - W(\tau)$ .

**Remark 5.8.** There is a slight difference with the 1D shear model I in the function  $W'(\tau)$ . Because of this difference, we do not need to restrict the positive number  $\varepsilon$  to get periodic orbits for the parameter  $q$  in proper range. The conclusions are completely the same when  $1 \geq \varepsilon > 0$  as in 1D shear model I when  $\varepsilon > 1$ .

We have the following:

**Proposition 5.9.** *When  $1 \geq \varepsilon > 0$ , the behavior of the function  $W'(\tau)$  is the same as that of the function  $W'(\tau)$  in the 1D shear model I when  $\varepsilon > 1$ . In fact,  $W''(\tau)$  is monotone increasing in this case for  $\tau > 0$ .*

**Remark 5.10.** For the range  $\varepsilon > 1$ , numerics suggest that the behaviors are also the same as we may show that the function  $W'(\tau)$  has exactly three solutions and two critical points. We have a small problem to verify this by direct computation though we just need to show that  $f(X) > 0$  evaluated at the larger root of  $f'(X)$  ( $f(X)$  is defined similar as in 1D shear model I as the second derivatives of the two potentials differ with a constant  $32\varepsilon^2$ ). Even without this, we still can conclude the existence of periodic orbits since the potential  $V(\tau; \tau_-)$  admits a minimum. Together with the existence obtained by variational argument, we know that there are still infinitely many nontrivial periodic waves for any  $\varepsilon > 0$ .

**5.3. 1D compressible model III.** For this model, we need to pay special attention to the physical restriction  $\tau_3 > 0$  when we do the phase-plane analysis. To find physical waves, we use a continuity argument and a simple comparison criterion.

In this case  $\tau_1 = \tau_2 \equiv 0$ ; let  $\tau = \tau_3$ . The potential and its derivatives are given below. The elastic potential becomes  $W(\tau) = \left(2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)\right)^2$ . Its first and second order derivative are

$$D_{\tau_3}W(\tau) = 8\tau_3(\tau_3^2 - 1 - \varepsilon^2)\{2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)\}$$

and

$$w_{33} := D_{\tau_3\tau_3}W(\tau) = 8(3\tau_3^2 - 1 - \varepsilon^2)\{2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)\} + 32\tau_3^2(\tau_3^2 - 1 - \varepsilon^2)^2.$$

We write  $V = V(\tau, q, \varepsilon) := q\tau - W(\tau)$  in this section to emphasize the analytical dependence of  $V$  on the parameters  $q$  and  $\varepsilon$  (because  $V$  is a polynomial). As in previous sections, we see:

**Proposition 5.11.** (1)  $W'(\tau) = q$  always has one, two or three roots when  $|q| > Q$ ,  $|q| = Q$  or  $|q| < Q$  for some positive  $Q$ . In the case that  $W'(\tau) = q$  has 3 distinct roots, we denote them from small to large by  $\tau_l, \tau_m$  and  $\tau_r$ .

(2)  $W'(\tau)$  has exactly two critical points.

As before, in order to analyze the existence of periodic, homoclinic or heteroclinic waves, we just need to consider the potential energy  $V(\tau, q, \varepsilon)$ . Further, in order to have physical waves, we need necessarily that  $-Q < q < 0$ . In this situation, the two roots  $\tau_m$  and  $\tau_r$  of  $W'(\tau) = q$  are positive. Noticing that  $\tau = \tau_m$  is a local minimizer of  $V(\tau, q, \varepsilon)$ , there is a periodic annulus around  $\tau_m$ . Hence we have the following proposition:

**Proposition 5.12.** When  $-Q < q < 0$ , there always exists a periodic annulus.

To show existence of physical homoclinic orbit, we just need to compare the values of  $V(0, q, \varepsilon) =: V(0)$  and  $V(\tau_r, q, \varepsilon) =: V(r)$ . We have

**Proposition 5.13.** Let  $-Q < q < 0$ . If  $V(0) > V(r)$ , there is a physical homoclinic orbit; If  $V(0) \leq V(r)$ , there is no physical homoclinic orbit.

In particular, for the case  $q=0$ , the 3 distinct roots of  $W'(\tau)=q$  are easily seen to be  $\tau_l = -\sqrt{1+\varepsilon^2}$ ,  $\tau_m = 0$  and  $\tau_r = \tau_l$ . So  $V(0, 0, \varepsilon) = -(2\varepsilon^2 + (1+\varepsilon^2)^2)^2 < V(\tau_r, 0, \varepsilon) = -(2\varepsilon^2)^2$ . By continuity, we have the following conclusion:

**Proposition 5.14.** There exists a constant  $\eta > 0$  such that if  $-\eta < q \leq 0$ , then there exists no physical homoclinic orbit.

*Proof.* When  $q = 0$ ,  $V(0, 0, \varepsilon) < V(\tau_r, 0, \varepsilon)$ . Thus, by continuous dependence and Propositions 8.11 and 8.13, we have the relation  $V(0) < V(m)$  holds when  $q < 0$  is small and the conclusion holds.  $\square$

Next, we study the existence of physical homoclinic waves when  $-Q < q < 0$  is large. For this purpose, we first set  $\varepsilon = 0$  then proceed by a perturbation argument. When  $\varepsilon = 0$ , the corresponding elastic energy function and its derivatives are:

$$\tilde{W}(\tau) = (\tau^2 - 1)^4; \quad \tilde{W}'(\tau) = 8\tau(\tau^2 - 1)^3; \quad \tilde{W}''(\tau) = 8(\tau^2 - 1)^2(7\tau^2 - 1).$$

Note that  $\tilde{W}''(\tau) = 0$  has roots  $\tau = \pm 1, \pm\sqrt{\frac{1}{7}}$  (this can be easily computed).

For this potential  $\tilde{V}(\tau, q, \varepsilon = 0) := q\tau - \tilde{W}(\tau)$ , we need  $|q| < \tilde{W}'\left(-\sqrt{\frac{1}{7}}\right) = 8\left(\frac{6}{7}\right)^3\sqrt{\frac{1}{7}}$  to have a homoclinic wave. For physical ones, we need  $-8\left(\frac{6}{7}\right)^3\sqrt{\frac{1}{7}} < q < 0$ . Consider the case  $q \rightarrow -8\left(\frac{6}{7}\right)^3\sqrt{\frac{1}{7}}$  from the right, we see the largest root  $\tau_m$  of  $\tilde{W}'(\tau) = q$  tends to  $\sqrt{\frac{1}{7}}$ . Consequently,  $\tilde{V}(\tau = 0, q, \varepsilon = 0) \rightarrow -\tilde{W}(0) = -1$  and the right local maximum value of  $\tilde{V}(\tau_m, q, \varepsilon = 0) \rightarrow -8\left(\frac{6}{7}\right)^3\sqrt{\frac{1}{7}} - \tilde{W}\left(\sqrt{\frac{1}{7}}\right) < -1$ . By Proposition 8.11 and Proposition 8.13, we have:

**Proposition 5.15.** *For the compressible 1D model, assume  $-Q < q < 0$  ( $Q$  as in Proposition 8.11). Then, when  $\varepsilon > 0$  and  $q + Q$  are small, we have a physical homoclinic orbit.*

## 6. Time-evolutionary stability

We conclude by discussing briefly the question of time-evolutionary stability of elastic traveling waves with strain-gradient effects.

**6.1. Spectral vs. nonlinear stability.** A very useful observation regarding the earlier phenomenological models  $\tau_t - u_x = 0$ ,  $u_t + dW(\tau)_x = bu_{xx} - d\tau_{xxx}$ ,  $b, d > 0$  constant, for 1D shear flow, made by Slemrod and used by Schechter and Shearer in [19], was that for a wide range of  $b, d$ , specifically,  $d < \frac{b^2}{4}$ , the system can be transformed by the change of independent variable  $u \rightarrow \tilde{u} := u - c\tau_x$ ,  $c(b - c) = d$  to the fully parabolic system

$$\begin{aligned} \tau_t - \tilde{u}_x &= c\tau_{xx} \\ \tilde{u}_t + dW(\tau)_x &= (b - c)\tilde{u}_{xx}, \end{aligned}$$

thus allowing the treatment of nonlinear stability by standard parabolic techniques, taking into account, for example, sectorial structure, parabolic smoothing, etc.

Quite recently, this observation has been profoundly generalized by M. Kotschote [15], who showed that a somewhat different transformation in similar spirit may be used to convert elasticity or fluid-dynamical equations with strain-gradient (resp. capillarity) effects to quasilinear fully parabolic form, *in complete generality*, not only to the cases  $4d > b^2$  previously uncovered for the phenomenological model, but to the entire class of physical models considered here. For further discussion/description of this transformation, see Section 6.1.1.

This reduces the question of nonlinear stability to a standard format already well studied. In particular, it follows that (except possibly in nongeneric boundary cases of neutrally stable spectrum) *nonlinear stability is equivalent to spectral stability*, appropriately defined. This follows for heteroclinic and homoclinic waves by the analysis of [13], and for periodic waves by the analysis of [14].<sup>2</sup> For precise definitions of the notions of spectral stability, we refer the reader to those references; in the shock wave (heteroclinic or homoclinic) case, see also the discussion of [8]. Spectral stability may be efficiently determined numerically by Evans function techniques, as in for example [4–8]. We intend to carry out such a numerical study in a followup work [9].

**6.1.1. Transformation to strictly parabolic form.** We now show how to apply the approach of Kotschote in our context and verify that we thereby obtain the structural properties needed to apply the general theory of [13]. So we shall in the following verify the structural properties of the elastic model with strain-gradient effect and related modified systems obtained by the approach of Kotschote [15]. Introducing the phase variable  $z := \tau_x$ , we may write (2.3) as a quasilinear second-order system

$$\begin{cases} \tau_t + z_x - u_x = \tau_{xx} \\ z_t = u_{xx} \\ u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(z)z_x)_x. \end{cases} \quad (6.1)$$

**Remark 6.1.** This transformation, introduced in [15], is similar in spirit to but more general than the one<sup>3</sup> introduced by Slemrod [20–22] and used in [16] for an artificial viscosity/capillarity model.

<sup>2</sup>The analysis of [14] concerns modulational stability, or stability with respect to localized perturbations on the whole line; co-periodic stability may be treated by standard semigroup techniques [12]. Spectral analyses of [9, 16, 18] suggest that modulational stability occurs rarely if ever for viscoelastic waves.

<sup>3</sup>A transformation  $(\tau, u) \rightarrow (\tau, u - c\tau_x)$  reducing the model to a parabolic system of the same size.



We can slightly modify the above system in the second equation. Then we have the following system

$$\begin{cases} \tau_t + z_x - u_x = \tau_{xx} \\ z_t + z_x = u_{xx} + \tau_{xx} \\ u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(z)z_x)_x. \end{cases} \quad (6.2)$$

If we write the above system (6.3) and (6.4) in matrix form

$$U_t + f(U)_x = (B(U)U_x)_x$$

using the variable  $U := \begin{pmatrix} \tau \\ z \\ u \end{pmatrix}$ , the corresponding matrix  $B$  becomes:

$$\begin{pmatrix} I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & I_3 \\ 0_3 & -d(z) & b(\tau) \end{pmatrix}; \quad \begin{pmatrix} I_3 & 0_3 & 0_3 \\ I_3 & 0_3 & I_3 \\ 0_3 & -d(z) & b(\tau) \end{pmatrix}.$$

Also, the corresponding matrices  $Df(U)$  for the two systems are:

$$\begin{pmatrix} 0_3 & I_3 & -I_3 \\ 0_3 & 0_3 & 0_3 \\ D\sigma(\tau) & 0_3 & 0_3 \end{pmatrix}; \quad \begin{pmatrix} 0_3 & I_3 & -I_3 \\ 0_3 & I_3 & 0_3 \\ D\sigma(\tau) & 0_3 & 0_3 \end{pmatrix}.$$

**Proposition 6.2** (Strict parabolicity). *Systems (6.3) and (6.4) are both strictly parabolic systems in the sense that the spectrum of  $B$  have positive real parts.*

*Proof.* Comparing the two matrices  $B$  above, we know they have the same spectrum. We prove this proposition for  $\tau, z, u \in \mathbb{R}^3$ . The lower dimension cases becomes easier and the computations are totally the same. Pick one of the  $B$ 's, say

$$\begin{pmatrix} I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & I_3 \\ 0_3 & -d(z) & b(\tau) \end{pmatrix}.$$

To compute the spectrum of  $B$ , consider the characteristic polynomial and  $\det(\lambda I_3 - B) = 0$ , which is

$$\det \begin{pmatrix} (\lambda - 1)I_3 & 0_3 & 0_3 \\ 0_3 & \lambda I_3 & -I_3 \\ 0_3 & +d(z) & \lambda - b(\tau) \end{pmatrix} = 0.$$

Doing Laplace expansion and elementary column transformation, we get

$$\det\{(\lambda - 1)I_3\} \det\{(\lambda^2 I_3 - \lambda b(\tau) + d(z))\}$$

$$\begin{aligned}
&= (\lambda - 1)^3 \left( \lambda^2 - \frac{\lambda}{\tau_3} + 1 \right)^2 \left( \lambda^2 - \frac{2\lambda}{\tau_3} + 1 \right) \\
&= 0.
\end{aligned}$$

From the first factor of the above degree 9 polynomial, we get three equal root 1 which has positive real parts. The other 6 roots also have positive real parts noticing that  $\tau_3 > 0$ .  $\square$

**Proposition 6.3** (Nonzero characteristic speeds). *The corresponding first order systems of (6.2) has nonzero characteristic speed at  $\tau$  where the matrix  $D^2W(\tau)$  are strictly positive definite.*

*Proof.* Again, we prove this for  $\tau, z, u \in \mathbb{R}^3$ . To prove the corresponding first order system is non-characteristic, we consider the spectrum of the matrix  $\begin{pmatrix} 0_3 & I_3 & -I_3 \\ 0_3 & I_3 & 0_3 \\ D\sigma(\tau) & 0_3 & 0_3 \end{pmatrix}$ . We get the following system for the characteristic speed:

$$\det(\lambda I_9 - Df(U)) = \begin{pmatrix} \lambda I_3 & -I_3 & I_3 \\ 0_3 & (\lambda - 1)I_3 & 0_3 \\ -D\sigma(\tau) & 0_3 & \lambda I_3 \end{pmatrix} = 0.$$

By direct computation, we know:

$$\det(\lambda I_9 - Df(U)) = \det((\lambda - 1)I_3) \det(-D\sigma(\tau) - \lambda^2 I_3) = 0. \quad (6.3)$$

It is easy to see we have three roots 1 which is not 0. The other roots satisfy the algebraic equation:  $\det(-D\sigma(\tau) - \lambda^2 I_3) = \det(D^2W(\tau) - \lambda^2 I_3) = 0$ . Hence the proposition follows.  $\square$

**Proposition 6.4** (Same spectrum). *For system (6.4) and its first order system, the matrix  $Df(U)$  and  $B^{-1}Df(U)$  have the same spectrum.*

*Proof.* We prove this for the variables  $\tau, z, u \in \mathbb{R}^3$ . It is easy to verify that

$$B^{-1} = \begin{pmatrix} I_3 & 0_3 & 0_3 \\ -b(\tau) & b(\tau) & -I_3 \\ -I_3 & I_3 & 0_3 \end{pmatrix}.$$

Since  $Df(U) = \begin{pmatrix} 0_3 & I_3 & -I_3 \\ 0_3 & I_3 & 0_3 \\ D\sigma(\tau) & 0_3 & 0_3 \end{pmatrix}$ , we immediately get

$$B^{-1}Df(U) = \begin{pmatrix} 0_3 & I_3 & -I_3 \\ -D\sigma(\tau) & 0_3 & b(\tau) \\ 0_3 & 0_3 & I_3 \end{pmatrix}.$$

Considering the corresponding eigenvalue problem, we have:

$$\det \left( \lambda I_3 - B^{-1} Df(U) \right) = \det \begin{pmatrix} \lambda I_3 & -I_3 & I_3 \\ D\sigma(\tau) & \lambda I_3 & -b(\tau) \\ 0_3 & 0_3 & (\lambda - 1)I_3 \end{pmatrix} = 0.$$

Doing Laplace expansion and performing basic transformation, we get:

$$\det(\lambda I_3 - I_3) \det(\lambda^2 I_3 + D\sigma(\tau)) = \det(\lambda I_3 - I_3) \det(\lambda^2 I_3 - D^2 W(\tau)) = 0,$$

which implies the conclusion by noticing (6.3).  $\square$

**6.2. Variational vs. time-evolutionary stability.** More fundamentally, perhaps, there is a relation between variational stability of periodic waves and their time-evolutionary stability as solutions of (2.1). In particular, the energy functional that we minimized in constructing periodic solutions is essentially the functional that defines the mechanical energy of the system, a Lyapunov functional that decreases with the flow of (2.1). This gives a strong link between the two notions of stability.

Moreover, as discussed further in [9], the indirect spectral arguments of [24] on 1D heteroclinic and homoclinic waves extend to the general-dimensional and or periodic case, yielding the much stronger result that variational stability in each of these contexts is *necessary and sufficient* for time-evolutionary stability (co-periodic stability, in the case of periodic waves, and variational stability constrained by a prescribed mean). Moreover, these arguments yield at the same time the curious fact that unstable spectra of the linearized operator about the wave *must, if it exists, be real*. These properties give additional insight, and additional avenues by which time-evolutionary stability may be studied.

*In particular, this shows that the waves we have constructed are the (co-periodically) stable ones.* However, these are not necessarily the only stable waves, as we did not construct all minimizers of the variational problem, but only those with mean satisfying a nonconvexity condition. Just recently (in particular, after the completion of the analysis of this paper), there has been introduced in [18] a different, more direct argument showing equivalence of variational and time-evolutionary stability, which yields at the same time concise conditions for variational stability. These yield in particular that the sharp condition for stability is not the condition of nonconvexity of  $W$  at  $m$ , defined as the mean over one period of  $\tau$ , but rather the “averaged” condition of nonconvexity of the Jacobian with respect to  $m$  of the mean over one period of  $DW(\tau)$ . See [18] for further details.

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