Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 33 (2014), 247–258
DOI: 10.4171/ZAA/1509


# Blow-Up Solutions and Global Existence for a Kind of Quasilinear Reaction-Diffusion Equations

*Lingling Zhang, Na Zhang and Lixiang Li*

**Abstract.** In this paper, we study the blow-up solutions and global existence for a quasilinear reaction-diffusion equation including a gradient term and nonlinear boundary condition:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)\nabla u) + f(x,u,|\nabla u|^2,t) & \text{in } D \times (0,T) \\ \frac{\partial u}{\partial n} = r(u) & \text{on } \partial \mathrm{D} \times (0,\mathrm{T}) \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\mathrm{D}}, \end{cases}$$

where $D \subset R^N$ is a bounded domain with smooth boundary $\partial D$. The sufficient conditions are obtained for the existence of a global solution and a blow-up solution. An upper bound for the "blow-up time", an upper estimate of the "blow-up rate", and an upper estimate of the global solution are specified under some appropriate assumptions for the nonlinear system functions $f, g, r, a$, and initial value $u_0$ by constructing suitable auxiliary functions and using maximum principles.

**Keywords.** Reaction-diffusion equation, blow-up solution, global solution

**Mathematics Subject Classification (2010).** Primary 35K57, secondary 35B44

## 1. Introduction

Everyone knows that blow-up solutions and global existence for reaction diffusion equations had played an important role in many fields. So many authors always focus on the study of this field ([4–6, 9, 12]). In this paper, we study the blow-up solution and global existence for the following initial-boundary-value problem of quasilinear reaction-diffusion equation with a gradient term

---

L. Zhang, N. Zhang, L. Li: Department of Mathematics, Taiyuan University of Technology, Taiyuan, P. R. China; zllww@126.com; znmath@126.com; 704761476@qq.com

and nonlinear boundary condition:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)\nabla u) + f(x,u,q,t) & \text{in D} \times (0,\text{T}) \\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial\text{D} \times (0,\text{T}) \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\text{D}}, \end{cases} \tag{1}$$

where $q = |\nabla u|^2$, $D \subset R^N$ is a bounded domain with smooth boundary $\partial D$, $\frac{\partial}{\partial n}$ represents the outward normal derivative on $\partial D$, $u_0$ is the initial value, $T$ is the maximal existence time of $u$. Set $R^+ = (0, +\infty)$, we assume that $f(x,s,d,t)$ is a nonnegative $C^1(\overline{D} \times R^+ \times R^+ \times R^+)$ function, $g(s)$ is a $C^2(R^+)$ function, $g'(s) > 0$ for any $s > 0$, $r(s)$ is a positive $C^2(R^+)$ function, $a(s)$ is a $C^2(R^+)$ function. Under the assumptions above, the problem (1) has a unique classical solution $u(x,t)$ with some $T > 0$, and the solution is positive over $\overline{D} \times [0,T)$.

The problem (1) describes many physical phenomena in mechanics, physics and biology, etc. We refer to [5, 10] and the reference therein for many other applications. Souplet et al. [13] deal with the blow-up and global solutions of initial value problems for the reaction-diffusion equations with a gradient term. [1, 3, 14] study the existence of blow-up and global solutions for the reaction-diffusion equations with a gradient term and initial-Dirichlet-boundary-value. Zhang [16] and Ding and Guo [7] investigate the blow-up and global solutions for the reaction-diffusion equations with gradient terms and initial-Neumann-boundary-values.

Some special cases of (1) are also treated. Walter [15] studies the following problem:

$$\begin{cases} u_t = \triangle u & \text{in D} \times (0,\text{T}) \\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial\text{D} \times (0,\text{T}) \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\text{D}}, \end{cases}$$

where $D \subset R^N$ is a bounded domain with smooth boundary. The sufficient conditions characterized by function $r$ are given for the existence of blow-up and global solutions. Zhang [17] considers the following problem:

$$\begin{cases} (g(u))_t = \triangle u + f(u) & \text{in D} \times (0,\text{T}) \\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial\text{D} \times (0,\text{T}) \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\text{D}}, \end{cases}$$

where $D \subset R^N$ is a bounded domain with smooth boundary. The sufficient conditions are obtained there for the existence of a global solution and a blow-up solution. Meanwhile, the upper estimate of the global solution, the upper bound of the "blow-up time", and the upper estimate of the "blow-up rate" are also given. Ding and Guo [8] consider the following problem:

$$\begin{cases} (g(u))_t = \triangle u + f(x,u,q,t) & \text{in } \mathrm{D} \times (0,\mathrm{T}) \\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial \mathrm{D} \times (0,\mathrm{T}) \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\mathrm{D}}, \end{cases}$$

where $q = |\nabla u|^2$, $D \subset R^N$ is a bounded domain with smooth boundary. The sufficient conditions for the existence of a blow-up solution, an upper bound for the "blow-up time", an upper estimate of the "blow-up rate", the sufficient conditions for the existence of the global solution, and an upper estimate of the global solution are specified under some appropriate assumptions on the nonlinear functions $f, g, r$, and initial value $u_0$.

In this paper, we consider blow-up solutions and global solutions of (1). We obtain some existence theorems for blow-up solutions, upper bounds of the blow-up time, upper estimates of the blow-up rate, existence theorems for global solutions, and upper estimates of global solutions. The results generalize and deepen ones from corresponding work in [8, 15, 17].

The plan of this paper is as follows. In Section 2 we give the proofs for the main results. A few examples are presented in Section 3 to illustrate the applications of the abstract results.

## 2. Main results

Our first result Theorem 2.1 is about the existence of a blow-up solution.

**Theorem 2.1.** *Let $u$ be a solution of* (1)*. Assume that the following conditions* (i)–(iii) *are satisfied:*

(i) *the initial value condition:*

$$\begin{aligned} &\beta = \min_{\overline{D}} \frac{a(u_0)[a'(u_0)q_0 + a(u_0)\Delta u_0 + f(x,u_0,q_0,0)]}{r(u_0)g'(u_0)} > 0, \\ &q_0 = |\nabla u_0|^2; \end{aligned} \tag{2}$$

(ii) *further restrictions for functions involved: for any $(x,s,d,t) \in D \times R^+ \times \overline{R^+} \times R^+$,*

$$\begin{aligned} 2f_d(x,s,d,t)\left(\frac{r(s)}{a(s)}\right)' - \frac{r'^2(s)}{a(s)}\left(\frac{a(s)}{r'(s)}\right)' &\geq 0, \\ \left(\frac{a(s)}{g'(s)}\right)' \geq 0, \quad \left(\frac{f(x,s,d,t)}{r(s)}\right)'_s &\geq 0, \\ f_t(x,s,d,t) \geq 0, \qquad\qquad a'(s) &\geq 0; \end{aligned} \tag{3}$$

(iii) *the integration condition:*

$$\int_{M_0}^{+\infty} \frac{a(s)}{r(s)} ds < \infty, \quad M_0 = \max_{\overline{D}} u_0(x). \tag{4}$$

*Then the solution $u$ of* (1) *must blow up in a finite time $T$, and*

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{r(s)} ds, \quad u(x,t) \leq H^{-1}(\beta(T-t)),$$

*where $H(z) = \int_z^{+\infty} \frac{a(s)}{r(s)} ds$, $z > 0$, and $H^{-1}$ is the inverse function of $H$.*

*Proof.* Consider the auxiliary function

$$\Psi(x,t) = -\frac{1}{r(u)} u_t + \beta \frac{1}{a(u)}. \tag{5}$$

We find that

$$\nabla\Psi = \frac{r'}{r^2} u_t \nabla u - \frac{\nabla u_t}{r} - \frac{\beta a' \nabla u}{a^2}, \tag{6}$$

$$\begin{aligned} \Delta\Psi &= \nabla \cdot (\nabla\Psi) \\ &= \frac{\nabla \cdot (u_t r' \nabla u) r^2 - 2rr'^2 u_t |\nabla u|^2}{r^4} - \frac{r\nabla \cdot (\nabla u_t) - r' \nabla u \cdot \nabla u_t}{r^2} \\ &\quad - \frac{a^2 \beta \nabla \cdot (a' \nabla u) - 2a \cdot a'^2 \beta |\nabla u|^2}{a^4} \\ &= \frac{r'' |\nabla u|^2 u_t + r' u_t \Delta u + 2r' \nabla u \cdot \nabla u_t}{r^2} - \frac{2r'^2 |\nabla u|^2 u_t}{r^3} - \frac{\Delta u_t}{r} \\ &\quad - \frac{\beta(a'' |\nabla u|^2 + a' \Delta u)}{a^2} + \frac{2a'^2 |\nabla u|^2 \beta}{a^3}, \end{aligned} \tag{7}$$

$$\Psi_t = \frac{r'}{r^2}(u_t)^2 - \frac{(u_t)_t}{r} - \frac{\beta a' u_t}{a^2}.$$

By (1) we have $u_t = \frac{a'|\nabla u|^2 + a\Delta u}{g'} + \frac{f}{g'}$,

$$\begin{aligned} \Psi_t &= \frac{r'}{r^2}(u_t)^2 - \frac{(u_t)_t}{r} - \frac{\beta a' u_t}{a^2} \\ &= \frac{r'}{r^2}(u_t)^2 - \frac{1}{r}\left(\frac{a'|\nabla u|^2 + a\Delta u}{g'} + \frac{f}{g'}\right)_t - \frac{\beta a'}{a^2}\left(\frac{a'|\nabla u|^2 + a\Delta u}{g'} + \frac{f}{g'}\right) \\ &= \frac{r'}{r^2}(u_t)^2 - \frac{a\Delta u_t}{rg'} + \left(\frac{a'g''}{rg'^2} - \frac{a''}{rg'}\right) u_t |\nabla u|^2 + \left(\frac{ag''}{rg'^2} - \frac{a'}{rg'}\right) u_t \Delta u \\ &\quad + \left(\frac{fg''}{rg'^2} - \frac{f_u}{rg'}\right) u_t - \frac{2a' + 2f_q}{rg'} \nabla u_t \nabla u - \frac{f_t}{rg'} - \frac{\beta a'^2}{a^2 g'} |\nabla u|^2 - \frac{\beta a'}{ag'} \Delta u - \frac{a' \beta f}{a^2 g'}. \end{aligned} \tag{8}$$

It follows from (7) and (8) that

$$\frac{a}{g'}\Delta\Psi = \left(\frac{ar''}{g'r^2} - \frac{2ar'^2}{g'r^3}\right)|\nabla u|^2 u_t + \frac{ar'}{g'r^2}u_t\Delta u + \frac{2ar'}{g'r^2}\nabla u\cdot\nabla u_t$$
$$- \frac{a}{g'r}\Delta u_t + \left(\frac{2\beta a'^2}{g'a^2} - \frac{\beta a''}{g'a}\right)|\nabla u|^2 - \frac{\beta a'}{g'a}\Delta u,$$

$$\begin{aligned}
&\frac{a}{g'}\Delta\Psi - \Psi_t \\
&= \left(\frac{ar''}{g'r^2} - \frac{2ar'^2}{g'r^3}\right)|\nabla u|^2 u_t + \frac{ar'}{g'r^2}u_t\Delta u + \frac{2ar'}{g'r^2}\nabla u\cdot\nabla u_t - \frac{a}{g'r}\Delta u_t \\
&\quad + \left(\frac{2\beta a'^2}{g'a^2} - \frac{\beta a''}{g'a}\right)|\nabla u|^2 - \frac{\beta a'}{g'a}\Delta u - \frac{r'}{r^2}(u_t)^2 + \frac{a\Delta u_t}{rg'} \\
&\quad - \left(\frac{a'g''}{rg'^2} - \frac{a''}{rg'}\right)u_t|\nabla u|^2 - \left(\frac{ag''}{rg'^2} - \frac{a'}{rg'}\right)u_t\Delta u - \left(\frac{fg''}{rg'^2} - \frac{f_u}{rg'}\right)u_t \\
&\quad + \frac{2a' + 2f_q}{rg'}\nabla u_t\cdot\nabla u + \frac{f_t}{rg'} + \frac{\beta a'^2}{a^2 g'}|\nabla u|^2 + \frac{\beta a'}{ag'}\Delta u + \frac{a'\beta f}{a^2 g} \\
&= \left(\frac{ar''}{g'r^2} - \frac{2ar'^2}{g'r^3} - \frac{a'g''}{rg'^2} + \frac{a''}{rg'}\right)u_t|\nabla u|^2 + \left(\frac{ar'}{g'r^2} - \frac{ag''}{rg'^2} + \frac{a'}{rg'}\right)u_t\Delta u \\
&\quad + \left(\frac{2ar'}{g'r^2} + \frac{2a' + 2f_q}{rg'}\right)\nabla u\cdot\nabla u_t + \left(\frac{2\beta a'^2}{g'a^2} - \frac{\beta a''}{g'a} + \frac{\beta a'^2}{a^2 g'}\right)|\nabla u|^2 \\
&\quad - \frac{r'}{r^2}(u_t)^2 - \left(\frac{fg''}{rg'^2} - \frac{f_u}{rg'}\right)u_t + \frac{f_t}{rg'} + \frac{a'\beta f}{a^2 g'}.
\end{aligned} \tag{9}$$

In view of (6), we have

$$\nabla u_t = \frac{r'}{r}u_t\nabla u - \frac{r\beta a'\nabla u}{a^2} - r\nabla\Psi. \tag{10}$$

Substitute (10) into (9) to obtain

$$\begin{aligned}
&\frac{a}{g'}\Delta\Psi - \Psi_t \\
&= \left(\frac{ar'' + (2a + 2f_q)r'}{r^2 g'} - \frac{a'g''}{rg'^2} + \frac{a''}{rg'}\right)u_t|\nabla u|^2 + \left(\frac{ar'}{g'r^2} - \frac{ag''}{rg'^2} + \frac{a'}{rg'}\right)u_t\Delta u \\
&\quad + \left(\frac{\beta a'^2}{g'a^2} - \frac{2f_q\beta a'}{g'a^2} - \frac{2\beta a'r'}{g'ra} - \frac{\beta a''}{g'a}\right)|\nabla u|^2 - \left(\frac{2ar'}{g'r} + \frac{2a' + 2f_q}{g'}\right)\nabla u\cdot\nabla\Psi \\
&\quad - \frac{r'}{r^2}(u_t)^2 - \left(\frac{fg''}{rg'^2} - \frac{f_u}{rg'}\right)u_t + \frac{f_t}{rg'} + \frac{a'\beta f}{a^2 g'},
\end{aligned}$$

$$
\begin{aligned}
&\frac{a}{g'}\Delta\Psi+\left(\frac{2ar'}{g'r}+\frac{2a'+2f_q}{g'}\right)\nabla u\cdot\nabla\Psi-\Psi_t\\
&=\left(\frac{ar''+(2a'+2f_q)r'}{r^2g'}-\frac{a'g''}{rg'^2}+\frac{a''}{rg'}\right)u_t|\nabla u|^2\\
&\quad+\left(\frac{ar'}{g'r^2}-\frac{ag''}{rg'^2}+\frac{a'}{rg'}\right)u_t\Delta u+\left(\frac{\beta a'^2}{g'a^2}-\frac{2f_q\beta a'}{g'a^2}-\frac{2\beta a'r'}{g'ra}-\frac{\beta a''}{g'a}\right)|\nabla u|^2\\
&\quad-\frac{r'}{r^2}(u_t)^2-\left(\frac{fg''}{rg'^2}-\frac{f_u}{rg'}\right)u_t+\frac{f_t}{rg'}+\frac{a'\beta f}{a^2g'}.
\end{aligned}
\tag{11}
$$

By (1) we have

$$
\Delta u=\frac{g'u_t-f}{a}-\frac{a'}{a}|\nabla u|^2. \tag{12}
$$

Substitute (12) into (11), to get

$$
\begin{aligned}
&\frac{a}{g'}\Delta\Psi+\left(\frac{2ar'}{g'r}+\frac{2a'+2f_q}{g'}\right)\nabla u\cdot\nabla\Psi-\Psi_t\\
&=\left(\frac{ar''+(a'+2f_q)r'}{r^2g'}-\frac{a'^2}{rag'}+\frac{a''}{rg'}\right)u_t|\nabla u|^2+\left(\frac{a'}{ra}-\frac{g''}{rg'}\right)u_t^2\\
&\quad+\left(\frac{f_u}{rg'}-\frac{fr'}{g'r^2}-\frac{fa'}{arg'}\right)u_t+\left(\frac{\beta a'^2}{g'a^2}-\frac{\beta a''}{g'a}-\frac{2\beta a'r'}{g'ra}-\frac{2\beta a'f_q}{g'a^2}\right)|\nabla u|^2\\
&\quad+\frac{f_t}{rg'}+\frac{a'\beta f}{a^2g'}.
\end{aligned}
\tag{13}
$$

With (5), we have

$$
u_t=-r\Psi+\frac{r\beta}{a}. \tag{14}
$$

Substitution of (14) into (13) gives

$$
\begin{aligned}
&\frac{a}{g'}\Delta\Psi+\left(\frac{2ar'}{g'r}+\frac{2a'+2f_q}{g'}\right)\nabla u\cdot\nabla\Psi\\
&\quad-\left[\left(-\frac{ar''+(a'+2f_q)r'}{rg'}-\frac{a''}{g'}+\frac{a'^2}{ag'}\right)|\nabla u|^2\right.\\
&\quad\left.+\frac{2\beta rg''}{ag'}-\frac{2a'r\beta}{a^2}+\frac{fr'}{g'r}+\frac{fa'}{ag'}-\frac{f_u}{g'}+\left(\frac{ra'}{a}-\frac{rg''}{g'}\right)\Psi\right]\Psi-\Psi_t\\
&=\frac{\beta}{rg'}\left(2f_q\left(\frac{r}{a}\right)'-\frac{r'^2}{a}\left(\frac{a}{r'}\right)'\right)|\nabla u|^2+\frac{\beta^2rg'}{a^3}\left(\frac{a}{g'}\right)'+\frac{\beta r}{ag'}\left(\frac{f}{r}\right)'_u+\frac{f_t}{rg'}.
\end{aligned}
\tag{15}
$$

From assumptions (2) and (3), the right-hand side of (15) is nonnegative, i.e.

$$
\begin{aligned}
&\frac{a}{g'}\Delta\Psi+\left(\frac{2ar'}{g'r}+\frac{2a'+2f_q}{g'}\right)\nabla u\cdot\nabla\Psi-\left[\left(-\frac{ar''+(a'+2f_q)r'}{rg'}-\frac{a''}{g'}+\frac{a'^2}{ag'}\right)|\nabla u|^2\right.\\
&\left.+\frac{2\beta rg''}{ag'}-\frac{2a'r\beta}{a^2}-\frac{f_u}{g'}+\frac{fr'}{g'r}+\frac{fa'}{ag'}+\left(\frac{ra'}{a}-\frac{rg''}{g'}\right)\Psi\right]\Psi-\Psi_t\\
&\geq 0.
\end{aligned}
\tag{16}
$$

Now by (2), we have

$$\begin{aligned}\max_{\overline{D}} \Psi(x,0) &= \max_{\overline{D}} \left(-\frac{1}{r(u_0)}u_t + \beta\frac{1}{a(u_0)}\right)\\ &= \max_{\overline{D}} \left(-\frac{a'(u_0)q_0 + a(u_0)\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} + \frac{\beta}{a(u_0)}\right)\\ &= 0.\end{aligned} \tag{17}$$

It follows from (1) that, on $\partial D \times (0,T)$,

$$\begin{aligned}\frac{\partial\Psi}{\partial n} &= \frac{r'}{r^2}u_t\frac{\partial u}{\partial n} - \frac{1}{r}\frac{\partial u_t}{\partial n} - \frac{\beta a'}{a^2}\frac{\partial u}{\partial n}\\ &= \frac{r'}{r}u_t - \frac{1}{r}r_t - \frac{\beta a' r}{a^2}\\ &= \frac{r'}{r}u_t - \frac{r'}{r}u_t - \frac{\beta a'}{a^2}r\\ &= -\frac{\beta a'}{a^2}r\\ &\le 0.\end{aligned} \tag{18}$$

Combining (16)–(18), and applying the maximum principles [11], we know that the maximum of $\Psi$ in $D \times [0,T)$ is zero. Thus $\Psi \le 0$, in $\overline{D} \times [0,T)$, and

$$\frac{a(u)}{\beta r(u)}u_t \ge 1. \tag{19}$$

At the point $x_0 \in \overline{D}$ where $u_0(x_0) = M_0$, integrate (19) over $[0,t]$ to produce

$$\frac{1}{\beta}\int_0^t \frac{a(u)}{r(u)}u_t dt = \frac{1}{\beta}\int_{M_0}^{u(x_0,t)} \frac{a(s)}{r(s)}ds \ge \int_0^t ds = t.$$

This together with assumption (4) shows that $u$ must blow up in the finite time $T$ and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{a(s)}{r(s)}ds.$$

By integrating the inequality (19) over $[t,s](0 < t < s < T)$, one has, for each fixed $x$, that

$$H(u(x,t)) \ge H(u(x,t)) - H(u(x,s)) = \int_{u(x,t)}^{u(x,s)} \frac{a(s)}{r(s)}ds \ge \beta(s-t),$$

passing to the limit as $s \to T$ yields $H(u(x,t)) \ge \beta(T-t)$, which implies that

$$u(x,t) \le H^{-1}(\beta(T-t)).$$

The proof is complete. □

The result on the global solution is stated as Theorem 2.2 below.

**Theorem 2.2.** *Let* $u$ *be a solution of* (1). *Assume that the following conditions are satisfied:*

(i) *the initial value condition:*

$$\mu = \max_{\overline{D}} \frac{a(u_0)[a'(u_0)q_0 + a(u_0)\Delta u_0 + f(x, u_0, q_0, 0)]}{r(u_0)g'(u_0)} > 0, \qquad q_0 = |\nabla u_0|^2 \tag{20}$$

(ii) *further restrictions on functions involved:*
*for any* $(x, s, d, t) \in D \times R^+ \times \overline{R^+} \times R^+$,

$$\begin{aligned} 2f_d(x,s,d,t)\left(\frac{r(s)}{a(s)}\right)' - \frac{r'^2(s)}{a(s)}\left(\frac{a(s)}{r'(s)}\right)' &\le 0, \\ \left(\frac{a(s)}{g'(s)}\right)' \le 0, \qquad \left(\frac{f(x,s,d,t)}{r(s)}\right)'_s &\le 0, \\ f_t(x,s,d,t) \le 0, \qquad\qquad a'(s) &\le 0; \end{aligned} \tag{21}$$

(iii) *the integration condition:*

$$\int_{m_0}^{+\infty} \frac{a(s)}{r(s)} ds = +\infty, \quad m_0 = \min_{\overline{D}} u_0(x). \tag{22}$$

*Then the solution* $u$ *of* (1) *must be a global solution and*

$$u(x,t) \le G^{-1}(\mu t + G(u_0(x))),$$

*where* $G(z) = \int_{m_0}^{z} \frac{a(s)}{r(s)} ds$, $z \ge m_0$, *and* $G^{-1}$ *is the inverse function of* $G$.

*Proof.* Construct an auxiliary function

$$\Phi(x,t) = -\frac{1}{r(u)} u_t + \mu \frac{1}{a(u)}. \tag{23}$$

Replacing $\Psi$ and $\beta$ with $\Phi$ and $\mu$ in (15), we have

$$\begin{aligned} &\frac{a}{g'}\Delta\Phi + \left(\frac{2ar'}{g'r} + \frac{2a' + 2f_q}{g'}\right)\nabla u \cdot \nabla\Phi \\ &- \left[\left(-\frac{ar'' + (a' + 2f_q)r'}{rg'} - \frac{a''}{g'} + \frac{a'^2}{ag'}\right)|\nabla u|^2 \right. \\ &\left. + \frac{2\mu r g''}{ag'} - \frac{2a'r\mu}{a^2} - \frac{f_u}{g'} + \frac{fr'}{g'r} + \frac{fa'}{ag'} + \left(\frac{ra'}{a} - \frac{rg''}{g'}\right)\Phi\right]\Phi - \Phi_t \\ =& \frac{\mu}{rg'}\left(2f_q\left(\frac{r}{a}\right)' - \frac{r'^2}{a}\left(\frac{a}{r'}\right)'\right)|\nabla u|^2 + \frac{\mu^2 rg'}{a^3}\left(\frac{a}{g'}\right)' + \frac{\mu r}{ag'}\left(\frac{f}{r}\right)'_u + \frac{f_t}{rg'}. \end{aligned} \tag{24}$$

It is seen from assumptions (20) and (21) that the right-hand side of (24) is nonpositive, i.e.

$$\begin{aligned}
&\frac{a}{g'}\Delta\Phi + \left(\frac{2ar'}{g'r} + \frac{2a'+2f_q}{g'}\right)\nabla u\cdot\nabla\Phi \\
&-\Bigg[\left(-\frac{ar''+(a'+2f_q)r'}{rg'} - \frac{a''}{g'} + \frac{a'^2}{ag'}\right)|\nabla u|^2 \\
&+\frac{2\mu r g''}{ag'} - \frac{2a'r\mu}{a^2} - \frac{f_u}{g'} + \frac{fr'}{g'r} + \frac{fa'}{ag'} + \left(\frac{ra'}{a} - \frac{rg''}{g'}\right)\Phi\Bigg]\Phi - \Phi_t \\
&\le 0.
\end{aligned} \tag{25}$$

By (20), we have

$$\min_{\overline{D}}\Phi(x,0) = \min_{\overline{D}}\left(-\frac{a'(u_0)q_0 + a(u_0)\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} + \frac{\mu}{a(u_0)}\right) = 0. \tag{26}$$

From (1) it follows that

$$\frac{\partial\Phi}{\partial n} = \frac{r'}{r^2}u_t\frac{\partial u}{\partial n} - \frac{1}{r}\frac{\partial u_t}{\partial n} - \frac{\mu a'}{a^2}\frac{\partial u}{\partial n} = \frac{r'}{r}u_t - \frac{1}{r}r_t - \frac{\mu a' r}{a^2} = -\frac{\mu a'}{a^2}r \ge 0 \tag{27}$$

on $\partial D\times(0,T)$. Combining (25)–(27) and applying the maximum principles, we know that the minimum of $\Phi$ in $\overline{D}\times[0,T)$ is zero. Hence $\Phi\ge 0$ in $\overline{D}\times[0,T)$, i.e.

$$\frac{a(u)}{\mu r(u)}u_t \le 1. \tag{28}$$

For each fixed $x\in\overline{D}$, integrate (28) over $[0,t]$ to get

$$\frac{1}{\mu}\int_0^t \frac{a(u)}{r(u)}u_t dt = \frac{1}{\mu}\int_{u_0(x)}^{u(x,t)}\frac{a(s)}{r(s)}ds \le t.$$

This together with (22) shows that $u$ must be a global solution. Moreover, (28) implies that

$$G(u(x,t)) - G(u_0(x)) = \int_{m_0}^{u(x,t)}\frac{a(s)}{r(s)}ds - \int_{m_0}^{u_0(x)}\frac{a(s)}{r(s)}ds = \int_0^t\frac{a(u)}{r(u)}u_t dt \le \mu t.$$

Therefore

$$u(x,t) \le G^{-1}(\mu t + G(u_0(x))).$$

The proof is complete. □

## 3. Applications

**Example 3.1.** Let $u$ be a solution of the following problem:

$$\begin{cases} (\sqrt{u}+u)_t = \nabla\cdot(u\nabla u) + \left(tu^2 + q + \sum_{i=1}^{3} x_i^2\right)u^3 & \text{in } \mathrm{D}\times(0,\mathrm{T}) \\ \dfrac{\partial u}{\partial n} = \dfrac{u^3}{4} & \text{on } \partial\mathrm{D}\times(0,\mathrm{T}) \\ u(x,0) = 1 + \sum_{i=1}^{3} x_i^2 & \text{in } \overline{\mathrm{D}}, \end{cases} \tag{29}$$

where $q = |\nabla u|^2$, $D = \left\{x = (x_1, x_2, x_3) \,\middle|\, \sum_{i=1}^{3} x_i^2 < 1\right\}$ is the unit ball of $R^3$. Here

$$g(u) = \sqrt{u} + u, \qquad f(x,u,q,t) = \left(tu^2 + q + \sum_{i=1}^{3} x_i^2\right)u^3,$$

$$a(u) = u, \qquad r(u) = \frac{u^3}{4},$$

and

$$\begin{aligned} \beta &= \min_{\overline{D}} \frac{a(u_0)[a'(u_0)q_0 + a(u_0)\Delta u_0 + f(x,u_0,q_0,0)]}{r(u_0)g'(u_0)} \\ &= 8 \min_{1\le u_0\le 2} \frac{5u_0^4 - 5u_0^3 + 10u_0 - 4}{u_0^{\frac{3}{2}} + 2u_0^2} \\ &= 16. \end{aligned}$$

It is easy to check that (3) and (4) hold. It follows from Theorem 2.1 that $u$ must blow up in a finite time $T$, and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{a(s)}{r(s)}ds = \frac{1}{8}, \quad u(x,t) \le H^{-1}(\beta(T-t)) = \frac{1}{4(T-t)}.$$

**Example 3.2.** Let $u$ be a solution of the following problem:

$$\begin{cases} (ue^u)_t = \nabla\cdot(e^{-u}\nabla u) + e^{-u}\left(e^{-t} + q + \sum_{i=1}^{3} x_i^2\right) & \text{in } \mathrm{D}\times(0,\mathrm{T}) \\ \dfrac{\partial u}{\partial n} = 2e^{2-u} & \text{on } \partial\mathrm{D}\times(0,\mathrm{T}) \\ u(x,0) = 1 + \sum_{i=1}^{3} x_i^2 & \text{in } \overline{\mathrm{D}}, \end{cases} \tag{30}$$

where $q = |\nabla u|^2$, $D = \left\{x = (x_1, x_2, x_3) \,\middle|\, \sum_{i=1}^{3} x_i^2 < 1\right\}$ is the unit ball of $R^3$. Now we have

$$g(u) = ue^u, \qquad f(x,u,q,t) = e^{-u}\left(e^{-t} + q + \sum_{i=1}^{3} x_i^2\right),$$
$$a(u) = e^{-u}, \qquad r(u) = 2e^{2-u},$$

and

$$\begin{aligned}\mu &= \max_{\overline{D}} \frac{a(u_0)[a'(u_0)q_0 + a(u_0)\Delta u_0 + f(x,u_0,q_0,0)]}{r(u_0)g'(u_0)} \\ &= \max_{1\le u_0\le 2} \frac{u_0e^{-u_0} + 6e^{-u_0}}{2e^2(e^{u_0} + u_0e^{u_0})} \\ &= \frac{7}{4}e^{-4}.\end{aligned}$$

It is easy to check that (21) and (22) hold. It follows from Theorem 2.2 that $u$ must be a global solution and

$$u(x,t) \le G^{-1}(\mu t + G(u_0(x))) = \frac{7}{2}e^{-2}t + u_0(x).$$

**Acknowledgement.** This work was supported by the National Natural Science Foundation of China (No. 61250011 ), the Natural Science Foundation of Shanxi Province (No. 2012011004-4), China Postdoctoral Science Foundation (No. 2012M510786) and Research Project supported by Shanxi Scholarship Council of China (No. 2011-011).

## References

[1] Chen, S., Global existence and blow-up of solutions for a parabolic equation with a gradient term. *Proc. Amer. Math. Soc.* 129 (2001), 975 – 981.

[2] Chen, S., Boundedness and blow-up for nonlinear degenerate parabolic equations. *Nonlinear Anal.* 70 (2009), 1087 – 1095.

[3] Chipot, M. and Weissler, F., Some blow-up results for a nonlinear parabolic equation with a gradient term. *SIAM J. Math. Anal.* 20 (1989), 886 – 907.

[4] Deng, K. and Levine, H., The role of critical exponents in blow-up theorems: the sequel. *J. Math. Anal. Appl.* 243 (2000), 85 – 126.

[5] Diaz, J. and Thélin, F., On a nonlinear parabolic problem arising in some model related to turbulent ows. *SIAM J. Math. Anal.* 25 (1994), 1085 – 1111.

[6] Ding, J. and Guo, B., Blow-up and global existence for nonlinear parabolic equations with Neumann boundary conditions. *Comput. Math. Appl.* 60 (2010), 670 – 679.

[7] Ding, J. and Guo, B., Global and blow-up solutions for nonlinear parabolic equations with a gradient term. *Houston J. Math.* 37 (2011), 1265 – 1277.

[8] Ding, J. and Guo, B., Global existence and blow-up solutions for quasilinear reaction-diffusion equations with a gradient term. *Appl. Math. Lett.* 24 (2011), 936 – 942.

[9] Hollis, S. and Morgan, J., On the blow-up of solutions to some semilinear and quasilinear reaction-diffusion systems. *Rocky Mountain J. Math.* 14 (1994), 1447 – 1465.

[10] Jüngel, A., *Quasi-Hydrodynamic Semiconductor Equations.* Basel: Birkhäuser 2001.

[11] Protter, M. and Weinberger, H., *Maximum Principles in Differential Equations.* Englewood Cliffs (NJ): Prentice-Hall 1967.

[12] Qi, Y., Wang, M. and Wang, Z., Existence and non-existence of global solutions of diffusion systems with nonlinear boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A* 134 (2004), 1199 – 1217.

[13] Souplet, P., Tayachi, S. and Weissler, F., Exact self-similar blow-up of solutions of a semilinear parabolic equation with a nonlinear gradient term. *Indiana Univ. Math. J.* 45 (1996), 655 – 682.

[14] Souplet, P. and Weissler, F., Self-similar subsolutions and blow-up for nonlinear parabolic equations. *J. Math. Anal. Appl.* 212 (1997), 60 – 74.

[15] Walter, W., On existence and nonexistence in the large of solutions of parabolic differential equation with nonlinear boundary condition. *SIAM J. Math. Anal.* 24 (1975), 85 – 90.

[16] Zhang, L., Blow-up of solutions for a class of nonlinear parabolic equations. *Z. Anal. Anwend.* 25 (2006), 479 – 486.

[17] Zhang, H., Blow-up solutions and global solutions for nonlinear parabolic problems. *Nonlinear Anal.* 69 (2008), 4567 – 4575.

Received April 23, 2013; revised February 3, 2014