Asymptotic Behaviour of Solutions for p-Laplacian Wave Equation with m-Laplacian Dissipation

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Abstract. In this paper we study decay properties of solutions to the p-Laplacian wave equation with m-Laplacian weak dissipation and source term. Meanwhile, we investigate the decay estimate of the energy of the global solutions to this problem by using a different inequalities.

Keywords. Wave equation of p-Laplacian type, dissipation of m-Laplacian type, global existence, decay estimate

Mathematics Subject Classification (2010). Primary 35B40, secondary 35A01, 35L70

1. Introduction

We consider the initial boundary value problem for the nonlinear wave equation of p-Laplacian type with a weak nonlinear dissipation, that is

$$(P) \begin{cases} u_{tt} - \operatorname{div}(|\nabla_x u|^{p-2}\nabla_x u) - a\operatorname{div}(|\nabla_x u_t|^{m-2}\nabla_x u_t) = b|u|^{r-2}u & \text{in } \Omega \times [0, +\infty[\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \Gamma \times [0, +\infty[\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega. \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$, $a, b > 0$ and $p, m, r \geq 2$ are real numbers.

Problem (P) can be considered as a system governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear voight model. In this context we can cite the works of Andrews [1], Andrews and Ball [2], Kawashima and Shibata [8], Ang and Dinh [3], and Messaoudi [14], among others.

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The global existence was discussed in Nakao and Nanbu [15]. Later Biazutti [4] extended this result and gave the global existence and uniqueness of a weak solution by using monotonicity and compactness arguments. The asymptotic behavior of solutions have been studied by Yang [18]. Moreover, energy decay property ($E(t) \leq (1+t)^{-\frac{1}{2p}}$ for $t \geq 0$) is also given.

More recently Messaoudi [13] improved the result of Yang by giving more precise decay rates. His proof is based on perturbed energy methods. He showed that the energy related to the problem (P) decays exponentially when $p = 2$. On the other hand and when $p > 2$, $E(t) \leq C(1+t)^{-\frac{2}{p-2}}$ for $t \geq 0$.

In the same context and with considering $\delta|u'|^{m-2}u'$ ($\delta > 0, m \geq 2$) instead of $-\mathrm{div}(|\nabla_x u'|^{m-2}\nabla_x u')$, Ye [19] proved that the energy decay rate of problem (P) is $E(t) \leq (1+t)^{-\frac{p}{mp-m-1}}$ for $t \geq 0$, for which he used the general method of energy decay introduced by Nakao [16].

Chen, Yao and Shao [5] investigated the global existence and uniqueness of a solution to an initial boundary problem $u_{tt} - \mathrm{div}(|\nabla_x u|^{p-2}\nabla_x u) - \Delta u_t + g(x, u) = f(x)$ under certain assumptions on g where $2 \leq p < n$. In the same direction and considering $f = 0$ it is important to mention the result of Ma and Soriano in [12].

Our purpose in this paper is to extend the results obtained by Ye ([20] and [21]) to the case of m-Laplacian weak dissipation equations. On the one hand, by the argument in [21], as well as combining it with the potential well theory introduced by sattinger [17], we proved that the global solution for the problem (P) exists as long as $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ and u_0 belong the stable set H. On the other hand we show the asymptotic behavior of global solutions through the use the integral inequality given by Komornik [9].

Our paper is organized as follows. In Section 2, some assumptions and the main result are stated. And the proof of the global existence of solution and asymptotic behavior is given in Section 3.

Throughout this paper the functions considered are all real valued. For simplicity of notation, hereafter we denote by $\| \cdot \|_p$ the Lebesgue space $L^p(\Omega)$ norm, and $\| \cdot \|_2$ denotes $L^2(\Omega)$ norm and we write equivalent norm $\|\nabla \cdot \|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\| \cdot \|_{W_0^{1,p}(\Omega)}$. We also denote by $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$. As usual, we write respectively $u(t)$ and $u_t(t)$ instead $u(x,t)$ and $u_t(t, x)$. All along this paper we denote C various positive constants which may be different at different occurrences.

2. Preliminaries and main result

We begin by introducing some definition that will be used throught this work. We first define the following functionals:

$$K(u) = \|\nabla u\|_p^p - b\|u\|_r^r, \quad J(u) = \frac{1}{p}\|\nabla_x u\|_p^p - \frac{b}{r}\|u\|_r^r,$$

for $u \in W_0^{1,p}(\Omega)$. Then, for the problem (P), we are able to define the stable set

$$H \equiv \{u \in W_0^{1,p}(\Omega), \, K(u) > 0\} \cup \{0\}.$$

We define the total energy associated to the solution of the problem (P) by the formula

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{b}{r}\|u\|_r^r = \frac{1}{2}\|u_t\|_2^2 + J(u),$$

for $u \in W_0^{1,p}(\Omega)$ and $t \geq 0$. We first state some well-known lemmas.

Lemma 2.1 (Energy identity). *Let $u(t,x)$ be a solution to the problem (P) on $[0,\infty)$. Then we have*

$$E(t) + \int_\Omega \int_0^t a|\nabla u_t(s)|^m \, ds \, dx = E(0),$$

for all $t \in [0,\infty)$.

Remark 2.2. It is clear that $E(t)$ is a non-increasing function for $t > 0$ and we have

$$\frac{d}{dt}E(t) = -a\|\nabla u_t\|_m^m \leq 0.$$

Lemma 2.3 (Sobolev-Poincaré inequality). *Let r be a number with $2 \leq r < +\infty$ $(n = 1,2,\ldots,p)$ or $2 \leq r \leq \frac{np}{n-p}$ $(n \geq p+1)$. Then there is a constant $c_* = c_*(\Omega,r)$ such that*

$$\|u\|_r \leq c_*\|\nabla u\|_p \quad \text{for } u \in W_0^{1,p}(\Omega).$$

Lemma 2.4 ([9]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that there are nonnegative constants β and $A > 0$ with*

$$\int_S^{+\infty} E(t)^{\beta+1} \, dt \leq A \, E(S), \quad 0 \leq S < +\infty,$$

then we have

$$E(t) \leq \left(A\left(1+\frac{1}{\beta}\right)\right)^{\frac{1}{\beta}} t^{\frac{-1}{\beta}} \quad \forall t > 0, \quad \text{if } \beta > 0,$$

and

$$E(t) \leq E(0)\exp\left(1-\frac{t}{A}\right) \quad \forall t \geq 0, \quad \text{if } \beta = 0.$$

This lemma is due to Haraux and its proof can be found in [6,7] or [9–11].

Now we recall the following local existence theorem, which can be established by using the argument in [18].

Theorem 2.5. *Let* $2 < p < r < \frac{np}{n-p}$, $n > p$ *and* $2 < p < r < \infty$, $n \leq p$ *and assume that* $2 \leq m \leq p$, $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ *and* u_0 *belong the stable set* H. *Then there exists* $T > 0$ *such that the problem* (P) *has a unique local solution* $u(t)$ *in the class*

$$u \in L^\infty\left([0,T); W_0^{1,p}(\Omega)\right),$$
$$u_t \in L^\infty\left([0,T); L^2(\Omega)\right) \cap L^m\left([0,T); L^m(\Omega)\right).$$

We list up some useful lemmas before stating the global existence theorem and decay property. From now on, we denote the life span of the solution $u(t)$ of the problem (P) by T_{max}.

Lemma 2.6. *Assume that the hypotheses in Theorem* 2.5 *hold, then*

$$\frac{r-p}{rp}\|\nabla_x u\|_p^p \leq E(t), \tag{1}$$

for $u \in H$.

Proof. The definition of $K(u)$ and $J(u)$ assume that

$$K(u) + \frac{r-p}{p}\|\nabla_x u\|_p^p = rJ(u). \tag{2}$$

Since $u \in H$, so we have $K(u) \geq 0$. Hence we deduce from (2) that

$$\frac{r-p}{rp}\|\nabla_x u\|_p^p \leq J(u) \leq E(t). \qquad \square$$

Lemma 2.7 ([21]). *Let* $u(t)$ *be a solution to problem* (P) *on* $[0, T_{\max})$. *Suppose that* $2 \leq p < r \leq \frac{np}{n-p}$, $n \geq p$ *and* $2 < p < r < +\infty$, $n \leq p$. *If* $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$ *satisfy*

$$\theta = bC^r\left(\frac{r-p}{rp}E(0)\right)^{\frac{r-p}{p}} < 1, \tag{3}$$

then $u(t) \in H$, *for each* $t \in [0, T_{max})$.

Now we are in position to state our main result.

Theorem 2.8. *Let $u(t, x)$ be a local solution of problem (P) on $[0, T_{\max})$ with initial data $u_0 \in H$, $u_1 \in L^2(\Omega)$ and sufficiently small initial energy $E(0)$ so that*

$$bC^r \left(\frac{r - p}{rp} E(0) \right)^{\frac{r-p}{p}} < 1.$$

If the hypotheses in Theorem 2.5 are valid and $2 < m < \frac{np}{n-p}$, $n > p$ and $2 < m < \infty$, $n \leq p$, then $T_{\max} = \infty$. Furthermore, the global solution of the problem (P) satisfies the following energy decay estimates

  (i) *If $p = m = 2$, then there exists a positive constant ω independent of $E(0)$ such that*

$$E(t) \leq E(0) \exp(1 - \omega t) \quad \forall t > 0.$$

 (ii) *If $m = 2, p > 2$, then there exists a positive constant τ depending continuously on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau}{t} \right)^{\frac{p}{p-2}} \quad \forall t > 0.$$

(iii) *If $p \geq m \geq 3$, then there exists a positive constant τ depending continuously on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau}{t} \right)^{\frac{2}{m-2}} \quad \forall t > 0.$$

(iv) *If $2 < m < 3$, then there exists a positive constant τ depending continuously on $E(0)$ such that*

$$E(t) \leq \left(\frac{\tau}{t} \right)^{\frac{2}{m-2}} \quad \forall t > 0 \quad if \ m \leq p \leq \frac{2}{3-m},$$

$$E(t) \leq \left(\frac{\tau}{t} \right)^{\frac{p-m}{p(m-1)}} \quad \forall t > 0 \quad if \ p > \frac{2}{3-m}.$$

3. Proof of main result

3.1. The global existence. Since $E(t)$ in a nonincreasing function on t, we have from (1) that

$$\frac{1}{2}\|u_t\|^2 + \frac{r - p}{rp}\|\nabla u\|_p^p \leq \frac{1}{2}\|u_t\|^2 + J(u) = E(t) \leq E(0). \tag{4}$$

Hence, we get

$$\|u_t\|^2 + \|\nabla u\|_p^p \leq \max\left(2, \frac{rp}{r - p} \right) E(0) < +\infty.$$

Then the above inequality and the continuation principle complete the proof of the global existence of solution, that is $T_{\max} = \infty$.

3.2. Decay estimate. Now, we shall derive the decay estimate for the solutions in Theorem 2.8. For this we use the method of multipliers. From now on, C denotes various positive constants depending on the known constants and may be different at each appearance.

Multiplying the first equation in (P) by $E(t)^q u$ and integrating over $\Omega \times [T, S]$, where $0 \le S \le T \le \infty$. We obtain that

$$0 = \int_S^T \int_\Omega E(t)^q u [u_{tt} - \mathrm{div}(|\nabla_x u|^{p-2}\nabla_x u) - a\,\mathrm{div}(|\nabla_x u_t|^{m-2}\nabla_x u_t) - bu|u|^{r-2}]dxdt.$$

Since

$$\int_S^T \int_\Omega E(t)^q u u_{tt}\, dx\, dt$$

$$= \int_\Omega E(t)^q u u_t dx \Big|_S^T - \int_S^T \int_\Omega E(t)^q |u_t|^2 dx\, dt - q \int_S^T \int_\Omega E(t)^{q-1} E'(t) u u_t\, dx\, dt,$$

we deduce that

$$
\begin{aligned}
0 = {}& \int_S^T \int_\Omega E(t)^q \left(|u_t|^2 + \frac{2}{p}|\nabla u|^p - \frac{2b}{r}|u|^r \right) dx\, dt \\
& - 2\int_S^T \int_\Omega E(t)^q |u_t|^2\, dx\, dt + \int_\Omega E(t)^q u u_t\, dx \Big|_S^T \\
& - q\int_S^T \int_\Omega E(t)^{q-1} E'(t) u u_t\, dx\, dt + \left(1 - \frac{2}{p}\right)\int_S^T \int_\Omega E(t)^q |\nabla u|^p dx\, dt \\
& + \left(\frac{2b}{r} - b\right)\int_S^T E(t)^q \int_\Omega |u|^r\, dx\, dt + a\int_S^T \int_\Omega E(t)^q |\nabla u_t|^{m-2}\nabla u_t \nabla u\, dx\, dt.
\end{aligned}
\tag{5}
$$

From Lemma 2.3, (1) and (3) we get

$$
\begin{aligned}
& b\left(1 - \frac{2}{r}\right)\int_S^T E(t)^q \|u\|_r^r\, dt \\
& \le b\left(1 - \frac{2}{r}\right)\int_S^T E(t)^q C^r \|\nabla u\|_p^r\, dt \\
& = b\left(1 - \frac{2}{r}\right)\int_S^T E(t)^q C^r \|\nabla u\|_p^{r-p}\|\nabla u\|_p^p\, dt \\
& \le b\left(1 - \frac{2}{r}\right)\int_S^T E(t)^q C^r \left(\frac{rp}{r-p}E(0)\right)^{\frac{r-p}{p}} \frac{rp}{r-p} E(t) dt \\
& = \theta\frac{(r-2)p}{r-p}\int_S^T E(t)^{q+1} dt,
\end{aligned}
\tag{6}
$$

and

$$\frac{p-2}{p}\int_T^S E(t)^q \|\nabla u\|_p^p\, dt \le \frac{r(p-2)}{r-p}\int_T^S E(t)^{q+1} dt. \tag{7}$$

Consequently, from (5)–(7), it follows that

$$\frac{4r - p[(r-2)\theta + r + 2]}{r - p} \int_T^S E(t)^{q+1} dt$$

$$\leq 2 \int_S^T \int_\Omega E(t)^q |u_t|^2 \, dx \, dt - \int_\Omega E(t)^q u u_t \, dx \Big|_S^T \tag{8}$$

$$+ q \int_S^T \int_\Omega E(t)^{q-1} E'(t) u u_t \, dx \, dt - a \int_S^T \int_\Omega E(t)^q |\nabla u_t|^{m-2} \nabla u_t \nabla u \, dx \, dt.$$

Here, we have $\frac{4r - p[(r-2)\theta + r + 2]}{r-p} > 0$ as long as $0 < \theta < 1$.

Using the Hölder inequality, we get the estimate

$$\left| -a \int_S^T \int_\Omega E(t)^q |\nabla u_t|^{m-2} \nabla u_t \nabla u \, dx \right| \leq a \int_S^T E(t)^q \|\nabla u_t\|_{\frac{p(m-1)}{p-1}}^{m-1} \|\nabla u\|_p,$$

and

$$\|\nabla u_t\|_{\frac{p(m-1)}{p-1}}^{m-1} \leq |\Omega|^{\frac{p-m}{pm}} \|\nabla u_t\|_m^{m-1}.$$

Set $\varepsilon_1 > 0$; thanks to Young's inequality, Lemma 2.3 and the energy identity from Lemma 2.1, we get

$$\left| -a \int_S^T \int_\Omega E(t)^q |\nabla u_t|^{m-2} \nabla u_t \nabla u \, dx \right|$$

$$\leq a \int_S^T E(t)^q \left(\frac{rp}{r-p} E(t) \right)^{\frac{1}{p}} \left(\frac{-E'(t)}{a} \right)^{\frac{m-1}{m}} dt \tag{9}$$

$$\leq \frac{1}{a^{\frac{1}{m}}} C \left(\frac{rp}{r-p} \right)^{\frac{1}{p}} \varepsilon_1^m \frac{1}{m} \int_S^T E(t)^{(q+\frac{1}{p})m} dt + C \frac{1}{\varepsilon_1^{\frac{m}{m-1}}} \frac{m-1}{m} E(S).$$

Therefore, we have

$$2 \int_S^T \int_\Omega E(t)^q |u_t|^2 \, dx \, dt$$

$$\leq C \int_S^T E(t)^q \|\nabla u_t\|_m^2 dt$$

$$\leq C \int_S^T E(t)^q \left(-\frac{E'}{a} \right)^{\frac{2}{m}} dt$$

$$\leq C \varepsilon_2^{\frac{m}{m-2}} \frac{m-2}{m} \int_S^T E(t)^{q\frac{m}{m-1}} dt + C \frac{1}{\varepsilon_2^{\frac{m}{2}}} \frac{2}{m} \int_S^T (-E') \, dt \tag{10}$$

$$\leq C \varepsilon_2^{\frac{m}{m-2}} \frac{m-2}{m} \int_S^T E(t)^{q\frac{m}{m-1}} dt + C \frac{1}{\varepsilon_2^{\frac{m}{2}}} \frac{2}{m} (E(S) - E(T))$$

$$\leq C \varepsilon_2^{\frac{m}{m-2}} \frac{m-2}{m} \int_S^T E(t)^{q\frac{m}{m-1}} dt + C \frac{1}{\varepsilon_2^{\frac{m}{2}}} \frac{2}{m} E(S),$$

where we have also used Lemma 2.1 and the Young inequality for $\varepsilon_2 > 0$. Also, using Hölder's inequality, Lemma 2.1 and (4) we have

$$
\begin{aligned}
&\left| q \int_S^T \!\!\! \int_\Omega E(t)^{q-1} E'(t) u u_t \, dx \, dt \right| \\
&\leq q \int_S^T E(t)^{q-1} |E'(t)| \left(\frac{C^p r p}{r-p} \frac{r-p}{rp} \|\nabla u\|_p^p + \frac{1}{2} \|u_t\|_2^2 \right) dt \\
&\leq -q \max \left(\frac{C^p r p}{r-p}, 1 \right) \int_S^T E(t)^q E'(t) dt \\
&\leq C E(S)^{q+1},
\end{aligned}
\tag{11}
$$

and

$$
\left| - \int_\Omega E(t)^q u u_t \, dx \Big|_S^T \right| \leq \max \left(\frac{C^p r p}{r-p}, 1 \right) E(t)^{q+1} \Big|_S^T \leq C E(S)^{q+1}.
\tag{12}
$$

We have five cases related to the parameters p and m.

Case 1 : $\mathbf{p \geq m \geq 3}$. We choose q such that

$$
q + 1 = q \frac{m}{m-2}.
$$

Giving $q = \frac{m-2}{2}$ and hence $q + 1 + \alpha = (q + 1/p) m$ with

$$
\alpha = \frac{(m-2)(m-1)}{2} + \frac{m}{p} - 1 > 0.
$$

Set $\varepsilon_1 = \varepsilon E(0)^{-\frac{\alpha}{m}}$. Choosing $\varepsilon, \varepsilon_1$ small enough, then substituting the estimates (9)–(12) into (8) we get

$$
\begin{aligned}
\int_S^T E(t)^{1+q} dt &\leq C E(S)^{q+1} + C' E(S) + C'' E(0)^{\frac{(m-2)}{2} + \frac{m-p}{p \, (m-1)}} E(S) \\
&\leq \left(C' + C E(0)^q + C'' E(0)^{\frac{(m-2)}{2} + \frac{m-p}{p \, (m-1)}} \right) E(S),
\end{aligned}
$$

where C, C' and C'' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 2.4 that

$$
E(t) \leq \left(C' + C E(0)^q + C'' E(0)^{\frac{(m-2)}{2} + \frac{m-p}{p \, (m-1)}} \right)^{\frac{1}{q}} \left(1 + \frac{1}{q} \right)^{\frac{1}{q}} t^{-\frac{1}{q}}.
$$

Case 2 : $2 < \mathbf{m} < 3$ and $\mathbf{m} \le \mathbf{p} \le \frac{2}{3-\mathbf{m}}$. We have $\frac{m-2}{2} \ge \frac{p-m}{p(m-1)}$. Thus we take $q = \frac{m-2}{2}$ and

$$\alpha = \frac{(m-2)(m-1)}{2} + \frac{m}{p} - 1 \ge 0.$$

Set $\varepsilon_1 = \varepsilon E(0)^{-\frac{\alpha}{m}}$. Choosing $\varepsilon, \varepsilon_1$ small enough, then substituting the estimates (9)–(12) into (8) we get

$$\int_S^T E(t)^{1+q}dt \le CE(S)^{q+1} + C'E(S) + C''E(0)^{\frac{(m-2)}{2}+\frac{m-p}{p(m-1)}}E(S)$$

$$\le \left(C' + CE(0)^{q+1} + C''E(0)^{\frac{(m-2)}{2}+\frac{m-p}{p\ (m-1)}}\right)E(S),$$

where C, C' and C'' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 2.4 that

$$E(t) \le \left(C' + CE(0)^q + C''E(0)^{\frac{(m-2)}{2}+\frac{m-p}{p\ (m-1)}}\right)^{\frac{1}{q}}\left(1+\frac{1}{q}\right)^{\frac{1}{q}}t^{-\frac{1}{q}}.$$

Case 3 : $2 < \mathbf{m} < 3$ and $\mathbf{p} > \frac{2}{3-\mathbf{m}}$. We have $\frac{m-2}{2} < \frac{p-m}{p(m-1)}$. We choose q such that

$$q + 1 = \left(q + \frac{1}{p}\right)m.$$

Giving $q = \frac{p-m}{p(m-1)}$ and hence $q + 1 + \alpha_1 = q\frac{m}{m-2}$ with

$$\alpha_1 = \frac{m(3p - pm - 2)}{p(m-1)(m-2)} > 0.$$

Set $\varepsilon_2 = \varepsilon E(0)^{-\frac{(m-2)\alpha_1}{m}}$. Choosing $\varepsilon, \varepsilon_2$ small enough, then substituting the estimates (9)–(12) into (8) we get

$$\int_S^T E(t)^{1+q}dt \le CE(S)^{q+1} + C'E(S) + C''E(0)^{\frac{m(3p-p\ m-2)}{2p(m-1)}}E(S)$$

$$\le \left(C' + CE(0)^q + C''E(0)^{\frac{m(3p-p\ m-2)}{2p(m-1)}}\right)E(S),$$

where C, C' and C'' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 2.4 that

$$E(t) \le \left(C' + CE(0)^q + C''E(0)^{\frac{m(3p-pm-2)}{2p(m-1)}}\right)^{\frac{1}{q}}\left(1+\frac{1}{q}\right)^{\frac{1}{q}}t^{-\frac{1}{q}}.$$

Case 4 : m = 2 and p > 2. We obtain instead of (10) that

$$2\int_S^T E^q \int_\Omega |u'|^2\,dxdt \le C \int_S^T E(t)^q \left(-\frac{E'}{a}\right) dt \le \frac{C}{q+1} E(S)^{q+1}. \qquad (13)$$

So, we choose q such that

$$q+1 = 2\left(q+\frac{1}{p}\right),$$

so that $q = \frac{p-2}{p}$. Choosing ε_1 small enough, then substituting the estimates (9), (11)–(13) into (8) we get

$$\int_S^T E(t)^{1+q}dt \le CE(S) + C'E(S)^{q+1} \le (C + C'E(0)^q)\,E(S),$$

where C, C' and C'' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 2.4 that

$$E(t) \le (C + C'E(0)^q)^{\frac{1}{q}} \left(1+\frac{1}{q}\right)^{\frac{1}{q}} t^{-\frac{1}{q}}.$$

Case 5 : m = p = 2. We choose $q = 0$. Choosing ε_1 small enough, then substituting the estimates (9)–(12) into (8) we get $\int_S^T E(t)dt \le CE(S)$, where C is a positive constant independent of $E(0)$. Hence, we deduce from Lemma 2.4, that

$$E(t) \le E(0) \exp\left(1-\frac{t}{C}\right).$$

This ends the proof of Theorem 2.8.

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