

# On the Convergence of Successive Approximations for a Fractional Differential Equation in Banach Spaces

*Aldona Dutkiewicz*

**Abstract.** In this paper we present a convergence result for the successive approximations for a nonlinear fractional differential equation of Caputo type in Banach spaces. Also an example is given to illustrate our result.

**Keywords.** Fractional differential equations, initial value problem, successive approximations, Caputo's derivative, measures of noncompactness, Banach spaces

**Mathematics Subject Classification (2010).** Primary 34G20, secondary 45N05

## 1. Introduction

Assume that  $I = [0, a]$ ,  $E$  is a Banach space,  $B = \{x \in E : \|x - x_0\| \leq b\}$  and  $f : I \times B \mapsto E$  is a bounded continuous function. We consider the Cauchy problem

$$\begin{aligned} D^\beta x &= f(t, x) \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where  $0 < \beta < 1$  and  $D^\beta$  denotes the fractional derivative of order  $\beta$  in the Caputo sense (cf. [1, 5]).

Recently, the theory of fractional differential equations has gained considerable popularity and importance, and as a result, several research papers and monographs have been published in this field (see, for example, [7, 12, 13] and the references therein).

It is our object in this paper to establish a convergence theorem for the successive approximations for the nonlinear fractional initial value problem (1) under the generalized Osgood type condition.

In what follows we shall need the following result of W. Mydlarczyk given in [11].

---

A. Dutkiewicz: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland; szukala@amu.edu.pl

**Theorem 1.1.** *Let  $\alpha > 0$  and let  $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a nondecreasing function such that  $\omega(0) = 0$ ,  $\omega(t) > 0$  for  $t > 0$ . Then the equation*

$$u(t) = \int_0^t (t-s)^{\alpha-1} \omega(u(s)) ds \quad (t \geq 0)$$

*has a nontrivial continuous solution if and only if*

$$\int_0^\delta \frac{1}{s} \left[ \frac{s}{\omega(s)} \right]^\frac{1}{\alpha} ds < \infty \quad (\delta > 0).$$

## 2. Results

Let  $M = \sup\{\|f(t, x)\| : t \in I, x \in B\}$ . We choose a positive number  $d$  such that  $d \leq a$  and  $M \frac{d^\beta}{\Gamma(\beta+1)} \leq b$ . Denote by  $C = C(J, E)$  the Banach space of continuous functions  $x : J \mapsto E$  with usual supremum norm  $\|\cdot\|$ , where  $J = [0, d]$ . Let  $\tilde{B} \subset C(J, E)$  be the subset of those functions with values in  $B$ .

The problem (1) is equivalent to the Volterra integral equation (cf. [6])

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \quad (t \in I).$$

We define a mapping  $F$  by

$$F(x)(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s)) ds \quad (t \in J, x \in \tilde{B}).$$

It is known (cf. [2, 10]) that  $F$  is a continuous mapping  $\tilde{B} \mapsto \tilde{B}$ . Moreover, if  $z = F(x)$ , then  $z'(t) = \frac{\beta-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-2} f(s, x(s)) ds$ , so that

$$\|z'(t)\| \leq K \quad \text{for } t \in J,$$

where  $K = \frac{Md^{\beta-1}}{\Gamma(\beta)}$ . By the mean value theorem this implies that

$$\|F(x)(t) - F(x)(\tau)\| \leq K|t - \tau| \quad \text{for } t, \tau \in J \text{ and } x \in \tilde{B}. \quad (2)$$

We shall now state our main result:

**Theorem 2.1.** Let  $\omega : [0, 2b] \mapsto \mathbb{R}_+$  be a continuous nondecreasing function such that  $\omega(0) = 0$ ,  $\omega(r) > 0$  for  $r > 0$  and

$$\int_0^\delta \frac{1}{s} \left[ \frac{s}{\omega(s)} \right]^{\frac{1}{\beta}} ds = \infty \quad (\delta > 0). \quad (3)$$

If

$$\|f(t, x) - f(t, y)\| \leq \omega(\|x - y\|) \quad \text{for } t \in I, \ x, y \in B, \quad (4)$$

then the successive approximations  $u_n$ , defined by

$$u_0 = x_0, \quad u_{n+1} = F(u_n) \quad \text{for } n \in N, \quad (5)$$

converge uniformly on  $J$  to the unique solution  $u$  of (1).

*Proof.* We first show (similarly as in the proof of [8, Theorem 9.1 III]), that

$$\lim_{n \rightarrow \infty} \|u_n(t) - u_{n-1}(t)\| = 0 \quad \text{for } t \in J. \quad (6)$$

Put  $\phi(t) = \overline{\lim}_{n \rightarrow \infty} \|u_n(t) - u_{n-1}(t)\|$ . From (1) and (2) it is clear that

$$\|u_n(t_1) - u_{n-1}(t_1)\| \leq \|u_n(t_2) - u_{n-1}(t_2)\| + 2K|t_1 - t_2|.$$

For any  $\varepsilon > 0$  there is  $n_0 \in N$  such that

$$\|u_n(t_2) - u_{n-1}(t_2)\| \leq \phi(t_2) + \varepsilon \quad \text{for } n \geq n_0.$$

Therefore

$$\|u_n(t_1) - u_{n-1}(t_1)\| \leq \phi(t_2) + \varepsilon + 2K|t_1 - t_2| \quad \text{for } n \geq n_0$$

and consequently,  $\phi(t_1) \leq \phi(t_2) + \varepsilon + 2K|t_1 - t_2|$ . As  $\varepsilon$  is arbitrary, we get

$$\phi(t_1) \leq \phi(t_2) + 2K|t_1 - t_2|.$$

Since  $t_1, t_2$  can be interchanged, we obtain

$$|\phi(t_1) - \phi(t_2)| \leq 2K|t_1 - t_2| \quad \text{for } t_1, t_2 \in J$$

which proves the continuity of  $\phi(\cdot)$ . Further, from (5) it follows that

$$\begin{aligned} \|u_{n+1}(t) - u_n(t)\| &= \|F(u_n)(t) - F(u_{n-1})(t)\| \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|f(s, u_n(s)) - f(s, u_{n-1}(s))\| ds. \end{aligned}$$

By (4) this implies

$$\|u_{n+1}(t) - u_n(t)\| \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(\|u_n(s) - u_{n-1}(s)\|) ds. \quad (7)$$

Since the sequence  $(\|u_n(\cdot) - u_{n-1}(\cdot)\|)$  is equicontinuous and uniformly bounded, from the definition of  $\phi(\cdot)$  and Arzela's Lemma we deduce that for fixed  $t \in J$  there exists a subsequence  $(n_k)$  such that  $\lim_{k \rightarrow \infty} \|u_{n_k+1}(t) - u_{n_k}(t)\| = \phi(t)$  and  $\|u_{n_k}(s) - u_{n_k-1}(s)\| \rightarrow \phi_1(s)$  uniformly in  $s \in J$ . Replacing  $n$  by  $n_k$  in (7) and passing to the limit as  $k \rightarrow \infty$ , we obtain the inequality

$$\phi(t) \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(\phi_1(s)) ds.$$

As  $\phi_1(s) \leq \overline{\lim} \|u_n(s) - u_{n-1}(s)\| = \phi(s)$  and  $\omega(r)$  is nondecreasing, we see that

$$0 \leq \phi(t) \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(\phi(s)) ds \quad \text{for } t \in J. \tag{8}$$

By Theorem 1.1 and assumption (3) the integral equation

$$z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(z(s)) ds \quad \text{for } t \in J$$

has the unique solution  $z(t) \equiv 0$ , which is also the maximal solution. Applying now the theorem on integral inequalities ([3, Theorem 2]), from (8) we deduce that  $\phi(t) \leq z(t)$  for  $t \in J$ . Thus  $\phi(t) \equiv 0$  for  $t \in J$ , which proves (6).

On the other hand, (4) implies that

$$\alpha(f(t, X)) \leq \omega(\alpha(X)) \quad \text{for } t \in J \text{ and } X \subset B, \tag{9}$$

where  $\alpha$  is the Kuratowski measure of noncompactness (cf. [4]). Now we shall show that the sequence  $(u_n)$  has a limit point.

Let  $V = \{u_n : n \in N\}$ . Then, by (2),  $V$  is a bounded equicontinuous subset of  $\tilde{B}$ . Denote by  $v$  the function defined by  $v(t) = \alpha(V(t))$  for  $t \in J$ , where  $V(t) = \{u_n(t) : n \in N\}$ . It is well known that the function  $v$  is continuous. As  $V = F(V) \cup \{0\}$ , we have

$$V(t) = F(V)(t) \cup \{0\}$$

and consequently  $\alpha(V(t)) = \alpha(F(V)(t))$ . Since

$$F(V)(t) \subset \frac{1}{\Gamma(\beta)} \left\{ \int_0^t (t-s)^{\beta-1} f(s, u_n(s)) ds : n \in N \right\},$$

Heinz's lemma [9] proves that

$$\begin{aligned} \alpha(F(V)(t)) &\leq \frac{1}{\Gamma(\beta)} \alpha \left( \left\{ \int_0^t (t-s)^{\beta-1} f(s, u_n(s)) ds : n \in N \right\} \right) \\ &\leq \frac{2}{\Gamma(\beta)} \int_0^t \alpha(\{(t-s)^{\beta-1} f(s, u_n(s)) : n \in N\}) ds \\ &\leq \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \alpha(\{f(s, u_n(s)) : n \in N\}) ds. \end{aligned}$$

Moreover, in view of (9), we have  $\alpha(\{f(s, u_n(s)) : n \in N\}) \leq \omega(\alpha(V(s)))$ . Hence

$$v(t) \leq \alpha(F(V)(t)) \leq \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(v(s)) ds \quad \text{for } t \in J.$$

Applying Theorem 1.1, assumption (3) and the theorem on integral inequalities ([3, Theorem 2]), repeating arguments from the above we deduce that  $v(t) = 0$  for  $t \in J$ . Consequently  $\alpha(V(t)) = 0$  for  $t \in J$ . Therefore for each  $t \in J$  the set  $V(t)$  is relatively compact in  $E$  and by Ascoli's theorem the set  $V$  is relatively compact in  $C$ . Hence the sequence  $(u_n)$  has a subsequence  $(u_{n_k})$  which converges to a limit  $u$ . This fact, together with (5) and (6), implies that  $u = F(u)$ , i.e.  $u$  is a solution of (1).

Suppose that  $\bar{u}$  is another solution of (1). Then

$$\begin{aligned} \|u(t) - \bar{u}(t)\| &= \|F(u)(t) - F(\bar{u})(t)\| \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \omega(\|u(s) - \bar{u}(s)\|) ds \quad \text{for } t \in J, \end{aligned}$$

and using once more Theorem 1.1 and the theorem on integral inequalities we get  $\|u(t) - \bar{u}(t)\| \equiv 0$  on  $J$ . Thus  $u = \bar{u}$ .

From the above considerations it is clear that the sequence  $(u_n)$  has a unique limit point  $u$ , and hence  $\lim_{n \rightarrow \infty} u_n(t) = u(t)$  uniformly on  $J$ .  $\square$

**Example 2.2.** As an example, which illustrate the assumptions related to (1), we consider the function  $\omega(\xi) = \xi |\ln \xi|^\beta$  for  $0 < \xi \leq e^{-\beta}$ ,  $0 < \beta < 1$  and  $\omega(0) = 0$ . It can be easily verified that  $\omega$  is continuous, nondecreasing and

$$|\omega(\xi) - \omega(\eta)| \leq \omega(|\xi - \eta|) \quad \text{for } 0 \leq \xi, \eta \leq e^{-\beta}. \tag{10}$$

Moreover,

$$\int_{0+} \frac{1}{s} \left[ \frac{s}{\omega(s)} \right]^{\frac{1}{\beta}} = \int_{0+} \frac{ds}{s |\ln s|} = \infty.$$

Let  $E = C(0, 1)$  and  $B = \{x \in E : \|x\| \leq \frac{1}{2} e^{-\beta}\}$ .

We define a function  $f : B \mapsto E$  by

$$f(x)(\tau) = \omega(|x(\tau)|) \quad \text{for } \tau \in [0, 1] \text{ and } x \in B.$$

By (10) we get  $\|f(x) - f(y)\| \leq \omega(\|x - y\|)$  for  $x, y \in B$ . Therefore, our equation has the form:

$$D^\beta x = f(x), \quad 0 < \beta < 1.$$

By Theorem 2.1, it follows that the corresponding sequence of successive approximations  $(u_n(t))$  converges uniformly to the unique solution  $u(t) \equiv 0$ .

## References

- [1] Abel, N. H., Résolution d'un problème de mécanique (in French). *J. Reine Angew. Math.* 1 (1826), 153 – 157, (Oeuvres I, 97 – 101).
- [2] Appell, J. and Zabreiko, P. P., *Nonlinear Superposition Operators*. Cambridge: Cambridge Univ. Press 1990.
- [3] Azbelev, N. B. and Tsalyuk, Z. B., On integral inequalities I (in Russian). *Mat. Sb.* 56 (1962), 325 – 342.
- [4] Banaś, J. and Goebel, K., *Measures of Noncompactness in Banach Spaces*. New York: Marcel Dekker 1980.
- [5] Caputo, M., Linear models of dissipation whose  $Q$  is almost frequency independent II. *Geophys. J. Roy. Astronom. Soc.* 13 (1967), 529 – 539.
- [6] Deng, W., Short memory principle and a predictor-corrector for fractional differential equations. *J. Comp. Appl. Math.* 206 (2007), 174 – 188.
- [7] Dutkiewicz, A. and Szufła, S., On the Aronszajn property for an implicit differential equation of fractional order. *Z. Anal. Anwend.* 29 (2010), 429 – 435.
- [8] Hartman, P., *Ordinary Differential Equations*. New York: John Wiley 1964.
- [9] Heinz, H. P., On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions. *Nonlinear Anal.* 7 (1983), 1351 – 1371.
- [10] Januszewski, J., On Volterra integral equations with weakly singular kernel in Banach spaces. *Demonstr. Math.* 26 (1993), 131 – 136.
- [11] Mydlarczyk, W., The existence of nontrivial solutions of Volterra equations. *Math. Scand.* 68 (1991), 83 – 88.
- [12] Oldham, K. B. and Spanier, J., *The Fractional Calculus*. New York: Academic Press 1974.
- [13] Podlubny, I., *Fractional Differential Equations*. London: Academic Press 1999.

Received September 14, 2012; revised November 9, 2013