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Existence and Multiplicity of Solutions for Kirchhoff Type Problems Involving $p(x)$-Biharmonic Operators

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Abstract. This paper is concerned with the existence and multiplicity of weak solutions for a $p(x)$-Kirchhoff type problem of the following form

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle in two cases when the Carathéodory function $f(x,u)$ having special structure.

Keywords. $p(x)$-biharmonic operators, Kirchhoff type problems, mountain pass theorem, Ekeland's variational principle.
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1. Introduction

In this paper, we study the following problem

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x) \in C(\overline{\Omega})$ with $1 < \inf_{\overline{\Omega}} p(x) \le \sup_{\overline{\Omega}} p(x) < +\infty$, $\Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic

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operator and $M(t)$ is a continuous real-valued function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function having special structure.

Problem (1) is called a nonlocal one because of the presence of the term M, which implies that the equation in (1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particularly interesting. Nonlocal differential equations are also called Kirchhoff type equations because Kirchhoff [19] has investigated an equation of the form

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of equation (2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, and hence the equation is no longer a pointwise identity. The parameters in (2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [20] has proposed an abstract framework for the Kirchhoff type equations. After the work of Lions [20], various equations of Kirchhoff type have been studied extensively, see e.g. [3–10]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [5,6], [9,10]) and $p(x)$-Laplacian (see [7,8,17,21]).

Fourth order elliptic equations arise in many applications such as: Micro Electro Mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, and phase field models of multiphase systems (see [18, 23]) and the references therein. There is also another important class of physical problems leading to higher order partial differential equations. An example of this is Kuramoto-Sivashinsky equation which models pattern formation in different physical contexts, such as chemical reaction diffusion systems and a cellular gas flame in the presence of external stabilizing factors (see [25]).

We assume throughout this paper that the Kirchhoff function M satisfies the following hypotheses:

(M_1) there exists a positive constant m_0 such that $M(t) \geq m_0$,

(M_2) there exists $\mu \in (0,1)$ such that $\widehat{M}(t) \geq (1-\mu)M(t)t$, where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

There are many functions M satisfying the conditions (M_1) and (M_2), for example, $M(t) = a + bt$, $a, b > 0$. Inspired by the ideas in [7,21,22] and the results in [2,11], we study (1) in two distinct situations.

First, we consider the case when $f(x,u) = \lambda(x)|u|^{q(x)-2}u$ in which the weight function $\lambda(x)$ does not change sign, i.e.,

$$\begin{cases} M\left(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda(x)|u|^{q(x)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The function λ satisfies

(Λ_1) $\lambda \in L^\infty(\Omega)$,

(Λ_2) there exists an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $\lambda(x) = 0$ for $x \in \overline{B_R(x_0) \backslash B_r(x_0)}$ while $\lambda(x) > 0$ for $x \in \Omega \backslash \overline{B_R(x_0) \setminus B_r(x_0)}$,

and the function q is assumed to satisfy

(Q_1) $q \in C_+(\overline{\Omega})$ and $1 \leq q(x) < p_2^*(x)$ for any $x \in \overline{\Omega}$,

(Q_2) either $\max_{\overline{B_r(x_0)}} q(x) < p^- < \frac{p^-}{1-\mu} < p^+ < \frac{p^+}{1-\mu} < \min_{\overline{\Omega\backslash B_R(x_0)}} q(x)$ or $\max_{\overline{\Omega\backslash B_R(x_0)}} q(x) < p^- < \frac{p^-}{1-\mu} < p^+ < \frac{p^+}{1-\mu} < \min_{\overline{B_r(x_0)}} q(x)$.

Our main result concerning problem (3) is given by the following theorem.

Theorem 1.1. *Assume that conditions* (M_1)–(M_2), (Λ_1)–(Λ_2) *and* (Q_1)–(Q_2) *are fulfilled. Then there exists* $\nu^* > 0$ *such that problem* (3) *has at least two positive non-trivial weak solutions, provided that* $|\lambda|_{L^\infty(\Omega)} < \nu^*$.

For example, the functions

$$\lambda(x) = \begin{cases} \frac{1}{r}(r - |x - x_0|), & \text{for } x \in B_r(x_0) \\ 0, & \text{for } x \in \overline{B_R(x_0)\backslash B_r(x_0)} \\ \frac{1}{|x-x_0|}(|x - x_0| - R), & \text{for } x \in \Omega\backslash\overline{B_R(x_0) \setminus B_r(x_0)} \end{cases}$$

and

$$q(x) = \begin{cases} t_1, & \text{for } x \in B_r(x_0) \\ \frac{t_1(R-|x-x_0|)}{R-r} + \frac{t_2(|x-x_0|-r)}{R-r}, & \text{for } x \in \overline{B_R(x_0)\backslash B_r(x_0)} \\ t_2, & \text{for } x \in \Omega\backslash\overline{B_R(x_0) \setminus B_r(x_0)} \end{cases}$$

satisfy the above conditions, where the positive numbers t_1, t_2 can be chosen in a suitable manner such as $t_1 < p^- < \frac{p^+}{1-\mu} < t_2$ for the first case in (Q_2) and $t_2 < p^- < \frac{p^+}{1-\mu} < t_1$ for the second one.

Next, we consider the case when $f(x,u) = \lambda|u|^{q(x)-2}u$, λ is a positive parameter, that is,

$$\begin{cases} M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\Big)\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{q(x)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

More exactly, we study the existence of solutions for (4) under the hypotheses (M_1), (M_2) and $q(x)$ is a assumed to satisfy the following condition

(Q_3) $q^- < \frac{p^-}{1-\mu} < p^+$ and $q^+ < p_2^*(x)$ for all $x \in \overline{\Omega}$, μ is given by (M_2).

Our main result concerning problem (4) in this case is given by the following theorem.

Theorem 1.2. *Assume that the conditions* (M_1), (M_2) *and* (Q_3) *are fulfilled. Then there exists a positive constant* λ^* *such that for any* $\lambda \in (0, \lambda^*)$, *problem* (4) *has at least one non-trivial weak solution.*

2. Notations and preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to the papers [12, 13, 15, 16].

Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$\begin{aligned} C_+(\overline{\Omega}) &= \left\{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \text{for all } x \in \overline{\Omega}\right\}, \\ p^+ &= \max\{p(x);\ x \in \overline{\Omega}\}, \\ p^- &= \min\{p(x);\ x \in \overline{\Omega}\}, \\ L^{p(x)}(\Omega) &= \left\{u;\ u \text{ measurable real-valued function s.th.} \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}, \end{aligned}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Proposition 2.1 (see Fan and Zhao [16])**.** *The space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is separable, uniformly convex, reflexive and its conjugate space is* $L^{q(x)}(\Omega)$ *where* $q(x)$ *is the conjugate function of* $p(x)$, *i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. *For* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [14, 16]. Denote

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N \\ +\infty & \text{if } kp(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \geq 1$.

Proposition 2.2 (see Fan and Zhao [16])**.** *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1) are considered in the generalized Sobolev space

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

equipped with the norm

$$\|u\| = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Remark 2.3. According to [26], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta \cdot|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta \cdot|_{p(x)}$ are equivalent.

We consider the functional

$$\rho(u) = \int_\Omega |\Delta u|^{p(x)}\, dx$$

and give the following fundamental proposition.

Proposition 2.4 (see El Amrouss et al. [11]). *For $u \in X$ and $u_n \subset X$, we have*

(1) $\|u\| < 1(respectively = 1; > 1) \Longleftrightarrow \rho(u) < 1(respectively = 1; > 1)$;
(2) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$;
(3) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$;
(4) $\|u_n\| \to 0$ $(respectively \to \infty) \Longleftrightarrow \rho(u_n) \to 0$ $(respectively \to \infty)$.

Let us define the functional

$$J(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx.$$

It is well known that J is well defined, even and C^1 in X. Moreover, the operator $L = J' : X \to X^*$ defined as

$$\langle L(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx$$

for all $u, v \in X$ satisfies the following assertions.

Proposition 2.5 (see El Amrouss et al. [11]).

(1) *L is continuous, bounded and strictly monotone.*
(2) *L is a mapping of (S_+) type, namely*
$u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} L(u_n)(u_n - u) \le 0$, implies $u_n \to u$.
(3) *L is a homeomorphism.*

3. Proof of Theorem 1.1

In this section we discuss the existence of two non-trivial weak solutions of (3) by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle. For simplicity, we use C, c_i, $i = 1, 2, \ldots$ to denote the general positive constant (the exact value may change from line to line).

We confine ourselves to the case where the former condition of (Q_2) holds true. A similar proof can be made if the later condition holds true. The Euler-Lagrange functional associated to (3) is given by

$$I(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx\right) - \int_\Omega \frac{\lambda(x)}{q(x)} |u|^{q(x)}\, dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau)\, d\tau$. It is easy to verify that $I \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous with the derivative given by

$$\langle I'(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx - \int_\Omega \lambda(x) |u|^{q(x)-2} uv\, dx,$$

for all u, $v \in X$. Thus, we notice that we can seek weak solutions of (3) as critical point of the energetic functional I.

Remark 3.1. From (M_1) and Proposition 2.5 we can easily see that ϕ', defined by

$$\langle \phi'(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\,dx$$

is of (S_+) type.

Lemma 3.2. *There exists $\nu^* > 0$ such that provided $|\lambda|_{L^\infty(\Omega)} < \nu^*$ there exist $\rho_1 > 0$ and $\delta_1 > 0$ such that $I(u) \geq \delta_1 > 0$ for any $u \in X$ with $\|u\| = \rho_1$.*

Proof. Let us define $q_1 : \overline{B_r(x_0)} \to [1,\infty)$, $q_1(x) = q(x)$ for any $x \in \overline{B_r(x_0)}$ and $q_2 : \overline{\Omega\backslash B_R(x_0)} \to [1,\infty)$, $q_2(x) = q(x)$ for any $x \in \overline{\Omega\backslash B_R(x_0)}$. We also introduce the notation

$$q_1^- = \min_{x\in\overline{B_r(x_0)}} q_1(x), \qquad q_1^+ = \max_{x\in\overline{B_r(x_0)}} q_1(x),$$
$$q_2^- = \min_{x\in\overline{\Omega\backslash B_R(x_0)}} q_2(x), \qquad q_2^+ = \max_{x\in\overline{\Omega\backslash B_R(x_0)}} q_2(x).$$

Then by relations (Q_1) and (Q_2) we have

$$1 \leq q_1^- \leq q_1^+ < p^- < \frac{p^-}{1-\mu} < p^+ < \frac{p^+}{1-\mu} < q_2^- \leq q_2^+ < p_2^*(x),$$

for any $x \in X$. Thus, we have $X \hookrightarrow L^{q_i^\pm}(\Omega)$, $i \in \{1,2\}$. So, there exists a positive constant C such that

$$\int_\Omega |u|^{q_i^\pm}\,dx \leq C\|u\|^{q_i^\pm}, \quad \text{for all } u \in X,\ i \in \{1,2\}.$$

It follows that there exist two positive constants c_1 and c_2 such that

$$\begin{aligned}\int_{B_r(x_0)} |u|^{q_1(x)}\,dx &\leq \int_{B_r(x_0)} |u|^{q_1^-}\,dx + \int_{B_r(x_0)} |u|^{q_1^+}\,dx\\ &\leq \int_\Omega |u|^{q_1^-}\,dx + \int_\Omega |u|^{q_1^+}\,dx\\ &\leq c_1\Big(\|u\|^{q_1^-} + \|u\|^{q_1^+}\Big),\end{aligned} \tag{5}$$

and

$$\begin{aligned}\int_{\Omega\backslash B_R(x_0)} |u|^{q_2(x)}\,dx &\leq \int_{\Omega\backslash B_R(x_0)} |u|^{q_2^-}\,dx + \int_{\Omega\backslash B_R(x_0)} |u|^{q_2^+}\,dx\\ &\leq \int_\Omega |u|^{q_2^-}\,dx + \int_\Omega |u|^{q_2^+}\,dx\\ &\leq c_2\Big(\|u\|^{q_2^-} + \|u\|^{q_2^+}\Big).\end{aligned} \tag{6}$$

In view of (M_1) and relations (5) and (6), for $\|u\|$ sufficiently small, noting Proposition 2.4, we have

$$\begin{aligned} I(u) &\geq \frac{m_0}{p^+}\int_\Omega |\Delta u|^{p(x)}\,dx - \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)}\,dx - \int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)}\,dx \\ &\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{|\lambda|_{L^\infty(\Omega)}}{q^-}c_3(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}) \\ &\geq \left[\frac{m_0}{2p^+}\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_1^-} + \|u\|^{q_1^+})\right] \\ &\quad + \left[\frac{m_0}{2p^+}\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_2^-} + \|u\|^{q_2^+})\right]. \end{aligned}$$

Since the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = \frac{m_0}{2p^+} - c_4 t^{q_2^- - p^+} - c_4 t^{q_2^+ - p^+}$$

is positive in a neighborhood of the origin, it follows that there exists $0 < \rho_1 < 1$ such that $g(\rho_1) > 0$. On the other hand, defining

$$\nu^* = \min\left\{1, \frac{m_0}{2c_4 p^+}\min\{\rho^{p^+ - q_1^-}, \rho^{p^+ - q_1^+}\}\right\}, \tag{7}$$

we deduce that there exists $\delta_1 > 0$ such that for any $u \in X$ with $\|u\| = \rho_1$ we have $I(u) \geq \delta_1 > 0$ provided $|\lambda|_{L^\infty(\Omega)} < \nu^*$. □

Lemma 3.3. *There exists* $\psi \in X$, $\psi \neq 0$ *such that* $\lim_{t\to+\infty} I(t\psi) \to -\infty$.

Proof. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and there exist $x_1 \in \Omega\setminus B_R(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_1) \subset (\Omega\setminus B_R(x_0))$ we have $\psi(x) > 0$. When $t > t_0$, from (M_2) we can easily obtain that $\widehat{M}(t) \leq \widehat{M}(t_0)t_0^{-\frac{1}{(1-\mu)}} := c_5 t^{\frac{1}{(1-\mu)}}$, where t_0 is an arbitrary positive constant. Thus, for $t > 1$ we have

$$\begin{aligned} I(t\psi) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta t\psi|^{p(x)}\,dx\right) - \int_\Omega \frac{\lambda(x)}{q(x)}|t\psi|^{q(x)}\,dx \\ &\leq c_5\left(\int_\Omega |t\Delta\psi|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - \int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|t\psi|^{q(x)}\,dx \\ &\leq c_5 t^{\frac{p^+}{1-\mu}}\left(\int_\Omega |\Delta\psi|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - t^{q_2^-}\int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|\psi|^{q(x)}\,dx \\ &\to -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

due to the fact that $\frac{p^+}{1-\mu} < q_2^-$. □

By Lemmas 3.2 and 3.3 and the mountain pass theorem of Ambrosetti and Rabinowitz [1], we deduce the existence of a sequence (u_n) such that

$$I(u_n) \to c_6 > 0 \quad \text{and} \quad I'(u_n) \to 0 \text{ in } X^* \quad \text{as } n \to \infty. \tag{8}$$

We prove that (u_n) is bounded in X. Assume for the sake of contradiction, if necessary to a subsequence, still denote by (u_n), $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n.

By Proposition 2.4, we may infer that for n large enough

$$\begin{aligned}
&1 + c_7 + \|u_n\| \\
&\geq I(u_n) - \frac{1}{q_2^-}\langle I'(u_n), u_n\rangle \\
&= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right) - \int_\Omega \frac{\lambda(x)}{q(x)}|u_n|^{q(x)}\,dx \\
&\quad - \frac{1}{q_2^-}\left[M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\Big)\int_\Omega |\Delta u_n|^{p(x)}\,dx - \int_\Omega \lambda(x)|u_n|^{q(x)}\,dx\right] \\
&\geq \frac{(1-\mu)}{p^+}M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int_\Omega |\Delta u_n|^{p(x)}\,dx - \int_\Omega \frac{\lambda(x)}{q(x)}|u_n|^{q(x)}\,dx \\
&\quad - \frac{1}{q_2^-}\left[M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\Big)\int_\Omega |\Delta u_n|^{p(x)}\,dx - \int_\Omega \lambda(x)|u_n|^{q(x)}\,dx\right] \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\int_\Omega |\Delta u_n|^{p(x)}\,dx + \int_{B_r(x_0)}\left(\frac{1}{q_2^-} - \frac{1}{q_1(x)}\right)\lambda(x)|u_n|^{q_1(x)}\,dx \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\|u_n\|^{p^-} - \nu^*\left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right)\int_{B_r(x_0)} |u_n|^{q_1(x)}\,dx \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\|u_n\|^{p^-} - c_1\nu^*\left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right)\left(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\right) \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\|u_n\|^{p^-} - c_8\Big(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\Big).
\end{aligned}$$

But, this cannot hold true since $p^- > 1$. Hence (u_n) is bounded in X. This information combined with the fact X is reflexive implies that there exists a subsequence, still denoted by (u_n), and $u_1 \in X$ such that $u_n \rightharpoonup u_1$ in X. Since X is compactly embedded in $L^{q(x)}(\Omega)$, it follows that $u_n \to u_1$ in $L^{q(x)}(\Omega)$. Using Proposition 2.2 we deduce

$$\lim_{n\to\infty} \int_\Omega \lambda(x)|u_n|^{q(x)-2}u_n(u_n - u_1)\,dx = 0.$$

This fact and relation (8) yield

$$\lim_{n\to\infty} M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\,dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u_1)\,dx = 0.$$

In view of (M_1), we have $\lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u_1)\,dx = 0$. Using Proposition 2.5, we find that $u_n \to u_1$ in X. Then by relation (8) we have

$$I(u_1) = c_6 > 0 \quad \text{and} \quad I'(u_1) = 0,$$

that is, u_1 is a non-trivial weak solution of (3).

We hope to apply Ekeland's variational principle [24] to get a nontrivial weak solution of problem (3).

Lemma 3.4. *There exists $\varphi_1 \in X$, $\varphi_1 \neq 0$ such that $I(t\varphi_1) < 0$ for $t > 0$ small enough.*

Proof. Let $\varphi_1 \in C_0^\infty(\Omega)$, $\varphi_1 \geq 0$ and there exist $x_2 \in B_r(x_0)$ and $\varepsilon > 0$ such that for any $x \in B_\varepsilon(x_2) \subset B_r(x_0)$ we have $\varphi_1(x) > 0$. For any $0 < t < 1$, we have

$$\begin{aligned} I(t\varphi_1) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta t\varphi_1|^{p(x)}\,dx\right) - \int_\Omega \frac{\lambda(x)}{q(x)}|t\varphi_1|^{q(x)}\,dx \\ &\leq c_9\left(\int_\Omega |t\Delta\varphi_1|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)}|t\varphi_1|^{q(x)}\,dx \\ &\leq c_9 t^{\frac{p^-}{1-\mu}}\left(\int_\Omega |\Delta\varphi_1|^{p(x)}\,dx\right)^{\frac{1}{1-\mu}} - t^{q_1^+}\int_{B_r(x_0)} \frac{\lambda(x)}{q_1(x)}|\varphi_1|^{q_1(x)}\,dx. \end{aligned}$$

So $I(t\varphi_1) < 0$ for $t < \theta^{\frac{1}{\frac{p^-}{1-\mu}-q_1^+}}$, where

$$0 < \theta < \min\left\{1, \frac{\int_{B_r(x_0)} \frac{\lambda(x)}{q_1(x)}|\varphi_1|^{q_1(x)}\,dx}{c_9(\int_\Omega |\nabla\varphi_1|^{p(x)}\,dx)^{\frac{1}{1-\mu}}}\right\}. \qquad \square$$

Let $\nu^* > 0$ be defined as in (7) and assume $|\lambda|_{L^\infty(\Omega)} < \nu^*$. By Lemma 3.2 it follows that on the boundary of the ball centered at the origin and of radius ρ_1 in X, denoted by $B_{\rho_1}(0) = \{\omega \in X;\ ||\omega|| < \rho_1\}$, we have

$$\inf_{\partial B_{\rho_1}(0)} I > 0.$$

By Lemma 3.2, there exists $\varphi_1 \in X$ such that $I(t\varphi_1) < 0$ for $t > 0$ small enough. Moreover, for $u \in B_{\rho_1}(0)$,

$$\begin{aligned} I(u) \geq &\left[\frac{m_0}{2p^+}\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_1^-} + \|u\|^{q_1^+})\right] \\ &+ \left[\frac{m_0}{2p^+}\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_2^-} + \|u\|^{q_2^+})\right]. \end{aligned}$$

It follows that

$$-\infty < c_{10} = \inf_{\overline{B_{\rho_1}(0)}} I < 0.$$

We let now $0 < \varepsilon < \inf_{\partial B_{\rho_1}(0)} I - \inf_{B_{\rho_1}(0)} I$. Applying Ekeland's variational principle [24] to the functional $I : \overline{B_{\rho_1}(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_{\rho_1}(0)}$ such that

$$I(u_\varepsilon) < \inf_{\overline{B_{\rho_1}(0)}} I + \varepsilon,$$
$$I(u_\varepsilon) < I(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

Since

$$I(u_\varepsilon) \leq \inf_{\overline{B_{\rho_1}(0)}} I + \varepsilon \leq \inf_{B_{\rho_1}(0)} I + \varepsilon < \inf_{\partial B_{\rho_1}(0)} I,$$

we deduce that $u_\varepsilon \in B_{\rho_1}(0)$. Now, we define $K : \overline{B_{\rho_1}(0)} \to \mathbb{R}$ by $K(u) = I(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of K and thus

$$\frac{K(u_\varepsilon + tv) - K(u_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and $v \in B_{\rho_1}(0)$. The above relation yields

$$\frac{I(u_\varepsilon + tv) - I(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle I'(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$ and we infer that $\|I'(u_\varepsilon)\| \leq \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_{\rho_1}(0)$ such that

$$I(v_n) \to c_{10} \quad \text{and} \quad I'(v_n) \to 0. \tag{9}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_2 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_2 in X. Actually, with similar arguments as those used in the proof that the sequence $u_n \to u_1$ in X we can show that $v_n \to u_2$ in X. Thus, by relation (9),

$$I(u_2) = c_{10} < 0 \quad \text{and} \quad I'(u_2) = 0,$$

i.e., u_2 is a non-trivial weak solution for problem (3).

Finally, since

$$I(u_1) = c_6 > 0 > c_{10} = I(u_2),$$

we see that $u_1 \neq u_2$. Thus, problem (3) has two non-trivial weak solutions.

4. Proof of Theorem 1.2

In this section, assume that we are under the hypotheses of Theorem 1.2, using the Ekeland's variational principle we get the result. We define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right) - \lambda\int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau)\,d\tau$. It is easy to verify that $I_\lambda \in C^1(X,\mathbb{R})$ is weakly lower semi-continuous with the derivative given by

$$\langle I'_\lambda(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\,dx - \lambda\int_\Omega |u|^{q(x)-2}uv\,dx,$$

for all $u, v \in X$. Thus, weak solutions of problem (4) are exactly critical points of the functional I_λ. For applying Ekeland's variational principle, we start with two auxiliary results.

Lemma 4.1. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho_2, \delta_2 > 0$ such that $I_\lambda(u) \geq \delta_2 > 0$ for any $u \in X$ with $\|u\| = \rho_2$.*

Proof. Since $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, it follows that X is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant c_{11} such that

$$|u|_{q(x)} \leq c_{11}\|u\|, \quad \text{for all } u \in X. \tag{10}$$

Now, let us fix $\rho_2 \in (0,1)$ such that $\rho_2 < \frac{1}{c_{11}}$. Then relation (10) implies $|u|_{q(x)} < 1$, for all $u \in X$ with $\|u\| = \rho_2$. Thus

$$\int_\Omega |u|^{q(x)}dx \leq |u|_{q(x)}^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\| = \rho_2. \tag{11}$$

Relations (10) and (11) imply

$$\int_\Omega |u|^{q(x)}dx \leq c_{11}^{q^-}\|u\|^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\| = \rho_2. \tag{12}$$

Taking into account relation (12) and the condition (M_1), we deduce that for any $u \in X$ with $\|u\| = \rho_2$, we have

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right) - \lambda\int_\Omega \frac{|u|^{q(x)}}{q(x)}\,dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\,dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}c_{11}^{q^-}\|u\|^{q^-} \\
&= \frac{m_0}{p^+}\rho_2^{p^+} - \frac{\lambda}{q^-}c_{11}^{q^-}\rho_2^{q^-} \\
&= \rho^{q^-}\left(\frac{m_0}{p^+}\rho_2^{p^+-q^-} - \frac{\lambda}{q^-}c_{11}^{q^-}\right).
\end{aligned} \tag{13}$$

By (13) if we define

$$\lambda^* = \frac{m_0 q^- \rho_2^{p^+ - q^-}}{2p^+ c_{11}^{q^-}},$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\| = \rho_2$, there exists $\delta_2 > 0$ such that $I_\lambda(u) \geq \delta_2 > 0$. This completes the proof. $\square$

Lemma 4.2. *There exists $\varphi_2 \in X$ such that $\varphi_2 \geq 0$, $\varphi_2 \neq 0$ and $I_\lambda(t\varphi_2) < 0$ for all $t > 0$ small enough.*

Proof. Since $q^- < \frac{p^-}{1-\mu}$, there exists $\epsilon_0 > 0$ such that $q^- + \epsilon_0 < \frac{p^-}{1-\mu}$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$, for all $u \in \Omega_0$. Thus, we conclude that

$$q(x) \leq q^- + \epsilon_0 < \frac{p^-}{1-\mu} \quad \text{for all } u \in \Omega_0. \tag{14}$$

Let $\varphi_2 \in C_0^\infty(\Omega)$ be such that $\overline{\Omega_0} \subset \operatorname{supp} \varphi_2$, $\varphi_2 = 1$ for $x \in \overline{\Omega_0}$ and $0 \leq \varphi_2(x) \leq 1$ in Ω. Without loss of generality, we may assume $\|\varphi_2\| = 1$, that is

$$\int_\Omega |\Delta \varphi_2|^{p(x)}\, dx = 1. \tag{15}$$

Using relations (14) and (15), (M_2) and $\int_{\Omega_0} |\varphi_2|^{q(x)}\, dx = \operatorname{meas}(\Omega_0)$, for all $t \in (0,1)$ we have

$$\begin{aligned}
I_\lambda(t\varphi_2) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\Delta t\varphi_2|^{p(x)}\, dx\right) - \lambda \int_\Omega \frac{|t\varphi_2|^{q(x)}}{q(x)}\, dx \\
&\leq \left(\int_\Omega \frac{1}{p(x)} |\Delta t\varphi_2|^{p(x)}\, dx\right)^{\frac{1}{1-\mu}} - \lambda \int_\Omega \frac{|t\varphi_2|^{q(x)}}{q(x)}\, dx \\
&\leq \frac{t^{\frac{p^-}{1-\mu}}}{p^-} \int_\Omega |\Delta \varphi_2|^{p(x)}\, dx - \frac{\lambda}{q^+} \int_{\Omega_0} t^{q(x)} |\varphi_2|^{q(x)}\, dx \\
&\leq \frac{t^{\frac{p^-}{1-\mu}}}{p^-} - \frac{\lambda t^{q^- + \epsilon_0}}{q^+} \operatorname{meas}(\Omega_0).
\end{aligned}$$

Therefore $I_\lambda(t\varphi_2) < 0$, for $t < \eta^{\frac{1}{\frac{p^-}{1-\mu} - q^- - \epsilon_0}}$, with $0 < \eta < \min\left\{1, \frac{\lambda p^- \operatorname{meas}(\Omega_0)}{q^+}\right\}$. $\square$

By Lemma 4.1, we have

$$\inf_{\partial B_{\rho_2}(0)} I_\lambda > 0.$$

On the other hand, from Lemma 4.2, there exists $\varphi_2 \in X$ such that $I_\lambda(t\varphi_2) < 0$ for $t > 0$ small enough. Relation (13) implies that for any $u \in B_{\rho_2}(0)$ we have

$$I_\lambda(u) \geq \frac{m_0}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} c_{11}^{q^-} \|u\|^{q^-}.$$

It follows that

$$-\infty < c_{12} = \inf_{B_{\rho_2}(0)} I_\lambda(u) < 0.$$

Using the Ekeland's variational principle and the similar argument as those used in the proof of Theorem 1.1, we can deduce that there exists a sequence $(v'_n) \subset B_{\rho_2}(0)$ such that

$$I_\lambda(v'_n) \to c_{12} \quad \text{and} \quad I'_\lambda(v'_n) \to 0, \tag{16}$$

and (v'_n) converges strongly to some u_3 in X. By (16), $I_\lambda(u_3) = c_{12} < 0$ and $I'_\lambda(u_3) = 0$, i.e., u_3 is a non-trivial weak solution of problem (4).

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