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On a Singular Class of Hamiltonian Systems in Dimension Two

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^2$. In this paper, we consider the following systems of semilinear elliptic equations

$$\text{(S)} \begin{cases} -\Delta u = \frac{g(v)}{|x|^a} & \text{in } \Omega \\ -\Delta v = \frac{f(u)}{|x|^b} & \text{in } \Omega \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b \in [0, 2)$ and the functions f and g are nonlinearities having an exponential growth on Ω. This nonlinearity is motivated by suitable Trudinger-Moser inequality with a singular weight. In fact, we prove the existence of a nontrivial solution to (S). For the proof we use a variational argument (a linking theorem).

Keywords. Variational method, Trudinger-Moser inequality, Hamiltonian systems

Mathematics Subject Classification (2010). 34J50, 35J75

1. Introduction

In this paper, we consider the following system of singular elliptic equations

$$\begin{cases} -\Delta u = \dfrac{g(v)}{|x|^a} & \text{in } \Omega \\ -\Delta v = \dfrac{f(u)}{|x|^b} & \text{in } \Omega \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ containing the origin, $a, b \in [0, 2)$, and the functions g and f have the maximal growth which allow us to treat

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problem (1.1) variationally in the Sobolev space $H_0^1(\Omega)$. We are interested in finding nontrivial solution of (1.1) in the space $E := H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the norm

$$\|(u,v)\|_E := (\|u\|^2 + \|v\|^2)^{\frac{1}{2}},$$

where $\|u\| = (\int_\Omega |\nabla u|^2\, dx)^{\frac{1}{2}}$ is the norm of the Sobolev space $H_0^1(\Omega)$.

Motivated by pioneer work of de Figueiredo et al. [6] we treat the so-called subcritical case and also the critical case, which we define next. We say that a function $f(t)$ has subcritical growth at $+\infty$ if for all $\beta > 0$

$$\lim_{t\to+\infty} \frac{f(t)}{e^{\beta t^2}} = 0 \tag{1.2}$$

and $f(t)$ has critical growth at $+\infty$ if there exists $\beta_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{f(t)}{e^{\beta t^2}} = \begin{cases} 0 & \text{for } \beta > \beta_0 \\ +\infty & \text{for } \beta < \beta_0. \end{cases} \tag{1.3}$$

This notion of criticality is motivated by Trudinger-Moser inequality (see [12, 18]) which says that if $u \in H_0^1(\Omega)$ then $e^{\beta u^2} \in L^1(\Omega)$. Moreover, there exists a constant $C > 0$ such that

$$\sup_{\|u\|\le 1} \int_\Omega e^{\beta u^2}\, dx \le C|\Omega|, \quad \text{if } \beta \le 4\pi.$$

We would like to point out that in our present case, we have the presence of a singular term $|x|^{-a}$ which prevents us to use the Trudinger-Moser inequality, so we have to use a version of the Trudinger-Moser inequality with singular weight due to Adimurthi-Sandeep [2](see Lemma 2.1 in the next section). Let us introduce the precise assumptions under which our problem is studied:

(H_1) f and g are continuous functions with $f(t) = o(t)$ and $g(t) = o(t)$ near the origin.

(H_2) There exist constants $\theta > 2$ and t_0 such that

$$0 < \theta F(t) \le tf(t) \quad \text{and} \quad 0 < \theta G(t) \le tg(t) \quad \forall\, |t| \ge t_0,$$

where $F(t) = \int_0^t f(s)\, ds$ and $G(t) = \int_0^t g(s)\, ds$.

It is natural to find solution of our problem by looking for critical points of the corresponding functional of system (1.1) which we define next. The functional associated to (1.1) is given by

$$I(u,v) = \int_\Omega \nabla u \nabla v\, dx - \int_\Omega \frac{F(u)}{|x|^b}\, dx - \int_\Omega \frac{G(v)}{|x|^a}\, dx,$$

in the space $E := H_0^1(\Omega) \times H_0^1(\Omega)$. Under our assumptions this functional is well defined and $C^1(E, \mathbb{R})$. Also, for all $(\varphi, \psi) \in E$, we have

$$I'(u,v)(\varphi,\psi) = \int_\Omega \nabla u \nabla \psi \, dx + \int_\Omega \nabla v \nabla \varphi \, dx - \int_\Omega \frac{f(u)\varphi}{|x|^b} \, dx - \int_\Omega \frac{g(v)\psi}{|x|^a} \, dx.$$

Note that the system (1.1) with $a = b = 0$ and for nonlinearity having polynomial growth have been studied by several authors: de Figuerido and Felmer [5], Dai and Gu [3] and Hulshof et al. [11]. The case $a \neq 0$ and $b \neq 0$ was studied in [4, 8, 10]. On the other hand, the problems of the above type involving critical or subcritical exponential growth and without weights have been investigated in [7, 9, 14].

Our paper is closely related to the recent works of de Figueiredo et al. [7] and Ruf [14]. Indeed, we extend the results in [7] from $a = b = 0$ to $a, b \in [0, 2)$. This limitation on a and b is due to Lemma 2.1.

Our main results are stated as follows.

Theorem 1.1. *If g has subcritical growth, f has subcritical or critical growth and* (H_1), (H_2) *are satisfied then problem* (1.1) *has a nontrivial weak solution* $(u, v) \in E$.

Theorem 1.2. *If g and f have critical growth, $a = b$ and furthermore suppose that*

(H_3) *There exist $M > 0$ and $R > 0$ such that for all $|t| \geq R$*

$$0 < F(t) \leq M f(t) \quad \text{and} \quad 0 < G(t) \leq M g(t).$$

(H_4) *There exists $\beta_0 > 0$ such that*

$$\lim_{t \to +\infty} \frac{t f(t)}{e^{\beta_0 t^2}} > \frac{(2-a)^2}{\beta_0 d^{2-a}} \quad \text{and} \quad \lim_{t \to +\infty} \frac{t g(t)}{e^{\beta_0 t^2}} > \frac{(2-a)^2}{\beta_0 d^{2-a}},$$

where d is the radius of the largest open ball centered at origin and contained in Ω.

Then problem (1.1) *has a nontrivial weak solution* $(u, v) \in E$.

2. Preliminaries

In this paper, we shall use the following version of Trudinger-Moser inequality with a singular weight due to Adimurthi-Sandeep [2].

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ containing* 0 *and $u \in H_0^1(\Omega)$. Then for every $\alpha > 0$ and $a \in [0, 2)$*

$$\int_\Omega \frac{e^{\alpha u^2}}{|x|^a} \, dx < \infty.$$

Moreover,

$$\sup_{\|u\|\leq 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^a}\, dx < \infty \tag{2.1}$$

if and only if $\frac{\alpha}{4\pi} + \frac{a}{2} \leq 1$.

To show that the Palais-Smale sequence is bounded in E, we will use the following inequality whose proof was given in [7].

Lemma 2.2. *The following inequality holds*

$$st \leq \begin{cases} \left(e^{t^2} - 1\right) + s\left(\log^+ s\right)^{\frac{1}{2}}, & \text{for } t \geq 0 \text{ and } s \geq e^{\frac{1}{4}} \\ \left(e^{t^2} - 1\right) + \dfrac{1}{2}s^2, & \text{for } t \geq 0 \text{ and } s \leq e^{\frac{1}{4}}. \end{cases} \tag{2.2}$$

Lemma 2.3. *Let* $u \in H_0^1(\Omega)$ *and* $a \in [0, 2)$*. Then there exist* $C > 0$ *such that*

$$\int_\Omega \frac{|u|^2}{|x|^a}\, dx \leq C\|u\|^2. \tag{2.3}$$

Proof. Using Hölder's inequality, we have

$$\int_\Omega \frac{|u|^2}{|x|^a}\, dx \leq \left(\int_\Omega |x|^{\frac{-ar}{r-2}}\, dx\right)^{\frac{r-2}{r}} \left(\int_\Omega |u|^r\, dx\right)^{\frac{2}{r}},$$

we can choose r such that $r > \frac{4}{2-a}$, therefore $\int_\Omega \frac{|u|^2}{|x|^a}\, dx \leq C\|u\|_r^2$. Finally, by the continuous embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$, we conclude that

$$\int_\Omega \frac{|u|^2}{|x|^a}\, dx \leq C\|u\|^2. \qquad \square$$

We also will use the following convergence result due to M. de Souza and J. Marcos do Ò [17]

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain and* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Then for any sequence* (u_n) *in* $L^1(\Omega)$ *such that*

$$u_n \to u \text{ in } L^1(\Omega), \quad \frac{f(x, u_n)}{|x|^b} \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \frac{|f(x, u_n)u_n|}{|x|^b}\, dx \leq C$$

up to a subsequence we have

$$\frac{f(x, u_n)}{|x|^b} \to \frac{f(x, u)}{|x|^b} \text{ in } L^1(\Omega).$$

Lemma 2.5. *Let (u_n, v_n) be a Palais-Smale sequence for the fonctional I such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. Then (u_n, v_n) has a subsequence, still denoted by (u_n, v_n) such that*

$$\frac{F(u_n)}{|x|^b} \to \frac{F(u)}{|x|^b} \textit{ in } L^1(\Omega) \quad \textit{and} \quad \frac{G(v_n)}{|x|^a} \to \frac{G(v)}{|x|^a} \textit{ in } L^1(\Omega).$$

Proof. From (H_3), we can conclude that

$$|F(u_n)| \le M_1 + M|f(u_n)| \quad \text{and} \quad |G(v_n)| \le M_2 + M|g(v_n)| \tag{2.4}$$

where $M_1 = \sup_{[-R,R]} |F(u_n)|$ and $M_2 = \sup_{[-R,R]} |G(v_n)|$.

On the other hand, from Lemma 2.4, we have

$$\frac{f(u_n)}{|x|^b} \to \frac{f(u)}{|x|^b} \text{ in } L^1(\Omega) \quad \text{and} \quad \frac{g(v_n)}{|x|^a} \to \frac{g(v)}{|x|^a} \text{ in } L^1(\Omega),$$

which implies that there exist h_1, $h_2 \in L^1(\Omega)$ such that

$$\frac{|f(u_n)|}{|x|^b} \le h_1 \quad \text{and} \quad \frac{|g(v_n)|}{|x|^a} \le h_2 \quad \text{almost everywhere in } \Omega. \tag{2.5}$$

Then, by (2.4), (2.5) and Lebesgue dominated convergence theorem, we get

$$\frac{F(u_n)}{|x|^b} \to \frac{F(u)}{|x|^b} \text{ in } L^1(\Omega) \quad \text{and} \quad \frac{G(v_n)}{|x|^a} \to \frac{G(v)}{|x|^a} \text{ in } L^1(\Omega). \qquad \square$$

Remark 2.6. C is a generic positive constant.

3. Linking structure and Palais-Smale sequences

3.1. The geometry of the linking theorem. In this subsection, we verify that the functional I has a linking structure in $(0, 0)$. We use the following notations

$$E^+ = \left\{(u, u) \mid u \in H_0^1(\Omega)\right\} \quad \text{and} \quad E^- = \left\{(u, -u) \mid u \in H_0^1(\Omega)\right\}.$$

Lemma 3.1. *There exist $\rho > 0$ and $\sigma > 0$ such that*

$$I(z) \ge \sigma \quad \forall\, z \in \partial B_\rho \cap E^+.$$

Proof. From (H_1), for given $\varepsilon > 0$ there exists t_0 such that

$$f(t) \le 2\varepsilon t \quad \text{and} \quad g(t) \le 2\varepsilon t \quad \forall\, t \le t_0. \tag{3.1}$$

On the other hand, it follows from (1.2) and (1.3) that, for a given $q > 2$, there exists a constant $C > 0$ and β such that

$$F(t) \leq Ct^q e^{\beta t^2} \quad \text{and} \quad G(t) \leq Ct^q e^{\beta t^2} \quad \forall\, t \geq t_0. \tag{3.2}$$

From (3.1) and (3.2), we get

$$F(t) \leq \varepsilon t^2 + Ct^q e^{\beta t^2} \quad \text{and} \quad G(t) \leq \varepsilon t^2 + Ct^q e^{\beta t^2} \quad \forall\, t \geq 0. \tag{3.3}$$

Notice that using (3.3), Hölder inequality and Lemma 2.3, we have

$$\begin{aligned}\int_\Omega \frac{F(u)}{|x|^b}\,dx &\leq C\varepsilon\|u\|^2 + C\left(\int_\Omega u^{qs'}\,dx\right)^{\frac{1}{s'}}\left(\int_\Omega \frac{e^{s\beta u^2}}{|x|^{bs}}\,dx\right)^{\frac{1}{s}} \\ &\leq C\varepsilon\|u\|^2 + C\|u\|_{qs'}^q\left(\int_\Omega \frac{e^{s\|u\|^2\beta\left(\frac{u}{\|u\|}\right)^2}}{|x|^{bs}}\,dx\right)^{\frac{1}{s}},\end{aligned}$$

where $\frac{1}{s'} + \frac{1}{s} = 1$ with s sufficiently close to 1 such that $bs < 2$ and $qs' > 1$. If $\|u\| \leq \delta$, with $\delta > 0$ such that $\frac{\beta s\delta^2}{4\pi} + \frac{bs}{2} \leq 1$. So, by Trudinger-Moser inequality (2.1) and Sobolev imbedding theorem we obtain

$$\int_\Omega \frac{F(u)}{|x|^b}\,dx \leq C\varepsilon\|u\|^2 + C\|u\|^q.$$

In a similar way one also can see that if $\|u\| \leq \delta$, with $\delta > 0$ such that $\frac{\beta s\delta^2}{4\pi} + \frac{as}{2} \leq 1$, it holds

$$\int_\Omega \frac{G(v)}{|x|^a}\,dx \leq C\varepsilon\|v\|^2 + C\|v\|^q.$$

Thus, for $z \in \partial B_\rho \cap E^+$, we have $I(z) \geq (1 - 2C\varepsilon)\|u\|^2 - 2C\|u\|^q$. Then, for ε small enough we can find $\rho, \sigma > 0$ such that $I(z) \geq \sigma > 0$ for $\|u\| = \rho$ sufficiently small. □

Let $e_1 \in H_0^1(\Omega)$ be a fixed nonnegative function with $\|e_1\| = 1$ and

$$Q = \left\{r(e_1, e_1) + \omega \mid \omega \in E^-,\ \|\omega\| \leq R_0 \text{ and } 0 \leq r \leq R_1\right\}.$$

Lemma 3.2. *There exist $R_0, R_1 > 0$ such that $I(z) \leq 0$ for all $z \in \partial Q$, where ∂Q denotes the boundary of Q in $\mathbb{R}(e_1, e_1) \oplus E^-$.*

Proof. For $z \in \partial Q$, we have three cases.

Case 1: $z \in \partial Q \cap E^-$, then we have $z = (u, -u)$ and hence

$$I(z) = -\int_\Omega |\nabla u|^2\,dx - \int_\Omega \frac{F(u)}{|x|^b}\,dx - \int_\Omega \frac{G(-u)}{|x|^a}\,dx \leq -\|u\|^2 \leq 0.$$

Case 2: $z = R_1(e_1, e_1) + (u, -u) \in \partial Q$ with $\|(u, -u)\| \le R_0$. Then

$$I(z) = R_1^2 - \int_\Omega |\nabla u|^2\, dx - \int_\Omega \frac{F(R_1 e_1 + u)}{|x|^b}\, dx - \int_\Omega \frac{G(R_1 e_1 - u)}{|x|^a}\, dx. \tag{3.4}$$

By the assumption (H_2), there exist $C > 0$ such that

$$F(t) \ge C\left(t^\theta - 1\right) \quad \text{and} \quad G(t) \ge C\left(t^\theta - 1\right).$$

Then we obtain from (3.4) that

$$\begin{aligned} I(z) &\le R_1^2 - C \int_\Omega \left(\frac{(R_1 e_1 + u)^\theta}{|x|^b} + \frac{(R_1 e_1 - u)^\theta}{|x|^a} \right) dx + C. \\ &\le R_1^2 - C \int_{\{x \in \Omega \mid |x| \ge 1\}} \left(\frac{(R_1 e_1 + u)^\theta + (R_1 e_1 - u)^\theta}{|x|^{\max\{a,b\}}} \right) dx \\ &\quad - C \int_{\{x \in \Omega \mid |x| \le 1\}} \left((R_1 e_1 + u)^\theta + (R_1 e_1 - u)^\theta \right) dx + C. \end{aligned}$$

Now, using the convexity of the function $\phi(t) = t^\theta$, it follows that

$$I(z) \le R_1^2 - 2C R_1^\theta \left(\int_{\{x \in \Omega \mid |x| \ge 1\}} \frac{e_1^\theta}{|x|^{\max\{a,b\}}}\, dx + \int_{\{x \in \Omega \mid |x| \le 1\}} e_1^\theta\, dx \right) + C.$$

Then, for R_1 sufficiently large, we get $I(z) \le 0$.

Case 3: $z = r(e_1, e_1) + (u, -u) \in \partial Q$ with $\|(u, -u)\| = R_0$ and $0 \le r \le R_1$. Then

$$I(z) = r^2 - \int_\Omega |\nabla u|^2\, dx - \int_\Omega \frac{F(re_1 + u)}{|x|^b}\, dx - \int_\Omega \frac{G(re_1 - u)}{|x|^a}\, dx \le R_1^2 - \frac{1}{2} R_0^2.$$

Thus $I(z) \le 0$ if $R_0 \ge \sqrt{2} R_1$. □

3.2. On Palais-Smale sequences. To prove that a Palais-Smale sequence converges to a weak solution of problem (1.1) we need to establish the following lemma

Lemma 3.3. *Let $(u_n, v_n) \in E$ such that $I(u_n, v_n) \to c$ and $I'(u_n, v_n) \to 0$. Then*

$$\|u_n\| \le C, \qquad \|v_n\| \le C \tag{3.5}$$

$$\int_\Omega \frac{f(u_n) u_n}{|x|^b}\, dx \le C, \qquad \int_\Omega \frac{g(v_n) v_n}{|x|^a}\, dx \le C \tag{3.6}$$

$$\int_\Omega \frac{F(u_n)}{|x|^b}\, dx \le C, \qquad \int_\Omega \frac{G(v_n)}{|x|^a}\, dx \le C. \tag{3.7}$$

Proof. Let $(u_n,v_n)\in E$ be a sequence such that $I(u_n,v_n)\to c$ and $I'(u_n,v_n)\to 0$, that is, for any $(\varphi,\psi)\in E$,

$$\int_\Omega \nabla u_n \nabla v_n\,dx - \int_\Omega \frac{F(u_n)}{|x|^b}\,dx - \int_\Omega \frac{G(v_n)}{|x|^a}\,dx = c+\delta_n \tag{3.8}$$

and

$$\left|\int_\Omega \nabla u_n\psi\,dx + \int_\Omega \nabla\varphi\nabla v_n\,dx - \int_\Omega \frac{f(u_n)\varphi}{|x|^b}\,dx - \int_\Omega \frac{g(v_n)\psi}{|x|^a}\,dx\right| \le \varepsilon_n\|(\varphi,\psi)\|. \tag{3.9}$$

Choosing $(\varphi,\psi)=(u_n,v_n)$ in (3.9) and using (H_2), we have

$$\begin{aligned}
&\int_\Omega \frac{f(u_n)u_n}{|x|^b}\,dx + \int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx\\
&\le 2\left|\int_\Omega \nabla u_n\nabla v_n\,dx\right| + \varepsilon_n\|(u_n,v_n)\|\\
&\le 2c + 2\int_\Omega \frac{F(u_n)}{|x|^b}\,dx + 2\int_\Omega \frac{G(v_n)}{|x|^a}\,dx + 2\delta_n + \varepsilon_n\|(u_n,v_n)\|\\
&\le 2c + \frac{2}{\theta}\int_\Omega \frac{f(u_n)u_n}{|x|^b}\,dx + \frac{2}{\theta}\int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx + 2\delta_n + \varepsilon_n\|(u_n,v_n)\|.
\end{aligned}$$

Thus

$$\int_\Omega \frac{f(u_n)u_n}{|x|^b}\,dx + \int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx \le C\left(1+2\delta_n+\varepsilon_n\|(u_n,v_n)\|\right). \tag{3.10}$$

Now, taking $(\varphi,\psi)=(v_n,0)$ and $(\varphi,\psi)=(0,u_n)$ in (3.9) we have

$$\|v_n\|^2-\varepsilon_n\|v_n\| \le \int_\Omega \frac{f(u_n)v_n}{|x|^b}\,dx \quad\text{and}\quad \|u_n\|^2-\varepsilon_n\|u_n\| \le \int_\Omega \frac{g(v_n)u_n}{|x|^a}\,dx.$$

Setting $V_n=\frac{v_n}{\|v_n\|}$ and $U_n=\frac{u_n}{\|u_n\|}$ we obtain

$$\|v_n\| \le \int_\Omega \frac{f(u_n)}{|x|^b}V_n\,dx + \varepsilon_n \quad\text{and}\quad \|u_n\| \le \int_\Omega \frac{g(v_n)}{|x|^a}U_n\,dx + \varepsilon_n. \tag{3.11}$$

We apply the inequality (2.2) with $t=V_n$ and $s=f(u_n)$ in the first estimate in (3.11), we obtain

$$\begin{aligned}
\int_\Omega \frac{f(u_n)}{|x|^b}V_n\,dx \le C\int_\Omega \frac{e^{V_n^2}}{|x|^b}\,dx &+ \int_{\left\{x\in\Omega|\ f(u_n)\ge e^{\frac14}\right\}} \frac{f(u_n)}{|x|^b}\left[\log\left(f(u_n)\right)\right]^{\frac12}\,dx\\
&+\frac12\int_{\left\{x\in\Omega|\ f(u_n)\le e^{\frac14}\right\}} \frac{\left[f(u_n)\right]^2}{|x|^b}\,dx.
\end{aligned}$$

Using Trudinger-Moser inequality and the fact $b < 2$, we get

$$\int_\Omega \frac{f(u_n)}{|x|^b} V_n\, dx \le C\left(1 + \beta^{\frac{1}{2}} \int_\Omega \frac{f(u_n)u_n}{|x|^b}\, dx\right).$$

This estimate together with the first inequality in (3.11) implies that

$$\|v_n\| \le C\left(1 + \int_\Omega \frac{f(u_n)u_n}{|x|^b}\, dx + \varepsilon_n\right). \tag{3.12}$$

Similarly, we get from the second estimate in (3.11)

$$\|u_n\| \le C\left(1 + \int_\Omega \frac{g(v_n)v_n}{|x|^a}\, dx + \varepsilon_n\right). \tag{3.13}$$

Adding the estimates (3.12) and (3.13) and using (3.10), we obtain

$$\|(u_n, v_n)\| \le C\left(1 + \delta_n + \varepsilon_n \|(u_n, v_n)\| + \varepsilon_n\right)$$

Then $\|(u_n, v_n)\| \le C$. From this estimate, inequality (3.10) and (H_2), we obtain the estimates (3.6) and (3.7). Thus, the proof of Lemma 3.3 is complete. □

4. Finite-dimensional approximation

Note that the functional I is strongly indefinite in an infinite dimensional space, and hence the standard linking theorems cannot be applied. We therefore approximate problem (1.1) with a sequence of finite dimensional problems (Galerkin approximation).

Denote by $(\phi_i)_{i\in\mathbb{N}}$ an orthonormal set of eigenfunctions corresponding to the eigenvalues $(\lambda_i), i \in \mathbb{N}$, of $(-\Delta, H_0^1(\Omega))$ and set

$$\begin{aligned} E_n^+ &= \operatorname{span}\{(\phi_i, \phi_i) \mid i = 1, \dots, n\} \\ E_n^- &= \operatorname{span}\{(\phi_i, -\phi_i) \mid i = 1, \dots, n\} \\ E_n &= E_n^+ \oplus E_n^- . \end{aligned}$$

Set now $Q_n = Q \cap E_n$ where Q as in previous section and define the class of mappings

$$\Gamma_n = \{\gamma \in C\left(Q_n, \mathbb{R}\left(e_1, e_1\right) \oplus E_n\right) \mid \gamma\left(z\right) = z \text{ on } \partial Q_n\}$$

and set

$$c_n = \inf_{\gamma\in\Gamma_n} \max_{z\in Q_n} I\left(\gamma\left(z\right)\right) \tag{4.1}$$

Using an intersection theorem [13], we have

$$\gamma(Q_n) \cap \left(\partial B_\rho \cap E^+\right) \ne \emptyset \quad \forall\, \gamma \in \Gamma_n,$$

which in combination with Lemma 3.1 implies that $c_n \geq \sigma > 0$. On the other hand, since the identity mapping $Id : Q_n \to \mathbb{R}(e_1, e_1) \oplus E_n$ belongs to Γ_n, it is easy to prove that $c_n \leq R_1^2$. Then we have

$$0 < \sigma \leq c_n \leq R_1^2.$$

Now, by Lemma 3.1 and 3.2, we see that the geometry of a linking theorem holds for the functional $I_n = I|_{E_n}$. Therefore, applying the linking theorem for I_n (see [13, Theorem 5.3]), we get the following result:

For each $n \in \mathbb{N}$ the functional I_n has a critical point $z_n = (u_n, v_n) \in E_n$ at level c_n such that

$$I_n(z_n) = c_n \in \left[\sigma, R_1^2\right] \quad \text{and} \quad I_n'(z_n) = 0.$$

Furthermore, $\|z_n\| \leq C$ where C does not depend of n.

5. Subcritical case

In this section we assume that g has subcritical growth.

5.1. Proof of Theorem 1.1. In previous section, we find a sequence $z_n = (u_n, v_n) \in E_n$ bounded in E and such that

$$\begin{aligned} I_n(z_n) &= c_n \in \left[\sigma, R_1^2\right], && (5.1)\\ I_n'(z_n) &= 0, && (5.2)\\ (u_n, v_n) &\rightharpoonup (u, v) \quad \text{in } E, &&\\ u_n \to u \quad &\text{and} \quad v_n \to v \quad \text{in } L^q(\Omega) \quad \forall\, q \geq 1, &&\\ u_n(x) \to u(x) \quad &\text{and} \quad v_n(x) \to v \quad \text{a.e. in } \Omega && \end{aligned}$$

By Lemma 3.3, we have

$$\int_\Omega \frac{f(u_n)u_n}{|x|^b}\,dx \leq C, \qquad \int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx \leq C \tag{5.3}$$

$$\int_\Omega \frac{F(u_n)}{|x|^b}\,dx \leq C, \qquad \int_\Omega \frac{G(v_n)}{|x|^a}\,dx \leq C. \tag{5.4}$$

Taking as test functions $(0, \psi)$ and $(\varphi, 0)$ in (5.2), where φ and ψ are arbitrary functions in $F_n := \operatorname{span}\{\phi_i \mid i = 1, \ldots, n\}$ we get

$$\int_\Omega \nabla u_n \nabla \psi\,dx = \int_\Omega \frac{g(v_n)\psi}{|x|^a}\,dx \quad \forall\, \psi \in F_n \tag{5.5}$$

$$\int_\Omega \nabla v_n \nabla \varphi\,dx = \int_\Omega \frac{f(u_n)\varphi}{|x|^b}\,dx \quad \forall\, \varphi \in F_n. \tag{5.6}$$

Consequently, by Lemmas 3.3 and 2.4, $\frac{f(u_n)}{|x|^b} \to \frac{f(u)}{|x|^b}$ and $\frac{g(v_n)}{|x|^a} \to \frac{g(v)}{|x|^a}$ in $L^1(\Omega)$. Passing to the limit in (5.5) and (5.6) and using the fact that $\cup_{n\in\mathbb{N}} F_n$ is dense in $H_0^1(\Omega)$, we see that

$$\int_\Omega \nabla u \nabla \psi \, dx = \int_\Omega \frac{g(v)\psi}{|x|^a} \, dx \quad \forall\, \psi \in H_0^1(\Omega) \tag{5.7}$$

$$\int_\Omega \nabla v \nabla \varphi \, dx = \int_\Omega \frac{f(u)\varphi}{|x|^b} \, dx \quad \forall\, \varphi \in H_0^1(\Omega). \tag{5.8}$$

Thus, we conclude that (u, v) is a weak solution of (1.1). Finally, we prove that $(u, v) \in E$ is nontrivial. Assume by contradiction that $u = 0$, which implies that also $v = 0$.

Since g is subcritical, we obtain for all $\beta > 0$

$$|g(t)| \le Ce^{\beta t^2} \quad \forall\, t \in \mathbb{R}. \tag{5.9}$$

Now, we choose $\psi = u_n$ in (5.5), using Hölder inequality and (5.9) we get

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 \, dx &= \int_\Omega \frac{g(v_n)u_n}{|x|^a} \, dx \\ &\le C\|u_n\|_{L^{q'}} \left(\int_\Omega \frac{e^{q\beta u^2}}{|x|^{aq}} \, dx \right)^{\frac{1}{q}} \\ &= C\|u_n\|_{L^{q'}} \left(\int_\Omega \frac{e^{q\|u\|^2 \beta \left(\frac{u}{\|u\|}\right)^2}}{|x|^{aq}} \, dx \right)^{\frac{1}{q}},\end{aligned}$$

where $q' = \frac{q}{q-1}$ with $q > 1$ sufficiently close to 1 such that $\frac{\beta q \|u\|^2}{4\pi} + \frac{qa}{2} \le 1$. Then $\int_\Omega |\nabla u_n|^2 \, dx \le C\|u_n\|_{L^{q'}}$ and so we conclude that $\|u_n\| \to 0$ because $\|u_n\|_{L^{q'}} \to 0$. This implies

$$\int_\Omega \nabla u_n \nabla v_n \, dx \to 0. \tag{5.10}$$

Then, from (5.5) and (5.6), we obtain

$$\int_\Omega \frac{f(u_n)u_n}{|x|^b} \, dx \to 0 \quad \text{and} \quad \int_\Omega \frac{g(v_n)v_n}{|x|^a} \, dx \to 0.$$

By assumption (H_2) we now conclude that

$$\int_\Omega \frac{F(u_n)}{|x|^b} \, dx \to 0 \quad \text{and} \quad \int_\Omega \frac{G(v_n)}{|x|^a} \, dx \to 0. \tag{5.11}$$

Finally, by (5.10) and (5.11) we obtain that

$$I(u_n, v_n) = \int_\Omega \nabla u_n \nabla v_n \, dx - \int_\Omega \frac{F(u_n)}{|x|^b} \, dx - \int_\Omega \frac{G(v_n)}{|x|^a} \, dx \to 0,$$

but this contradicts (5.1). Consequently, we have a nontrivial critical point of I. This completes the proof of the Theorem 1.1. □

6. Critical case

In this section we assume that f and g have critical growth with exposent critical β_0 and $a = b$.

6.1. On the minimax level. In order to get a more precise information about the minimax level, it was crucial in our argument to consider the following sequence:

$$\text{For } k \in \mathbb{N}, \quad \tilde{\psi}_k(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} (\log k)^{\frac{1}{2}} & \text{for } 0 \le |x| \le \frac{1}{k} \\ \dfrac{\log \frac{1}{|x|}}{(\log k)^{\frac{1}{2}}} & \text{for } \frac{1}{k} \le |x| \le 1 \\ 0 & \text{for } \quad |x| \ge 1. \end{cases}$$

Now, we define the sets

$$Q_{n,k} = \left\{ r(e_k, e_k) + \omega \mid \omega \in E_n^-, \ \|\omega\| \le R_0 \text{ and } 0 \le r \le R_1 \right\},$$

where $e_k(x) = \tilde{\psi}_k(\frac{x}{d})$.

Lemma 6.1. *There exists $k \in \mathbb{N}$ such that*

$$\sup_{\mathbb{R}_+(e_k,e_k) \oplus E^-} I < \frac{2\pi(2-a)}{\beta_0}.$$

Proof. Suppose by contradiction that for all $k \in \mathbb{N}$, we have

$$\sup_{\mathbb{R}_+(e_k,e_k) \oplus E^-} I \ge \frac{2\pi(2-a)}{\beta_0}.$$

This means that there exists $z_{n,k} = \tau_{n,k}(e_k, e_k) + (u_{n,k}, -u_{n,k}) \in Q_{n,k}$ such that $I(z_{n,k}) \ge \frac{2\pi(2-a)}{\beta_0} - \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.

Let $h(t) := I(tz_{n,k})$. We see that $h(0) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$. Then, there exists a maximum point $t_0 z_{n,k}$ with $I(t_0 z_{n,k}) \ge \frac{2\pi(2-a)}{\beta_0} - \varepsilon_n$. We may assume that $z_{n,k}$ is this point, and then we get

$$\begin{aligned}&\tau_{n,k}^2-\int_\Omega |\nabla u_{n,k}|^2\,dx-\int_\Omega \frac{F(\tau_{n,k}e_k+u_{n,k})}{|x|^a}\,dx-\int_\Omega \frac{G(\tau_{n,k}e_k-u_{n,k})}{|x|^a}\,dx\\ &\geq \frac{2\pi(2-a)}{\beta_0}-\varepsilon_n\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}&\tau_{n,k}^2-\int_\Omega |\nabla u_{n,k}|^2\,dx\\ &=\int_\Omega \frac{f(\tau_{n,k}e_k+u_{n,k})(\tau_{n,k}e_k+u_{n,k})-g(\tau_{n,k}e_k-u_{n,k})(\tau_{n,k}e_k-u_{n,k})}{|x|^a}\,dx\end{aligned} \tag{6.2}$$

Now, put $\tau_{n,k}^2=s_n+\frac{2\pi(2-a)}{\beta_0}$. So, from (6.1) we get $s_n+\frac{2\pi(2-a)}{\beta_0}\geq\frac{2\pi(2-a)}{\beta_0}-\varepsilon_n$. By assumption ($H_4$), there exists $\bar{t}>0$ and

$$\eta_0>\frac{(2-a)^2}{\beta_0 d^{2-a}} \tag{6.3}$$

such that

$$tf(t)\geq(\eta_0-\varepsilon)\,e^{\beta_0 t^2}\quad\text{and}\quad tg(t)\geq(\eta_0-\varepsilon)\,e^{\beta_0 t^2}, \tag{6.4}$$

for all $t\geq\bar{t}$ and ε is arbitrarily small.

Next, choosing k sufficiently large such that $\tau_{n,k}\sqrt{\frac{\log k}{2\pi}}\geq\bar{t}$, we get

$$\max\{\tau_{n,k}e_k+u_{n,k},\tau_{n,k}e_k-u_{n,k}\}\geq\bar{t}\quad\forall x\in B_{\frac{d}{k}}(0).$$

Now, using (6.2) and (6.4), we obtain

$$\begin{aligned}s_n+\frac{2\pi(2-a)}{\beta_0}&\geq(\eta_0-\varepsilon)\int_{B_{\frac{d}{k}}(0)}\frac{e^{\beta_0\tau_{n,k}^2\frac{\log k}{2\pi}}}{|x|^a}\,dx\\ &\geq(\eta_0-\varepsilon)\,2\pi e^{\beta_0\left(s_n+\frac{2\pi(2-a)}{\beta_0}\right)\frac{\log k}{2\pi}}\int_0^{\frac{d}{k}}\xi^{1-a}d\xi\\ &\geq(\eta_0-\varepsilon)\,2\pi e^{\beta_0 s_n\frac{(\log k)}{2\pi}}e^{(2-a)\log k}\left(\frac{d}{k}\right)^{2-a}\\ &\geq(\eta_0-\varepsilon)\,\frac{2\pi d^{2-a}e^{\beta_0 s_n\frac{\log k}{2\pi}}}{2-a}.\end{aligned}$$

This and (6.1) imply that $\lim_{n\to+\infty}s_n=0$. So, we see that $\eta_0-\varepsilon\leq\frac{2(2-a)^2}{\beta_0 d^{2-a}}$, which contradicts (6.3). $\square$

6.2. Proof of Theorem 1.2. Lemma 6.1 implies that there is $\delta > 0$ such that for all n we have

$$c_{n,e} := c_n \le \frac{2\pi(2-a)}{\beta_0} - \delta$$

where c_n is defined by (4.1). Next, using (6.1) and Lemma 3.3, we have $z_n = (u_n, v_n) \in E_n$ bounded in E such that

$$I_n(z_n) = c_{n,e} \in \left[\sigma, \frac{2\pi(2-a)}{\beta_0} - \delta\right], \tag{6.5}$$

$$I_n'(z_n) = 0, \tag{6.6}$$

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } E,$$
$$u_n \to u \quad \text{and} \quad v_n \to v \quad \text{in } L^q(\Omega) \quad \forall\, q \ge 1,$$
$$u_n(x) \to u(x) \quad \text{and} \quad v_n(x) \to v \quad \text{a.e. in } \Omega.$$

By Lemma 3.3, we have

$$\int_\Omega \frac{f(u_n)u_n}{|x|^a}\, dx \le C, \qquad \int_\Omega \frac{g(v_n)v_n}{|x|^a}\, dx \le C \tag{6.7}$$

$$\int_\Omega \frac{F(u_n)}{|x|^a}\, dx \le C, \qquad \int_\Omega \frac{G(v_n)}{|x|^a}\, dx \le C. \tag{6.8}$$

Taking as test functions $(0, \psi)$ and $(\varphi, 0)$ in (5.2), where φ and ψ are arbitrary functions in $F_n := \operatorname{span}\{\phi_i \mid i = 1, \dots, n\}$, we get

$$\int_\Omega \nabla u_n \nabla \psi\, dx = \int_\Omega \frac{g(v_n)\psi}{|x|^a}\, dx \quad \forall\, \psi \in F_n \tag{6.9}$$

$$\int_\Omega \nabla v_n \nabla \varphi\, dx = \int_\Omega \frac{f(u_n)\varphi}{|x|^a}\, dx \quad \forall\, \varphi \in F_n. \tag{6.10}$$

Consequently, by Lemmas 3.3 and 2.4, $\frac{f(u_n)}{|x|^a} \to \frac{f(u)}{|x|^a}$ and $\frac{g(v_n)}{|x|^a} \to \frac{g(v)}{|x|^a}$ in $L^1(\Omega)$. Passing to the limit in (6.9) and (6.10) and using the fact that $\cup_{n\in\mathbb{N}} F_n$ is dense in $H_0^1(\Omega)$, we see that

$$\int_\Omega \nabla u \nabla \psi\, dx = \int_\Omega \frac{g(v)\psi}{|x|^a}\, dx \quad \forall\, \psi \in H_0^1(\Omega) \tag{6.11}$$

$$\int_\Omega \nabla v \nabla \varphi\, dx = \int_\Omega \frac{f(u)\varphi}{|x|^a}\, dx \quad \forall\, \varphi \in H_0^1(\Omega). \tag{6.12}$$

Thus, we conclude that (u, v) is a weak solution of (1.1).

Finally, it only remains to prove that $(u, v) \in E$ is nontrivial. Assume by contradiction that $u = 0$, which implies that also $v = 0$. Now, if $\|u_n\| \to 0$, then we get directly (6.18) below, and then a contradiction. Thus, assume that $\|u_n\| \geq b > 0$ for all n and consider

$$\|u_n\|^2 = \int_\Omega \frac{g(v_n)u_n}{|x|^a}\, dx. \tag{6.13}$$

Setting $\overline{u}_n = \left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \frac{u_n}{\|u_n\|}$, and using inequality (2.2) with $s = \frac{g(v_n)}{\sqrt{\beta_0}}$ and $t = \sqrt{\beta_0}\overline{u}_n$, we have

$$\begin{aligned}\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \|u_n\| &= \int_\Omega \frac{g(v_n)\overline{u}_n}{|x|^a}\, dx \\ &\leq \int_\Omega \frac{e^{\beta_0 \overline{u}_n^2} - 1}{|x|^a}\, dx + \int_{\left\{x\in\Omega:\, \frac{g(v_n(x))}{\sqrt{\beta_0}} \leq e^{\frac{1}{4}}\right\}} \frac{(g(v_n))^2}{\beta_0|x|^a}\, dx \\ &\quad + \int_{\left\{x\in\Omega:\, \frac{g(v_n(x))}{\sqrt{\beta_0}} \geq e^{\frac{1}{4}}\right\}} \frac{g(v_n)}{\sqrt{\beta_0}|x|^a} \left(\log\left(\frac{g(v_n)}{\sqrt{\beta_0}}\right)\right)^{\frac{1}{2}} dx.\end{aligned} \tag{6.14}$$

Since $\|u_n\|^2 = \frac{2\pi(2-a)}{\beta_0} - \delta$, it is clear that the function $m(u_n) := e^{\beta_0 \overline{u}_n^2} - 1$ satisfies the conditions of Lemma 2.4 , so the first term tends to zero. By Lebesgues dominated convergence, we can see also that the second term tends to zero.

From Lemma 2.4 and the fact that g has critical growth with exposent critical β_0, we can estimate the third term by

$$\begin{aligned}\int_\Omega \frac{g(v_n)}{\sqrt{\beta_0}|x|^a} \left(\log\left(\frac{g(v_n)}{\sqrt{\beta_0}}\right)\right)^2 dx &\leq \int_\Omega \frac{g(v_n)}{\sqrt{\beta_0}|x|^a} \left(\log\left(\frac{C_\epsilon e^{(\beta_0+\epsilon)v_n^2}}{\sqrt{\beta_0}}\right)\right)^{\frac{1}{2}} dx \\ &\leq \int_\Omega \frac{g(v_n)}{\sqrt{\beta_0}|x|^a} \left(\log\left(\frac{C_\epsilon}{\sqrt{\beta_0}}\right)^{\frac{1}{2}} + (\beta_0+\epsilon)^{\frac{1}{2}}\, v_n\right) dx \\ &\leq o(1) + \left(1 + \frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}} \int_\Omega \frac{g(v_n)v_n}{|x|^a},\end{aligned}$$

and hence, by (6.14), we get

$$\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \|u_n\| \leq o(1) + \left(1 + \frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}} \int_\Omega \frac{g(v_n)v_n}{|x|^a}\, dx. \tag{6.15}$$

Similarly, with $\|v_n\|^2 \leq \int_\Omega \frac{f(u_n)v_n}{|x|^a}\, dx$, we get

$$\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \|v_n\| \leq o(1) + \left(1 + \frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}} \int_\Omega \frac{f(u_n)u_n}{|x|^a}\, dx. \tag{6.16}$$

On the other hand, by Lemma 2.5 and (6.5), we can conclude that

$$\int_\Omega \frac{F(u_n)}{|x|^a}\,dx \to 0, \quad \int_\Omega \frac{G(v_n)}{|x|^a}\,dx \to 0 \tag{6.17}$$

and

$$\left|\int_\Omega \nabla u_n \nabla v_n\,dx\right| \le o(1) + \frac{2\pi(2-a)}{\beta_0} - \delta,$$

which, together with (6.6), imply that

$$\int_\Omega \frac{f(u_n)u_n}{|x|^a}\,dx + \int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx \le o(1) + 2\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)$$

So, from (6.15) and (6.16) we obtain

$$\|u_n\| + \|v_n\| \le o(1) + 2\left(1+\frac{\epsilon}{\beta_0}\right)^{\frac{1}{2}}\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}} \le 2\left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}},$$

for ϵ sufficiently small and n sufficiently large. It follows that there is a subsequence of (u_n) or (v_n) (without loss of generality assume it is (v_n)) such that $\|v_n\| \le \left(\frac{2\pi(2-a)}{\beta_0} - \delta\right)^{\frac{1}{2}}$. Thus, using Lemma 2.1 and Hölder inequality with $q > 1$ such that $q\left(\frac{(\beta_0+\epsilon)\left(\frac{2\pi(2-a)}{\beta_0}-\delta\right)}{4\pi} + \frac{a}{2}\right) \le 1$ we get

$$\left|\int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx\right| \le C_\epsilon \|v_n\|_{L^{q'}(\Omega)} \int_\Omega \frac{e^{q(\beta_0+\epsilon)v_n^2}}{|x|^{qa}}\,dx \le C\|v_n\|_{L^{q'}(\Omega)}.$$

Since $\|v_n\|_{L^{q'}(\Omega)} \to 0$, we get $\int_\Omega \frac{g(v_n)v_n}{|x|^a}\,dx \to 0$. Hence,

$$\int_\Omega \nabla u_n \nabla v_n\,dx \to 0 \tag{6.18}$$

which, together with (6.17), imply that $c_{n,e} \to 0$, yielding a contradiction.

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References

[1] Adams, R. A., *Sobolev Spaces.* New York: Academic Press 1975.

[2] Adimurthi and Sandeep, K., A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Diff. Equ. Appl.* 13 (2007), 585 – 603.

[3] Dai, Q. and Gu, Y., Positive solutions for non-homogeneous semilinear elliptic equations with data that changes sign. *Proc. Roy. Soc. Edinburgh* 133A (2003), 297 – 306.

[4] Fang, L. and Jianfu, Y., Nontrivial solutions of Hardy-Hénon type elliptic systems. *Acta Math. Sci. Ser. B* 27 (2007), 673 – 688.

[5] de Figueiredo, D. G. and Felmer, P., On superquadratic elliptic systems. *Trans. Amer. Math. Soc.* 343 (1994), 99 – 116.

[6] de Figueiredo, D. G., Miyagaki, O. H. and Ruf, B., Elliptic equations in $\mathbb{R}^2 with$ nonlinearities in the critical growth range. *Calc. Var. Partial Diff. Equ.* 3 (1995), 139 – 153.

[7] de Figueiredo, D. G., do Ó, J. M. and Ruf, B., Critical and subcritical elliptic systems in dimension two. *Indiana Univ. Math. J.* 53 (2004), 1037 – 1054.

[8] de Figueiredo, D. G., Peral, I. and Rossi, J. D., The critical hyperbola for a Hamiltonian elliptic system with weights. *Ann. Mat. Pura Appl. (4)* 187 (2008), 531 – 545.

[9] de Figueiredo, D. G. and Ruf, B., Elliptic systems with nonlinearities of arbitrary growth. *Mediterr. J. Math.* 1 (2004), 417 – 431.

[10] Han, P., Strongly indefinite systems with critical Sobolev exponents and weight. *Appl. Math. Lett.* 17 (2004), 909 – 917.

[11] Hulshof, J., Mitidieri, E. and Van der Vorst, R. C. A. M., Strongly indefinite systems with critical Sobolev exponents. *Trans. Amer. Math. Soc.* 350 (1998), 2349 – 2365.

[12] Moser, J., A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.* 20 (1971), 1077 – 1092.

[13] Rabinowitz, P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. Math. 65. Providence (RI): Amer. Math. Soc. 1986.

[14] Ruf, B., Lorentz spaces and nonlinear elliptic systems. *Progr. Nonlinear Diff. Equ.* 66 (2005), 471 – 489.

[15] de Souza, M., On a singular class of elliptic systems involving critical grouth in $\mathbb{R}^2$. *Nonlinear Anal.* 12 (2011), 1072 – 1088.

[16] de Souza, M., On a singular Hamiltonian elliptic systems involving critical growth in dimension two. *Comm. Pure Appl. Anal.* 11 (2012), 1859 – 1874.

[17] de Souza, M. and do Ó, J. M., On a singular and nonhomogeneous N-Laplacian equation involving critical growth. *J. Math. Anal. Appl.* 280 (2011), 241 – 263.

[18] Trudinger, N. S., On embedding into Orlicz spaces some applications. *J. Math. Mech.* 17 (1967), 473 – 484.

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