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Regularity Results for a Cahn-Hilliard-Navier-Stokes System with Shear Dependent Viscosity

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Abstract. We analyze a diffuse interface model describing the behavior of a mixture of two incompressible fluids. More precisely, we consider Navier-Stokes type equations with power-law like shear dependent viscosity. Such equations are nonlinearly coupled with a convective Cahn-Hilliard equation for the order parameter. The resulting system is endowed with no-slip and no-flux boundary conditions. We prove some regularity properties of weak solutions under rather general conditions. This is a generalization of previous results already proven for single fluids. Some consequences of such results are also addressed.

Keywords. Incompressible viscous fluids, shear dependent viscosity, binary mixtures, diffuse interfaces, Cahn-Hilliard equations, regularity of solutions

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1. Introduction

Diffuse interface methods are widely used in Fluid Mechanics to describe multiphase flows (see, e.g., [4, 16]). A typical model, known as model H (see [12, 13, 20]), consists of the Navier-Stokes equations coupled with a convective Cahn-Hilliard equation. The goal is to model the behavior of an isothermal binary mixture of incompressible fluids which are partially miscible in the diffuse interface (cf., for instance, [11]). If we assume that the density of the mixture is

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constant (and equal to one), then the model reduces to the following system

$$
\begin{aligned}
\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\boldsymbol{\mathcal{T}}(\phi,\boldsymbol{e}(\boldsymbol{u})) + \nabla\pi &= k\mu\nabla\phi + \boldsymbol{f}(t),\\
\nabla\cdot\boldsymbol{u} &= 0,\\
\partial_t\phi + (\boldsymbol{u}\cdot\nabla)\phi - \Delta\mu &= 0,\\
\mu &= -\varepsilon\Delta\phi + \alpha F'(\phi),
\end{aligned}
\tag{1}
$$

in $\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain and $T>0$ is given. Here $\boldsymbol{u}$ is the (average) velocity, ϕ is the difference of the two concentrations, μ is the chemical potential and $F:\mathbb{R}\to\mathbb{R}$ is a (regular) double-well potential (for instance, $F(y)=(y^2-1)^2$). The constants κ, ε and α are all positive. We recall that ε represents the thickness of the diffuse interface, while κ and α are proportional to ε and ε^{-1}, respectively.

The stress tensor $\boldsymbol{\mathcal{T}}$ is defined by the constitutive relation

$$
\boldsymbol{\mathcal{T}}(\phi,\boldsymbol{e}(\boldsymbol{u})) = \big(\nu_1(\phi)+\nu_2(\phi)|\boldsymbol{e}(\boldsymbol{u})|^{p-2}\big)\boldsymbol{e}(\boldsymbol{u}). \tag{2}
$$

Here ν_i are strictly positive given functions, $\boldsymbol{e}(\boldsymbol{u})$ is the usual symmetric velocity gradient and $p>1$. Of course, in the case $p=2$ we obtain the so-called Cahn-Hilliard-Navier-Stokes system which has been theoretically analyzed in several contributions (see, e.g., [1, 6, 9, 23, 24]).

In the case $p\neq 2$ the model for a single fluid is known as Ladyzhenskaya model (cf. [15]). However, it seems that this kind of assumption was first proposed in [22] to describe the behavior of a large class of non-Newtonian fluids. We recall that the degenerate case $p\in(1,2)$ accounts for shear-thinning fluids, while $p>2$ corresponds to shear-thickening ones. Let us briefly review the results related to Ladyzhenskaya model. Basic existence theory, under the condition $p\geq\frac{11}{5}$, is due to Ladyzhenskaya [15] who also observed that uniqueness holds if $p\geq\frac{5}{2}$; more generally, any solution with the regularity (39) is unique in the class of weak solutions. The existence of a weak solution is obtained by the method of monotone operators in the space $(7)_1$. It is worth observing that even for $\frac{11}{5}>p>\frac{6}{5}$ it is possible to establish the existence of a weak solution by using the method of Lipschitz truncation (cf. [8]).

According to the current state-of-the-art, it seems that if $p>\frac{11}{5}$ strictly, one can get one (and only one) more derivative (or strong solution) both in time and space. Spatial regularity was proven in [17, Chapter 5] in the periodic setting. Apparently, in the no-slip setting, one needs the slightly stronger condition $p>\frac{9}{4}$ (see [18]). Higher time regularity was established in [7], for any $p>\frac{11}{5}$ and for the physically more relevant boundary conditions. All these additional regularity estimates entail the unique continuation property of weak solutions as well as the existence of a finite-dimensional exponential attractor (see [7]). However, further regularity (or even uniqueness) of weak solutions for smaller p seems out of reach or, perhaps, as difficult as for the Navier-Stokes equations.

On the other hand, the situation is rather satisfactory in dimension two, where the Ladyzhenskaya model enjoys full regularity.

The above results are mostly paralleled in the literature about the binary fluid mixtures; more precisely, system (1) with (2). Basic existence theory (for $p \geq \frac{11}{5}$) was obtained in [14], and it has been recently extended to $p > \frac{6}{5}$ in [2]. Higher regularity in space, for $p > \frac{11}{5}$, in the periodic setting, can be found in [10]. The same system with a singular potential has been recently analyzed in [5]. In this case F is defined on a finite interval with F' unbounded at the endpoints. This includes, for instance, the physically relevant logarithmic potential.

The main aim of the present paper is to extend the result of [7] to (1) with (2) subject to no-slip and no-flux boundary conditions for $\boldsymbol{u}$ and ϕ, μ, respectively. More precisely, we assume

$$\boldsymbol{u} = \boldsymbol{0}, \quad \frac{\partial \phi}{\partial \boldsymbol{n}} = \frac{\partial \mu}{\partial \boldsymbol{n}} = 0, \tag{3}$$

on $\partial\Omega \times (0, T)$, where $\boldsymbol{n}$ stands for its outward normal and $\partial\Omega$ is as smooth as needed.

This paper is organized as follows. In the next section we introduce the functional setting for weak solutions, recalling the existence result and state our main regularity theorem. In Section 3, we briefly review the Nikolskii spaces and we also observe that any weak solution is already endowed with certain fractional time regularity. The proof of the main Theorem 2.3 is carried out in Section 4 (regularity of ϕ) and Section 5 (regularity of $\boldsymbol{u}$). Some of its consequences (i.e. uniqueness, existence of a finite-dimensional attractor) are discussed in Section 6.

2. Problem setting and main result

Let us begin with the assumptions on the nonlinearities. In particular, the stress tensor will be allowed to be more general that the one given by relation (2). More precisely we require

- $\boldsymbol{\mathcal{T}}(z, \boldsymbol{\mathcal{S}})$ is (locally) Lipschitz with respect to z and p-elliptic in $\boldsymbol{\mathcal{S}}$, i.e.

$$\begin{gathered}
\big(\boldsymbol{\mathcal{T}}(z,\boldsymbol{\mathcal{S}}_1) - \boldsymbol{\mathcal{T}}(z,\boldsymbol{\mathcal{S}}_2)\big)\cdot(\boldsymbol{\mathcal{S}}_1 - \boldsymbol{\mathcal{S}}_2) \geq \begin{cases} c(1+|\boldsymbol{\mathcal{S}}_1|+|\boldsymbol{\mathcal{S}}_2|)^{p-2}|\boldsymbol{\mathcal{S}}_1-\boldsymbol{\mathcal{S}}_2|^2 \\ c|\boldsymbol{\mathcal{S}}_1-\boldsymbol{\mathcal{S}}_2|^2 + c|\boldsymbol{\mathcal{S}}_1-\boldsymbol{\mathcal{S}}_2|^p. \end{cases} \\
|\boldsymbol{\mathcal{T}}(z,\boldsymbol{\mathcal{S}}_1) - \boldsymbol{\mathcal{T}}(z,\boldsymbol{\mathcal{S}}_2)| \leq c(1+|\boldsymbol{\mathcal{S}}_1|+|\boldsymbol{\mathcal{S}}_2|)^{p-2}|\boldsymbol{\mathcal{S}}_1-\boldsymbol{\mathcal{S}}_2|, \\
|\boldsymbol{\mathcal{T}}(z_1,\boldsymbol{\mathcal{S}}) - \boldsymbol{\mathcal{T}}(z_2,\boldsymbol{\mathcal{S}})| \leq c(1+|\boldsymbol{\mathcal{S}}|)^{p-1}|z_1-z_2|,
\end{gathered} \tag{4}$$

- F has a polynomially controlled growth and is convex close to infinity, i.e.

$$F \in C^3(\mathbb{R}, \mathbb{R}), \quad \liminf_{|y|\to\infty} F''(y) > 0, \quad |F^{(3)}(y)| \leq c(1+|y|^{r-1}). \tag{5}$$

- The key assumptions in view of the growth of the nonlinearities read

$$p \ge \frac{11}{5}, \quad r \le 3. \tag{6}$$

Throughout the paper, we use L^p, $W^{1,s}$ to denote the standard Lebesgue and Sobolev spaces of (scalar, vector- or tensor-valued) functions on Ω. In particular, we set

$$H = L^2 \cap \{\mathrm{div} = 0\}, \quad V_p = W_0^{1,p} \cap \{\mathrm{div} = 0\},$$

where the subscript 0 indicates the vanishing of the trace. We will use $\langle \cdot, \cdot \rangle_X$ for the duality between X^* and X (in this order); without any subscript, the symbol stands for the scalar product in L^2.

Definition 2.1. Let $\boldsymbol{f} \in L^{p'}(0,T;V_p^*)$. A pair $(\boldsymbol{u},\phi)$ such that

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(0,T;H) \cap L^p(0,T;V_p), \\ &\phi \in L^\infty(0,T;W^{1,2}) \cap L^2(0,T;W^{3,2}), \end{aligned} \tag{7}$$

is called weak solution to (1), (3) if the following identities are satisfied

$$\begin{aligned} \frac{d}{dt}\langle \boldsymbol{u},\boldsymbol{v}\rangle + \int_\Omega \boldsymbol{\mathcal{T}}(\phi,\boldsymbol{e}(\boldsymbol{u})) : \boldsymbol{e}(\boldsymbol{v}) - (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{v} - k\mu\nabla\phi\cdot\boldsymbol{v}\,dx &= \langle \boldsymbol{f}(t),\boldsymbol{v}\rangle_{V_p}, \\ \frac{d}{dt}\langle \phi,\eta\rangle - \int_\Omega \phi\boldsymbol{u}\cdot\nabla\eta + \nabla\mu : \nabla\eta\,dx &= 0, \\ \int_\Omega \varepsilon\nabla\phi\cdot\nabla\eta + \alpha F'(\phi)\eta\,dx &= \langle \mu,\eta\rangle \end{aligned} \tag{8}$$

for any pair $(\boldsymbol{v},\eta) \in V_p \times W^{1,2}$, in the sense of distributions on $(0,T)$.

Some remarks are in order. Assumptions (4), (6) ensure that integrals in (8) are well defined. We will see – cf. (22), (24) in the proof of Theorem 3.2 below – that (7) and (8) actually entail that

$$\partial_t \boldsymbol{u} \in L^{p'}(0,T;V_p^*), \quad \partial_t\phi \in L^2(0,T;W^{-1,2}). \tag{9}$$

Thus $\boldsymbol{u}$ and ϕ, $\Delta\phi$ are admissible test functions for the first and third equation in (1), respectively. Hence, weak solutions have continuous representatives in the class

$$\boldsymbol{u} \in C([0,T];H), \quad \phi \in C([0,T];W^{1,2}), \tag{10}$$

and one is allowed to take $\boldsymbol{v} = k^{-1}\boldsymbol{u}(t)$, $\eta = \mu(t)$ in (8) to deduce the usual dissipative estimate (cf., e.g., [10, Theorem 3.1]).

Boundary condition for $\boldsymbol{u}$ is incorporated in the function space V_p, while boundary conditions for ϕ and μ follow from (8) by the usual argument based on the fact that *any* $\eta \in W^{1,2}$ can be used as a test function. We will employ this repeatedly without explicit mentioning, while integrating by parts. Observe also that taking $\eta \equiv 1$ implies that $\langle \phi(t), 1\rangle$ is a conserved quantity.

The existence of a weak solution is given by

Theorem 2.2. *Let $T > 0$, $\boldsymbol{u}_0 \in H$, $\phi_0 \in W^{1,2}$ and $\boldsymbol{f} \in L^{p'}(0,T;V_p^*)$ be given. Let* (4)–(6) *be satisfied. Then* (1)–(3) *has at least one weak solution in the sense of Definition* 2.1*, satisfying $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\phi(0) = \phi(0)$.*

We omit the proof. The reader can consult [14, Theorem 1] which deals with a slightly more general system. The key observation, namely, that sequences of weak solutions with convergent initial conditions are compact, is also proven in [10, Theorem 2.4] for the case of periodic boundary conditions. The argument uses standard tools – monotonicity of $D \mapsto \boldsymbol{\mathcal{T}}(\cdot, D)$ and compactness for the lower-order nonlinearities. We remark that, by replacing the first of (6) with the weaker condition $p > \frac{6}{5}$, the existence can still be established; however, since $\boldsymbol{u}$ is not longer an admissible test function, a much more refined argument is needed, see [2].

The main result of this contribution is the following – we refer to Section 3 below for definition of the spaces $N^{s,p}$.

Theorem 2.3. *Let $(\boldsymbol{u}, \phi)$ be an arbitrary weak solution in the sense of Definition* 2.1*. Let moreover $\boldsymbol{f} \in N^{\frac{1}{p'},p'}(0,T;V_p^*)$ and $p > \frac{11}{5}$. Then for almost every $t_0 \in (0,T)$, one also has*

$$\boldsymbol{u} \in N^{\frac{1}{2},\infty}(t_0,T;L^2) \cap N^{\frac{1}{2},2}(t_0,T;V_2) \cap N^{\frac{1}{p},p}(t_0;V_p), \tag{11}$$

$$\phi \in N^{\frac{1}{2},\infty}(t_0,T;W^{1,2}) \cap N^{\frac{1}{2},2}(t_0,T;W^{3,2}). \tag{12}$$

We mention in passing that analogous conclusions can be proven in the 2d setting, under the conditions $p > 2$ and $r < \infty$ in place of (6).

3. Nikolskii spaces

Let $I \subset \mathbb{R}$ be interval and X be a (real) Banach space. For arbitrary $s \in (0,1]$ and $p \geq 1$ the Nikolskii space $N^{s,p}(I;X)$ of functions $u : I \to X$ is defined through the norm

$$\|u\|_{L^p(I;X)} + \sup_{h>0} h^{-s} \|\mathrm{d}^h u\|_{L^p(I_h;X)}. \tag{13}$$

Here and in the sequel we set

$$\begin{aligned} I_h &= \{t \in I;\ t+h \in I\}, \\ (\mathrm{d}^h u)(t) &= u(t+h) - u(t), \quad t \in I_h, \\ (\tau^h u)(t) &= u(t+h), \quad t \in I_h. \end{aligned}$$

It is not difficult to check that for $s = 1$, the supremum in (13) is finite if and only if $\frac{d}{dt}u \in L^p(I;X)$. In other words, $N^{1,p} = W^{1,p}$. For $s \in (0,1)$ these spaces are a possible description of fractional regularity of the time derivative. The embedding theorems hold as expected, i.e.

$$\begin{aligned} N^{s,p}(I;X) &\subset L^q(I;X), && \text{if } \frac{1}{q} > \frac{1}{p} - s > 0, \\ N^{s,p}(I;X) &\subset C^{0,\alpha}(I;X), && \text{if } \alpha = s - \frac{1}{p} > 0. \end{aligned} \tag{14}$$

We remark that Nikolskii spaces are a special instance of the so-called Besov spaces, which are covered in a number of books, see, e.g., [3, Chapter 7]. It is worth mentioning [21], which develops the whole theory in a rather elementary fashion.

For a later reference, it will be useful to observe that weak solutions are already endowed with some fractional time regularity (cf. (7) and (9)). Let us formulate it in terms of the following abstract "interpolation" lemma which can be easily proven through Hölder's inequality.

Lemma 3.1. *Let $Y \subset X \subset Z$ be such that $\|u\|_X^2 \le c\|u\|_Y \|u\|_Z$ for some positive constant c. Let*

$$u \in N^{s_1,p_1}(I;Y) \cap N^{s_2,p_2}(I;Z)$$

Then

$$u \in N^{s,p}(I;X), \quad s = \frac{s_1 + s_2}{2}, \quad \frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

In particular, we have that any weak solution satisfies

$$\boldsymbol{u} \in N^{\frac{1}{2},2}(0,T;H), \quad \phi \in N^{\frac{1}{2},2}(0,T;W^{1,2}). \tag{15}$$

Indeed, just take $s_1 = 0$, $s_2 = 1$ and $p_1 = p$, $p_2 = p'$, $X = H$, $Y = V_p$, $Z = V_p^*$ or $p_1 = p_2 = 2$, $X = W^{1,2}$, $Y = W^{3,2}$, $Z = (W^{1,2})^*$, respectively.

We remark that one usually employs the duality between $\boldsymbol{u}$ and $\partial_t \boldsymbol{u}$ to deduce the continuity of solutions (10). Property (15) is slightly weaker, as $N^{\frac{1}{2},2}$ just falls short of being embedded into L^∞. On the other hand, it contains certain *quantitative* estimate on the time continuity of solutions, namely,

$$\int_0^{T-h} \|\boldsymbol{u}(t+h) - \boldsymbol{u}(t)\|_2^2 dt + \int_0^{T-h} \|\phi(t+h) - \phi(t)\|_2^2 dt = \mathcal{O}(h), \quad h \to 0. \tag{16}$$

Moreover, by employing the structure of the equation, that is to say, the functional relationship between $\boldsymbol{u}$, $\partial_t \boldsymbol{u}$, and ϕ, $\partial_t \phi$, respectively, we will be able to show that integrands in (16) actually admit analogous estimates for almost every t, cf. (20) below.

Let $\psi : I \to \mathbb{R}$ be a (scalar) integrable function. Extending ψ by zero outside I, the Hardy-Littlewood maximal function is defined as

$$M\psi(t) := \sup_{h>0} \frac{1}{2h} \int_{t-h}^{t+h} |\psi(s)|\, ds. \tag{17}$$

We recall that

$$\begin{aligned} |\psi(t_0)| &\le M\psi(t_0), && \text{for almost any } t_0 \in I, \\ \Big| \int_0^h \psi(t_0+s)\, ds \Big| &\le hM\psi(t_0), && \text{for any } t_0 \in I,\ h>0. \end{aligned} \tag{18}$$

Theorem 3.2. *Let* $(\boldsymbol{u}, \phi)$ *be an arbitrary weak solution and set*

$$\Phi(t) := 1 + \|\boldsymbol{u}(t)\|_{1,p}^p + \|\phi(t)\|_{3,2}^2 + \|\boldsymbol{f}(t)\|_{V_p^*}^{p'}. \tag{19}$$

Then

$$\|\boldsymbol{u}(t_0+h) - \boldsymbol{u}(t_0)\|_2^2 + \|\phi(t_0+h) - \phi(t_0)\|_{1,2}^2 \le chM\Phi(t_0), \tag{20}$$

where the constant c *only depends on the norms of*

$$\boldsymbol{u} \in L^\infty(0,T;H), \quad \phi \in L^\infty(0,T;W^{1,2}). \tag{21}$$

Proof. We first establish that

$$\|\partial_t \boldsymbol{u}(t)\|_{V_p^*} \le c\big(1 + \|\boldsymbol{u}(t)\|_{1,p}^{p-1} + \|\phi(t)\|_{3,2} + \|\boldsymbol{f}(t)\|_{V_p^*}\big). \tag{22}$$

Omitting t for the sake of brevity, we invoke (1) to bound

$$\begin{aligned} \langle \partial_t \boldsymbol{u}, \boldsymbol{v} \rangle_{V_p} &= \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} - \boldsymbol{\mathcal{T}}(\phi, \boldsymbol{e}(\boldsymbol{u})) : \boldsymbol{e}(\boldsymbol{v}) + k\mu \nabla\phi \cdot \boldsymbol{v}\, dx + \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V_p} \\ &= R_1 + R_2 + R_3 + R_4, \end{aligned}$$

independently of $\boldsymbol{v} \in V_p$ with unit norm. We start with the convective term, which is the critical term here. Using the interpolation inequality

$$\|v\|_{2p'} \le \|v\|_2^{1-a} \|v\|_{p^*}^a, \quad a = \frac{3}{5p-6}, \quad 1-a = \frac{5p-9}{5p-6}, \tag{23}$$

one has $|R_1| \le \|\boldsymbol{u}\|_{2p'}^2 \|\nabla \boldsymbol{v}\|_p \le c\|\boldsymbol{u}\|_2^{2(1-a)} \|\boldsymbol{u}\|_{1,p}^{2a} \le c\big(1 + \|\boldsymbol{u}\|_{1,p}^{p-1}\big)$. Here we have used that $2a \le p-1$ which is just the first condition in (6). The constant c depends on the norms in (21). Analogous estimate for R_2 is immediate by (4). To deal with R_3, observe that by (1), (5) and (7)

$$|\nabla\mu| \le c\big(|\nabla^3\phi| + (1+|\phi|^3)|\nabla\phi|\big), \quad \phi \in L^\infty(0,T;L^6) \cap L^2(0,T;W^{1,\infty}).$$

Thus it follows that $\|\nabla\mu\|_2 \le c\big(1+\|\phi\|_{3,2}\big)$, whence it is easy to obtain

$$|R_3| \le c\|\mu\|_6\|\nabla\phi\|_2\|\boldsymbol{v}\|_3 \le c\big(1+\|\phi\|_{3,2}\big).$$

The estimate of R_4 being trivial, (22) follows. Analogously and in a simpler manner, one gets

$$\|\partial_t\phi(t)\|_{(W^{1,2})^*} \le c\big(1+\|\phi(t)\|_{3,2}\big). \tag{24}$$

To finish the proof, we write

$$\|\boldsymbol{u}(t_0+h)-\boldsymbol{u}(t_0)\|_2^2 = \int_0^h \langle\partial_t\boldsymbol{u}(t_0+s),\boldsymbol{u}(t_0+s)-\boldsymbol{u}(t_0)\rangle_{V_p}\,ds.$$

Thus, owing to (22) and (18), we deduce

$$\begin{aligned}\|\boldsymbol{u}(t_0+h)-\boldsymbol{u}(t_0)\|_2^2 &\le \int_0^h \|\partial_t\boldsymbol{u}(t_0+s)\|_{V_p^*}\big(\|\boldsymbol{u}(t_0+s)\|_{V_p}+\|\boldsymbol{u}(t_0)\|_{V_p}\big)\,ds\\ &\le c\int_0^h \Phi(t_0+s)+\Phi(t_0)\,ds\\ &\le chM\Phi(t_0).\end{aligned}$$

In a similar way, one shows the estimate for $\|\phi(t_0+h)-\phi(t_0)\|_{1,2}^2$, starting from

$$\|\phi(t_0+h)-\phi(t_0)\|_{1,2}^2 = \int_0^h \langle\partial_t\phi(t_0+s),\phi(t_0+s)-\phi(t_0)\rangle_{W^{3,2}}\,ds.$$

This completes the proof. □

4. Improving the regularity of ϕ

Theorem 4.1. *Let $(\boldsymbol{u},\phi)$ be an arbitrary weak solution. Then, for almost any $t_0\in(0,T)$, one has*

$$\phi \in N^{\frac12,\infty}(t_0,T;W^{1,2}) \cap N^{\frac12,2}(t_0,T;W^{3,2}), \tag{25}$$

$$\phi \in L^\infty(t_0,T;W^{1,\infty}). \tag{26}$$

The bounds depend on the norms of (7) *as well as on $M\Phi(t_0)$, where Φ has been defined in* (19).

Proof. For $h>0$ and $t\in(0,T-h)$, we apply d^h to $(1)_3$ and test the resulting equation by $\mathrm{d}^h\phi$ to deduce

$$\frac12\frac{d}{dt}\|\mathrm{d}^h\phi\|_2^2+\varepsilon\|\mathrm{d}^h\Delta\phi\|_2^2 = \int_\Omega \mathrm{d}^h\big(\phi\boldsymbol{u}\big)\cdot\mathrm{d}^h\nabla\phi+\alpha\mathrm{d}^h\big(\nabla F'(\phi)\big)\cdot\mathrm{d}^h\nabla\phi\,dx =: D_1+D_2.$$

We observe that it is possible to integrate by parts in the terms D_1, D_2. We thus estimate

$$\begin{aligned} D_1 &= -\int_\Omega (\mathrm{d}^h \boldsymbol{u} \cdot \nabla)\tau^h \phi \mathrm{d}^h \phi \, dx \\ &= \int_\Omega \tau^h \phi \mathrm{d}^h \boldsymbol{u} \cdot \nabla \mathrm{d}^h \phi \, dx \\ &\le \|\tau^h \phi\|_6 \|\mathrm{d}^h \boldsymbol{u}\|_{\frac{3}{2}} \|\nabla \mathrm{d}^h \phi\|_6 \\ &\le c\|\tau^h \phi\|_{1,2} \|\mathrm{d}^h \boldsymbol{u}\|_2 \|\mathrm{d}^h \phi\|_{2,2} \\ &\le \frac{c_0 \varepsilon}{8} \|\mathrm{d}^h \phi\|_{2,2}^2 + C\|\mathrm{d}^h \boldsymbol{u}\|_2^2, \end{aligned}$$

using $(7)_2$ and standard embeddings. The constant c_0 is taken small enough such that (27) below holds. By $(5)_3$ one has

$$F'(\phi_1) - F'(\phi_2) = \int_{\phi_2}^{\phi_1} F''(s) ds \le c\big(1 + |\phi_1|^3 + |\phi_2|^3\big)|\phi_1 - \phi_2|.$$

Hence, arguing similarly, we have

$$D_2 = -\alpha \int_\Omega \mathrm{d}^h F'(\phi) \mathrm{d}^h \Delta\phi \, dx \le \frac{c_0 \varepsilon}{8} \|\mathrm{d}^h \phi\|_{2,2}^2 + C \int_\Omega \big(1 + |\phi|^6 + |\tau^h \phi|^6\big) |\mathrm{d}^h \phi|^2 \, dx.$$

The last integral can be estimated as $c\big(1+\|\phi\|_6^6+\|\tau^h\phi\|_6^6\big)\|\mathrm{d}^h\phi\|_\infty^2 \le c\|\mathrm{d}^h\phi\|_\infty^2 \le \|\mathrm{d}^h\phi\|_2^{2\epsilon}\|\mathrm{d}^h\phi\|_{2,2}^{2(1-\epsilon)} \le \frac{c_0\varepsilon}{8}\|\mathrm{d}^h\phi\|_{2,2}^2 + C\|\mathrm{d}^h\phi\|_2^2$ – cf. (31) below. Finally, for c_0 small enough

$$c_0 \|\psi\|_{2,2}^2 \le \|\Delta\psi\|_2^2 + \|\psi\|_2^2 \,; \tag{27}$$

this follows by the usual regularity for the Laplacian with homogeneous Neumann boundary condition. Therefore we deduce

$$\frac{d}{dt}\|\mathrm{d}^h\phi\|_2^2 + c_0\varepsilon\|\mathrm{d}^h\phi\|_{2,2}^2 \le C\big(\|\mathrm{d}^h\boldsymbol{u}\|_2^2 + \|\mathrm{d}^h\phi\|_2^2\big). \tag{28}$$

Integrating (28) over $(t_0, t) \subset (0, T-h)$ yields

$$\|\mathrm{d}^h\phi(t)\|_2^2 + c\int_{t_0}^t \|\mathrm{d}^h\phi\|_{2,2}^2 \le \|\mathrm{d}^h\phi(t_0)\|_2^2 + C\int_{t_0}^t \big(\|\mathrm{d}^h\boldsymbol{u}\|_2^2 + \|\mathrm{d}^h\phi\|_2^2\big). \tag{29}$$

Invoking (15) and (20), we see that the right hand side of (29) is $\mathcal{O}(h)$ for $h \to 0$, and thus

$$\phi \in N^{\frac{1}{2},\infty}(t_0, T; L^2) \cap N^{\frac{1}{2},2}(t_0, T; W^{2,2}). \tag{30}$$

Let us take ϵ sufficiently small so that $W^{2(1-\epsilon),2} \subset L^\infty$; that is to say, we need $2(1-\epsilon) > \frac{3}{2}$ or $0 \le \epsilon < \frac{1}{4}$. Interpolation inequality

$$\|v\|_{2(1-\epsilon),2} \le c\|v\|_2^\epsilon \|v\|_{2,2}^{1-\epsilon}, \tag{31}$$

elementary estimates and (14) yield

$$\phi \in N^{\frac{1}{2},\frac{2}{1-\epsilon}}(t_0,T;W^{2(1-\epsilon),2}) \subset L^\infty(t_0,T;L^\infty). \tag{32}$$

We will now repeat essentially the same argument with one more spatial derivative. Test the equation for $\mathrm{d}^h\phi$ by $\mathrm{d}^h\Delta\phi$. The latter is an admissible test function. Moreover, we can write

$$\nabla\mu = -\varepsilon\nabla\Delta\phi + \alpha F''(\phi)\nabla\phi.$$

We thus obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla \mathrm{d}^h\phi\|_2^2 + \varepsilon\|\mathrm{d}^h\nabla\Delta\phi\|_2^2 \\ &= \int_\Omega \mathrm{d}^h\big(\phi\boldsymbol{u}\big)\cdot \mathrm{d}^h(\nabla\Delta\phi) + \mathrm{d}^h\big(\alpha F''(\phi)\nabla\phi\big)\cdot \mathrm{d}^h(\nabla\Delta\phi)\,dx \\ &=: E_1 + E_2. \end{aligned} \tag{33}$$

With the help of (32), it is easy to find

$$\begin{aligned} E_1 - \frac{\varepsilon}{4}\|\mathrm{d}^h\nabla\Delta\phi\|_2^2 &\le C\int_\Omega |\mathrm{d}^h\phi|^2|\boldsymbol{u}|^2 + |\tau^h\phi|^2|\mathrm{d}^h\boldsymbol{u}|^2\,dx \\ &\le C\big(\|\mathrm{d}^h\phi\|_\infty^2\|\boldsymbol{u}\|_2^2 + \|\tau^h\phi\|_\infty^2\|\mathrm{d}^h\boldsymbol{u}\|_2^2\big) \\ &\le C\big(\|\mathrm{d}^h\phi\|_{2,2}^2 + \|\mathrm{d}^h\boldsymbol{u}\|_2^2\big). \end{aligned}$$

Similarly, invoking also (5), we have

$$\begin{aligned} E_2 - \frac{\varepsilon}{4}\|\mathrm{d}^h\nabla\Delta\phi\|_2^2 &\le C\int_\Omega |\mathrm{d}^h F''(\phi)|^2|\nabla\phi|^2 + |\tau^h F''(\phi)|^2|\mathrm{d}^h\nabla\phi|^2\,dx \\ &\le C\big(\|\mathrm{d}^h\phi\|_\infty^2\|\nabla\phi\|_2^2 + (1+\|\tau^h\phi\|_\infty^2)\|\nabla \mathrm{d}^h\phi\|_2^2\big) \\ &\le C\|\mathrm{d}^h\phi\|_{2,2}^2. \end{aligned}$$

In view of the left hand side of (33), we recall that, for some $c_0 > 0$,

$$c_0\|\psi\|_{1,2}^2 \le \|\nabla\psi\|_2^2 + \left|\int_\Omega \psi\,dx\right|^2, \tag{34}$$

$$c_0\|\psi\|_{3,2}^2 \le \|\nabla\Delta\psi\|_2^2 + \|\Delta\psi\|_2^2 + \left|\int_\Omega \psi\,dx\right|^2. \tag{35}$$

While (34) is simply a Poincaré type inequality (and, moreover, $\int_\Omega \phi\,dx$ is a constant of motion), (35) follows from standard regularity estimates for the Laplace equation.

Altogether, we deduce for any $t \in (t_0, T-h)$

$$\|\mathrm{d}^h\phi(t)\|_{1,2}^2 + c_0\varepsilon \int_{t_0}^t \|\mathrm{d}^h\phi\|_{3,2}^2 \le C\big(\|\mathrm{d}^h\phi(t_0)\|_{1,2}^2 + \int_{t_0}^t \|\mathrm{d}^h\phi\|_{2,2}^2 + \|\mathrm{d}^h\boldsymbol{u}\|_2^2\big).$$

By some previous estimates, namely (15), (20) and (30), the right hand side is $\mathcal{O}(h)$ for $h \to 0$. We have established (25). This also implies that $\nabla\phi \in L^\infty(t_0, T; L^\infty)$ along the lines of the estimates preceding (32). This finishes the proof. □

Remark 4.2. It follows from (32) that $\phi \in C^{0,\epsilon}(t_0, T; L^\infty)$, where we could have taken an arbitrary $\epsilon \in (0, \frac{1}{4})$. For future reference, it is worth recording that in fact we have

$$\phi \in C^{0,\epsilon}(t_0, T; L^\infty), \quad \text{for any } \epsilon \in \left[0, \frac{3}{4}\right). \tag{36}$$

Indeed, we just replace (31) by

$$\|v\|_{3-2\epsilon,2} \le c\|v\|_{1,2}^{\epsilon}\|v\|_{3,2}^{1-\epsilon}$$

in the above argument. It suffices to have $3 - 2\epsilon > \frac{3}{2}$, i.e. $\epsilon \in [0, \frac{3}{4})$.

5. Improving the regularity of $\boldsymbol{u}$

Theorem 5.1. *Let $(\boldsymbol{u}, \phi)$ be an arbitrary weak solution. Let moreover*

$$\boldsymbol{f} \in N^{\frac{1}{p'},p'}(0, T; V_p^*), \quad p > \frac{11}{5}. \tag{37}$$

Then for almost every $t_0 \in (0, T)$, one has

$$\boldsymbol{u} \in N^{\frac{1}{2},\infty}(t_0, T; L^2) \cap N^{\frac{1}{2},2}(t_0, T; V_2) \cap N^{\frac{1}{p},p}(t_0; V_p). \tag{38}$$

The bounds depend on the norms of (7) *as well as on $M\Phi(t_0)$, where Φ has been defined in* (19).

The key step towards the proof is improved time-integrability of $\|\nabla\boldsymbol{u}\|_p$ which is given by

Lemma 5.2. *Let $(\boldsymbol{u}, \phi)$ be arbitrary weak solution, let* (37) *hold. Then for almost every $t_0 \in (0, T)$, one has*

$$\boldsymbol{u} \in L^{p_{\mathrm{uniq}}}(t_0, T; V_p), \quad p_{\mathrm{uniq}} = \frac{2p}{2p-3}. \tag{39}$$

Proof. If $p \geq \frac{5}{2}$, then $p_{\mathrm{uniq}} \leq p$ and (39) is immediate for any $t_0 \in [0, T)$. So we can assume that $p \in (\frac{11}{5}, \frac{5}{2})$. Fix $t_0 \in (0, T)$ such that $M\Phi(t_0) < \infty$.

The proof is based on an iterative scheme. We start with

$$\boldsymbol{u} \in L^q(t_0, T; V_p), \quad \text{for some } q \in \left(\frac{11}{5}, p_{\mathrm{uniq}}\right), \tag{40}$$

and show that the integrability exponent q can be improved. To this end, test the equation for $\mathrm{d}^h\boldsymbol{u}$ by $\mathrm{d}^h\boldsymbol{u}$, deducing

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathrm{d}^h\boldsymbol{u}\|_2^2 + \int_\Omega \mathrm{d}^h\big(\boldsymbol{\mathcal{T}}(\phi, \boldsymbol{e}(\boldsymbol{u}))\big) : \mathrm{d}^h\boldsymbol{e}(\boldsymbol{u})\,dx \\
&= \int_\Omega -\mathrm{d}^h\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big)\cdot \mathrm{d}^h\boldsymbol{u} + k\mathrm{d}^h\big(\mu\nabla\phi\big)\cdot\mathrm{d}^h\boldsymbol{u} + \mathrm{d}^h\boldsymbol{f}\cdot\mathrm{d}^h\boldsymbol{u}\,dx \\
&=: D_1 + D_2 + D_3.
\end{aligned} \tag{41}$$

By (4), the Young and Korn inequalities, the integral on the left hand side is estimated from below as follows

$$\begin{aligned}
&\int_\Omega \big(\boldsymbol{\mathcal{T}}(\phi, \tau^h\boldsymbol{e}(\boldsymbol{u})) - \boldsymbol{\mathcal{T}}(\phi, \boldsymbol{e}(\boldsymbol{u}))\big)\cdot \mathrm{d}^h\boldsymbol{e}(\boldsymbol{u}) \\
&\quad + \big(\boldsymbol{\mathcal{T}}(\tau^h\phi, \tau^h\boldsymbol{e}(\boldsymbol{u})) - \boldsymbol{\mathcal{T}}(\phi, \tau^h\boldsymbol{e}(\boldsymbol{u}))\big)\cdot \mathrm{d}^h\boldsymbol{e}(\boldsymbol{u})\,dx \\
&\geq \int_\Omega c_1\big(1 + |\tau^h\boldsymbol{e}(\boldsymbol{u})| + |\boldsymbol{e}(\boldsymbol{u})|\big)^{p-2}|\mathrm{d}^h\boldsymbol{e}(\boldsymbol{u})|^2 \\
&\quad - c_2(1 + |\tau^h\boldsymbol{e}(\boldsymbol{u})|)^{p-1}|\mathrm{d}^h\phi||\mathrm{d}^h\boldsymbol{e}(\boldsymbol{u})|\,dx \\
&\geq \int_\Omega c_3\big(1 + |\tau^h\boldsymbol{e}(\boldsymbol{u})| + |\boldsymbol{e}(\boldsymbol{u})|\big)^{p-2}|\mathrm{d}^h\boldsymbol{e}(\boldsymbol{u})|^2 - c_4\big(1 + |\tau^h\boldsymbol{e}(\boldsymbol{u})|\big)^p|\mathrm{d}^h\phi|^2\,dx \\
&\geq c_0\big(\|\mathrm{d}^h\boldsymbol{u}\|_{1,2}^2 + \|\mathrm{d}^h\boldsymbol{u}\|_{1,p}^p\big) - C\big(1 + \|\tau^h\boldsymbol{u}\|_{1,p}^p\big)\|\mathrm{d}^h\phi\|_\infty^2.
\end{aligned} \tag{42}$$

We further deal with D_1, coming from the convective term. This is the most difficult term. We write

$$D_1 = \int_\Omega \big(\tau^h\boldsymbol{u}\cdot\tau^h\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big)\cdot\mathrm{d}^h\boldsymbol{u}\,dx = \int_\Omega (\mathrm{d}^h\boldsymbol{u}\cdot\tau^h\nabla\boldsymbol{u})\cdot\mathrm{d}^h\boldsymbol{u}\,dx \leq \|\tau^h\nabla\boldsymbol{u}\|_p\|\mathrm{d}^h\boldsymbol{u}\|_{2p'}^2.$$

We use interpolation inequality (23) to obtain

$$D_1 \leq c\|\tau^h\boldsymbol{u}\|_{1,p}\|\mathrm{d}^h\boldsymbol{u}\|_2^{2(1-a)}\|\mathrm{d}^h\nabla\boldsymbol{u}\|_p^{2a} \leq \frac{c_0}{4}\|\mathrm{d}^h\nabla\boldsymbol{u}\|_p^p + C\|\tau^h\boldsymbol{u}\|_{1,p}^{Q_1}\|\mathrm{d}^h\boldsymbol{u}\|_2^{Q_2},$$

where c_0 comes from last line of (42) and

$$Q_1 = \frac{p(5p-6)}{p(5p-6)-6}, \quad Q_2 = \frac{2p(5p-9)}{p(5p-6)-6}. \tag{43}$$

Observe that $Q_1 < p$ (this is equivalent to $p > \frac{11}{5}$) and $Q_2 < 2$. Secondly, we have

$$D_2 \le \int_\Omega |\mathrm{d}^h\mu||\nabla\phi||\mathrm{d}^h\boldsymbol{u}| + |\tau^h\mu||\mathrm{d}^h\nabla\phi||\mathrm{d}^h\boldsymbol{u}|\, dx. \tag{44}$$

On account of (26) and

$$|\mu| \le c\big(|\Delta\phi| + 1\big), \quad |\mathrm{d}^h\mu| \le c\big(|\mathrm{d}^h\Delta\phi| + |\mathrm{d}^h\phi|\big),$$

inequality (44) entails, together with obvious embeddings,

$$\begin{aligned} D_2 &\le c\big(\|\mathrm{d}^h\mu\|_2\|\mathrm{d}^h\boldsymbol{u}\|_2 + \|\tau^h\mu\|_4\|\mathrm{d}^h\nabla\phi\|_2\|\mathrm{d}^h\boldsymbol{u}\|_4\big) \\ &\le \frac{c_0}{2}\|\mathrm{d}^h\boldsymbol{u}\|_{1,2}^2 + C\big(\|\mathrm{d}^h\phi\|_{2,2}^2 + (1 + \|\tau^h\phi\|_{3,2}^2)\|\mathrm{d}^h\phi\|_{1,2}^2\big). \end{aligned} \tag{45}$$

Here c_0 again comes from (42). We also have

$$D_3 \le \|\mathrm{d}^h\boldsymbol{f}\|_{V_p^*}\|\mathrm{d}^h\boldsymbol{u}\|_{V_p} \le \frac{c_0}{2}\|\mathrm{d}^h\boldsymbol{u}\|_{1,p}^p + C\|\mathrm{d}^h\boldsymbol{f}\|_{V_p^*}^{p'}.$$

Taking the above estimates into account and integrating over time, from (41) we deduce

$$\begin{aligned} &\|\mathrm{d}^h\boldsymbol{u}(t)\|_2^2 + c_0\int_{t_0}^t \big(\|\mathrm{d}^h\boldsymbol{u}\|_{1,2}^2 + \|\mathrm{d}^h\boldsymbol{u}\|_{1,p}^p\big) \\ &\le \|\mathrm{d}^h\boldsymbol{u}(t_0)\|_2^2 + \int_{t_0}^t H(s)ds + c\int_{t_0}^t \|\tau^h\boldsymbol{u}\|_{1,p}^{Q_1}\|\mathrm{d}^h\boldsymbol{u}\|_2^{Q_2}, \end{aligned} \tag{46}$$

where

$$\begin{aligned} H(s) \le C\Big\{&\big(1 + \|\tau^h\boldsymbol{u}(s)\|_{1,p}^p\big)\|\mathrm{d}^h\phi(s)\|_\infty^2 \\ &+ \|\mathrm{d}^h\phi(s)\|_{2,2}^2 + (1 + \|\tau^h\phi(s)\|_{3,2}^2)\|\mathrm{d}^h\phi(s)\|_{1,2}^2 + \|\mathrm{d}^h\boldsymbol{f}(s)\|_{V_p^*}^{p'}\Big\}. \end{aligned}$$

It follows from previous results, namely (20), (36), (25) and (37), that the first two terms on the right hand side of (46) are $\mathcal{O}(h)$ for $h \to 0$.

Concerning the last term in (46), more care is needed. It will be convenient to set

$$q = \frac{p}{1-g}, \tag{47}$$

cf. (40). In view of our assumptions $0 \le g < \frac{5}{2} - p$, where the upper bound lies strictly between $\frac{10}{3}$ (for $p = \frac{11}{5}$) and 0 (if $p = \frac{5}{2}$). We apply Hölder's inequality with

$$r = \frac{p}{(1-g)Q_1} = \frac{p(5p-6)-6}{(1-g)(5p-6)}, \quad r' = \frac{2p(5p-9)}{p(5p-11) + g(5p-6)}, \tag{48}$$

to deduce

$$\int_{t_0}^{T-h} \|\tau^h \boldsymbol{u}\|_{1,p}^{Q_1} \|\mathrm{d}^h \boldsymbol{u}\|_2^{Q_2} \le c \left(\int_{t_0}^{T-h} \|\mathrm{d}^h \boldsymbol{u}\|_2^{R_2} \right)^{\frac{1}{r'}},$$

where

$$R_2 = Q_2 r' = \frac{2p(5p-9)}{p(5p-11) + g(5p-6)}.$$

We claim that $R_2 > 2$ – indeed, this is equivalent to $g < \frac{2p}{5p-6}$, which is in our situation weaker than the already mentioned condition $g < \frac{5}{2} - p$. On the other hand, $\|\mathrm{d}^h \boldsymbol{u}(s)\|_2$ is a bounded function, simply because $\boldsymbol{u} \in L^\infty(0,T;L^2)$. Hence we can further estimate

$$\left(\int_{t_0}^{T-h} \|\mathrm{d}^h \boldsymbol{u}\|_2^{R_2} \right)^{\frac{1}{r'}} \le C \left(\int_{t_0}^{T-h} \|\mathrm{d}^h \boldsymbol{u}\|_2^{2} \right)^{\frac{1}{r'}} \le C h^{\frac{1}{r'}}.$$

Obviously $\frac{1}{r'} < 1$, and thus we deduce $\boldsymbol{u} \in N^{\frac{1}{pr'},p}(t_0,T;V_p)$. In view of the embedding $(14)_1$ and recalling (47) and (48), we eventually arrive at

$$\boldsymbol{u} \in L^{\tilde{q}}(t_0,T;V_p), \quad \tilde{q} < \frac{qp}{Q_1}. \tag{49}$$

We recall that $\frac{p}{Q_1} > 1$ iff $p > \frac{11}{5}$ – cf. (43). Thus the conclusion of Lemma, namely (39), follows by iterating (40) and (49), starting from $q = p$ in the zero-th step. $\square$

Proof of Theorem 5.1. Take $t_0 \in (0,T)$ such that $M\Phi(t_0)$ is finite. We test the equation for $\mathrm{d}^h \boldsymbol{u}$ by $\mathrm{d}^h \boldsymbol{u}$, and proceed just as in the proof of the latter lemma. The only difference now is in the estimate of the convective term, namely,

$$\begin{aligned} D_1 &\le \|\tau^h \nabla \boldsymbol{u}\|_p \|\mathrm{d}^h \boldsymbol{u}\|_{2p'}^2 \\ &\le c\|\tau^h \nabla \boldsymbol{u}\|_p \|\mathrm{d}^h \boldsymbol{u}\|_2^{2a} \|\mathrm{d}^h \boldsymbol{u}\|_{1,2}^{2(1-a)} \\ &\le \frac{c_0}{4} \|\mathrm{d}^h \boldsymbol{u}\|_{1,2}^2 + C \|\tau^h \boldsymbol{u}\|_{1,p}^{p_{\mathrm{uniq}}} \|\mathrm{d}^h \boldsymbol{u}\|_2^2, \end{aligned}$$

where the interpolation inequality (23) was replaced by

$$\|v\|_{2p'} \le c\|v\|_2^a \|v\|_6^{1-a}, \quad a = \frac{2p-3}{2p}, \quad 1-a = \frac{3}{2p}.$$

We obtain

$$\begin{aligned} &\|\mathrm{d}^h \boldsymbol{u}(t)\|_2^2 + c_0 \int_{t_0}^t \left(\|\mathrm{d}^h \boldsymbol{u}\|_{1,2}^2 + \|\mathrm{d}^h \boldsymbol{u}\|_{1,p}^p \right) \\ &\le \|\mathrm{d}^h \boldsymbol{u}(t_0)\|_2^2 + \int_{t_0}^t H(s) ds + c \int_{t_0}^t \|\tau^h \boldsymbol{u}\|_{1,p}^{p_{\mathrm{uniq}}} \|\mathrm{d}^h \boldsymbol{u}\|_2^2. \end{aligned}$$

By (39), the function $\|\tau^h \boldsymbol{u}(s)\|_{1,p}^{p_{\mathrm{uniq}}}$ is integrable over $t_0, T-h$. Hence (38) follows by Gronwall's lemma. $\square$

6. Final remarks

Having established the main Theorem 2.3, here we discuss, without entering the details, some of its implications and possible extensions.

6.1. Unique continuation property. Let $(\boldsymbol{u},\phi)$, $(\tilde{\boldsymbol{u}},\tilde{\phi})$ be arbitrary weak solutions, and let $\ell\in(0,T)$ such that $(\boldsymbol{u},\phi)\equiv(\tilde{\boldsymbol{u}},\tilde{\phi})$ on $[0,\ell]$. Then $(\boldsymbol{u},\phi)\equiv(\tilde{\boldsymbol{u}},\tilde{\phi})$ on $[0,T]$. Indeed, it is enough to take $t_0\in(0,\ell)$ such that $M\Phi(t_0)<\infty$, cf. (19). It follows that $(\boldsymbol{u},\phi)$ has the regularity (11), (12). As a matter of fact, $M\tilde{\Phi}(t_0)=M\Phi(t_0)$, hence $(\tilde{\boldsymbol{u}},\tilde{\phi})$ has the same regularity. Thus it is straightforward to prove that the solutions coincide for all $t>t_0$. We remark that the critical regularity for the uniqueness in the velocity component is already contained in Lemma 5.2.

6.2. Uniform bounds. The maximal operator is of weak $(1,1)$ type, i.e.

$$\big|\{t\in(0,\ell);\ M\psi(t)>\lambda\}\big|\le\frac{c}{\lambda}\|\psi\|_{L^1(0,\ell)}.$$

Hence, one can choose $t_0\in(0,\ell)$ such that $M\Phi(t_0)$ is bounded in terms of ℓ and the basic norms of (7) only. More precisely, on account of Theorems 3.2, 4.1 and 5.1, it can be shown that the estimates obtained in Theorem 2.3 are uniform with respect to the norms (21).

6.3. Large time behavior. As $(\boldsymbol{u},\mu)$ is an admissible test function pair for the weak formulation, one can prove the existence of a bounded absorbing set in $H\times W^{1,2}$ (see, e.g., [10, Theorem 3.1] for the periodic case). Assuming for simplicity that the external force $\boldsymbol{f}$ is independent of time, i.e. $\boldsymbol{f}(t)=\boldsymbol{f}\in V_p^*$, one can investigate the global dynamics. This issue can be conveniently analyzed by using the so-called "method of ℓ-trajectories" (see [19]). More precisely, one introduces a dynamical system $(\mathcal{X}_\ell,L_t)$ in the following way

$$\mathcal{X}_\ell=\big\{(\boldsymbol{u},\phi):[0,\ell]\to H\times W^{1,2};\ \ (\boldsymbol{u},\phi)\text{ is weak solution on }[0,\ell]\big\},\tag{50}$$

$$L_t:\mathcal{X}_\ell\to\mathcal{X}_\ell,\quad \big(L_t\chi\big)(s)=\chi(t+s),\quad s\in[0,\ell].\tag{51}$$

Note that $\chi(t+s)=(\boldsymbol{u}(t+s),\phi(t+s))$ is well defined thanks to the unique continuation property (see above).

On account of the fact that the trajectories are regular in the sense of Theorem 2.3, except possibly for some (uniformly) small neighborhood of $t=0$, one proves that L_t are continuous, and even have the smoothing property for $t>0$. We omit further details, referring to [10, Section 3], where the same system with periodic boundary conditions is analyzed. We just point out that existence of an exponential attractor and, in particular, of the global attractor with finite fractal dimension can be proven (see [10, Theorem 3.3].

6.4. Further regularity. Similarly as in [7], we can further improve the regularity of $\boldsymbol{u}$ to what might be called *maximal time regularity*. As a matter of fact, we can employ the results of [7] directly, thus by-passing the proof of Theorem 5.1. Indeed, observe that $\boldsymbol{u}$ is a weak solution to a generalized Ladyzhenskaya system

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\tilde{\boldsymbol{\mathcal{T}}}(t,x,\boldsymbol{e}(\boldsymbol{u})) + \nabla\pi = \tilde{\boldsymbol{f}}(t), \quad \nabla\cdot\boldsymbol{u} = 0,$$

where we set

$$\begin{aligned}\tilde{\boldsymbol{\mathcal{T}}}(t,x,\boldsymbol{e}(\boldsymbol{u})) &:= \boldsymbol{\mathcal{T}}(\phi(t,x),\boldsymbol{e}(\boldsymbol{u})),\\ \tilde{\boldsymbol{f}}(t) &:= k\mu\nabla\phi(t) + \boldsymbol{f}(t).\end{aligned}$$

It follows from (36) that $\tilde{\boldsymbol{\mathcal{T}}}$ is uniformly κ-Hölder continuous with respect to time, for any $\kappa < \frac{3}{4}$, cf. [7, assumption (4)]. By arguments analogous to those preceding (36), one establishes that $\mu\nabla\phi \in N^{\kappa,2}_{loc}(0,T;L^2)$, for any $\kappa < \frac{3}{4}$ as well.[1] Thus, assuming in addition that

$$\boldsymbol{f} \in N^{\kappa,2}_{loc}(0,T;L^2) \tag{52}$$

for some $\kappa < \frac{3}{2}$, if follows from [7, Theorem 1.2] that

$$\boldsymbol{u} \in N^{\kappa,\infty}(t_0,T;H) \cap N^{\kappa,2}(t_0,T;V_2) \cap N^{2\frac{\kappa}{p},p}(t_0,T;V_p).$$

In particular, for $\kappa = \frac{1}{2}$ we have the conclusion of Theorem 5.1. However, we believe that it is of some interest to provide self-contained, if weaker result, also for the readers' benefit. In addition, note that our assumption (37) on $\boldsymbol{f}$ is weaker than (52) with regard to spatial regularity.

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