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Pointwise Limits of Sequences of Right-Continuous Functions and Measurability of Functions of Two Variables

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Abstract. In this article I prove that the pointwise limit $f\colon \mathbb{R} \to \mathbb{R}$ of a sequence of right-continuous functions has some special property (G) and that bounded functions of two variables $g\colon \mathbb{R}^2 \to \mathbb{R}$ whose vertical sections g_x, $x \in \mathbb{R}$, are derivatives and horizontal sections g^y, $y \in \mathbb{R}$, are pointwise limits of sequences of right-continuous functions, are measurable and sup-measurable in the sense of Lebesgue.

Keywords. Pointwise convergence, right-continuity, Baire 1 class, derivative, approximate continuity, measurability of functions of two variables

Mathematics Subject Classification (2010). Primary 26A15, secondary 26A21, 26B05, 26B35

1. Introduction

Denote by c_0 be the class of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, by r_0 the class of all right-continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, and by j_0 the class of all regulated functions $f\colon \mathbb{R} \to \mathbb{R}$ (a function $f\colon \mathbb{R} \to \mathbb{R}$ is regulated if for each point $x \in \mathbb{R}$ the both unilateral limits $f(x+)$ and $f(x-)$ exist and are finite). Moreover, let c_1 (resp. r_1 or j_1) be the class of all pointwise limits of sequences of functions from c_0 (resp. from r_0 or j_0). Similarly, if we take pointwise limits of sequences of functions from c_1, r_1 or j_1 we define the classes c_2, r_2 and j_2. In [10], Reed obtained very interesting characterizations of c_1, r_1 and j_1. He proved that $c_1 \subset r_1 \subset j_1$, $c_1 \neq r_1 \neq j_1$ and $c_2 = r_2 = j_2$. Note that Reed's considerations in [10] concern functions from $[0, 1]$ to $\mathbb{R}$, but his theorems are true for functions from $\mathbb{R}$ to $\mathbb{R}$.

In [6, 7] it is proved the Lebesgue measurability of bounded functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ whose vertical sections $g_x(t) = g(x, t)$, $x \in \mathbb{R}$, are derivatives and horizontal sections $g^y(t) = g(t, y)$, $y \in \mathbb{R}$, belong to c_1.

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In [3,4] it is shown that the Continuum Hypothesis (CH) implies that there is a Lebesgue nonmeasurable function $h\colon \mathbb{R}^2 \to [0,1]$ with approximately continuous sections h_x, $x \in \mathbb{R}$, and such that for each $y \in \mathbb{R}$ the set $\mathbb{R} \setminus (h^y)^{-1}(0)$ is countable. Since every function $f\colon \mathbb{R} \to \mathbb{R}$ with the countable set $\mathbb{R} \setminus f^{-1}(0)$ belong to j_1 and every approximately bounded function is a derivative [2], we obtain that CH implies that there is a Lebesgue nonmeasurable function $h\colon \mathbb{R}^2 \to [0,1]$ with vertical sections h_x, $x \in \mathbb{R}$, being derivatives and horizontal sections h^y, $y \in \mathbb{R}$, belonging to j_1.

Hence the following natural question arises.

Problem. Let a bounded function $h\colon \mathbb{R}^2 \to \mathbb{R}$ be such that the vertical sections h_x, $x \in \mathbb{R}$, are derivatives and the vertical sections h^y, $y \in \mathbb{R}$, belong to r_1. Is the function h Lebesgue measurable?

In this article I prove that the answer is affirmative.

2. Main results

In [7] I introduce the following property (G) for the investigation of the Lebesgue measurability of functions of two variables. This definition bases on the notion of the density topology T_d [2].

For a point $x \in \mathbb{R}$ and for a Lebesgue measurable set $A \subset \mathbb{R}$ we define the lower density $D_l(A,x)$ of A at x as

$$\liminf_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h},$$

where μ denotes the Lebesgue measure on $\mathbb{R}$. If $D_l(A,x) = 1$ then x is called a density point of A. If B is arbitrary subset of $\mathbb{R}$ then x is said a density point of B if there is a Lebesgue measurable subset $A \subset B$ with $D_l(A,x) = 1$. A nonempty set $B \subset \mathbb{R}$ belongs to the density topology T_d if every point $x \in B$ is a density point of B. All sets belonging to T_d are Lebesgue measurable [2].

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ has the property (G) if for each nonempty set $A \in T_d$ and each real $\eta > 0$ there is an open interval I with $I \cap A \neq \emptyset$ such that the diameter $d(f(I \cap A))$ of the image of $f(I \cap A)$ is less than η.

Theorem 2.1. *If a function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to r_1 then it has the property* (G)*.*

Proof. Let $A \in T_d$ be a nonempty set and let η be a positive real. Since $f \in r_1$, there is a sequence of continuous on the right functions $f_n\colon \mathbb{R} \to \mathbb{R}$ which pointwise converges to f. For each point $x \in \mathbb{R}$ we find a positive integer $n(x)$ such that

$$|f_k(x) - f(x)| < \frac{\eta}{30} \quad \text{for } k \geq n(x).$$

Denote by B the closure of the set A and for $k \geq 1$ let $A_k = \{x \in B : n(x) = k\}$. Since B is of the second category in itself and since $B = \bigcup_{k \geq 1} A_k$, there is a positive integer m such that A_m is of the second category in B. So there is an open interval I_1 such that $I_1 \cap B \neq \emptyset$ and the intersection $I_1 \cap A_m$ is dense in $I_1 \cap B$. Since $B = \mathrm{cl}(A)$, there is a point $u \in A \cap I_1$. Let $i = \max(m, n(u))$. From the continuity on the right of the function f_i it follows that there is an open interval $I \subset I_1$ for which u is the left endpoint, and such that $|f_i(t) - f_i(u)| < \frac{\eta}{30}$ for $t \in I$. Since u is a density point of A, the intersection $I \cap A \neq \emptyset$. For $w \in I \cap A_m$ we have

$$\begin{aligned} |f(w) - f(u)| &\leq |f(w) - f_i(w)| + |f_i(w) - f_i(u)| + |f_i(u) - f(u)| \\ &< \frac{\eta}{30} + \frac{\eta}{30} + \frac{\eta}{30} = \frac{\eta}{10}. \end{aligned} \tag{$\star$}$$

We will prove that

$$f(t) \in \left[f(u) - \frac{\eta}{3}, f(u) + \frac{\eta}{3}\right] \quad \text{for } t \in I \cap A.$$

Suppose, contrary to our claim, that $|f(s) - f(u)| > \frac{\eta}{3}$ for some point $s \in A \cap I$. Let $j > i$ be a positive integer such that $|f_k(s) - f(s)| < \frac{\eta}{30}$ for $k \geq j$. From the continuity on the right of the function f_j it follows that there is an open interval $K \subset I$ with the left endpoint s such that $|f_j(t) - f_j(s)| < \frac{\eta}{30}$ for $t \in K$. Since s is a density point of A, the set $K \cap A \neq \emptyset$. But $K \subset I$, so the intersection $K \cap A_m$ is dense in $K \cap A$. Consequently, there is a point $w_1 \in A_m \cap I$. We have

$$\begin{aligned} |f(w_1) - f(s)| &\leq |f(w_1) - f_j(w_1)| + |f_j(w_1) - f_j(s)| + |f_j(s) - f(s)| \\ &< \frac{\eta}{30} + \frac{\eta}{30} + \frac{\eta}{30} = \frac{\eta}{10}. \end{aligned}$$

Thus

$$\begin{aligned} |f(w_1) - f(u)| &= |(f(s) - f(u)) + (f(w_1) - f(s))| \\ &\geq |f(s) - f(u)| - |f(w_1) - f(s)| \\ &> \frac{\eta}{3} - \frac{\eta}{10} \\ &> \frac{\eta}{10}, \end{aligned}$$

contradicting ($\star$). So the oscillation of f on $I \cap A$ is $\leq \frac{2\eta}{3} < \eta$ and f has the property (G). □

From the above Theorem 2.1 and from [7, Theorem 4] we obtain the following theorem.

Theorem 2.2. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded function such that the vertical sections g_x, $x \in \mathbb{R}$, are derivatives and the horizontal sections g^y, $y \in \mathbb{R}$, belong to r_1. Then the function g is measurable in the sense of Lebesgue.*

The continuity of functions $f\colon \mathbb{R} \to \mathbb{R}$ considered as some applications from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$, where T_e denotes the natural topology in $\mathbb{R}$, is said approximate continuity [2]. Since bounded approximately continuous functions are derivatives [2], from the above Theorem 2.2 we obtain the following.

Theorem 2.3. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded function. If the vertical sections g_x, $x \in \mathbb{R}$, are approximately continuous and the horizontal sections g^y, $y \in \mathbb{R}$, belong to r_1, then g is measurable in the sense of Lebesgue.*

Observe that in Theorem 2.3 the condition of boundedness of the function g can be omitted, since the class of approximately continuous functions and r_1 are both invariant under outer homeomorphisms.

3. Final observations

3.1. Property (K). Earlier in [5] I introduce the property (K) which is more special than the property (G). A function $f\colon \mathbb{R} \to \mathbb{R}$ has the property (K) if for each nonempty closed set $A \subset \mathbb{R}$ such that for each open interval I with $I \cap A \neq \emptyset$ the intersection $I \cap A$ is of positive Lebesgue measure, the restricted function $f \restriction A$ is continuous at a point $x \in A$. Evidently all functions from c_1 have the property (K) and if a function f has the property (K) then it has also the property (G).

However, there are functions $f \in r_1$ without the property (K).

Example 3.1. Let $A \subset (0, 1)$ be a nonempty nowhere dense closed set such that for each open interval I with $I \cap A \neq \emptyset$ the intersection $I \cap A$ is of positive Lebesgue measure. If $x \in A$ is isolated in A from the right then we put $f(x) = 1$. For other points $x \in \mathbb{R}$ we put $f(x) = 0$. Then evidently $f \in r_1$, but f does not have the property (K).

3.2. Lebesgue sup-measurability. Recall that a function $g\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be Lebesgue sup-measurable if for each Lebesgue measurable function $f\colon \mathbb{R} \to \mathbb{R}$ the Carathéodory superposition $x \mapsto g(x, f(x))$ is Lebesgue measurable [7]. It is known that the Lebesgue measurability of a bounded function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with the vertical sections g_x, $x \in \mathbb{R}$, being derivatives implies its Lebesgue sup-measurability [8]. So from Theorem 2.2 we obtain the following.

Theorem 3.2. *Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded function such that the vertical sections g_x, $x \in \mathbb{R}$, are derivatives and the horizontal sections g^y, $y \in \mathbb{R}$, belong to r_1. Then the function g is Lebesgue sup-measurable.*

In [7] it is shown an example of Lebesgue measurable bounded function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with constant horizontal sections g^y, $y \in \mathbb{R}$, and almost everywhere continuous (so having the property (K)) vertical sections g_x, $x \in \mathbb{R}$, which is not Lebesgue sup-measurable.

On the other hand Borel functions are Lebesgue sup-measurable and bounded functions $g\colon \mathbb{R}^2 \to \mathbb{R}$, whose the vertical sections g_x, $x \in \mathbb{R}$, belong to c_1 and whose the horizontal sections g^y, $y \in \mathbb{R}$, are approximately continuous, are Borel functions of Baire class 2 [9]. On applying the same argument as in the proof of Theorem 1 from [9] we obtain that bounded functions $g\colon \mathbb{R}^2 \to \mathbb{R}$, whose the vertical sections g_x, $x \in \mathbb{R}$, belong to c_1 and whose the horizontal sections g^y, $y \in \mathbb{R}$, are derivatives, are Borel functions of Baire class 2.

In this situation the following natural problems are open.

Problems. Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded function whose the vertical sections $g_x \in r_1$ for $x \in \mathbb{R}$ and the horizontal sections g^y, $y \in \mathbb{R}$, are derivative. Is g

(1) a Borel function?
(2) a Lebesgue sup-measurable function?

In the investigation of the sup-measurability very important role play the numerous contributions of Isaak V. Shragin. In particular, Shragin obtained many closely related results on sup-measurable functions which should be compared with Theorem 3.2 (see for example [11]). Moreover, the book [1] contains a whole chapter dedicated to this topic.

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