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Approximation by Polyhedral G Chains in Banach Spaces

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Abstract. In a Banach space with the metric approximation property, each compactly supported rectifiable G chain whose boundary is rectifiable as well, is approximatable in the flat norm by a polyhedral G chain of nearly the same normal Hausdorff mass.

Keywords. Rectifiable chains, polyhedral chains

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1. Introduction

Integral currents in Euclidean space are strongly approximated by polyhedral integral currents, according to H. Federer and W. H. Fleming, see e.g. [7, 4.2.20]. Specifically, if $Y \cong \ell_2^N$ is a Euclidean space, $T \in \mathbf{I}_n(Y)$ and $\varepsilon > 0$, there exist $P \in \mathscr{P}_n(Y)$ and a C^1 diffeomorphism $f : Y \to Y$ such that

$$\begin{aligned} \operatorname{spt} P &\subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon), \\ \mathscr{N}(P - f_\# T) &< \varepsilon, \\ \max\{\operatorname{Lip} f, \operatorname{Lip} f^{-1}\} &< 1 + \varepsilon, \\ f(x) = x \text{ for all } x &\notin \mathbf{B}(\operatorname{spt} T, \varepsilon), \\ \|f - \mathrm{id}_Y\|_\infty &< \varepsilon. \end{aligned}$$

One step in proving this consists in showing that the carrying n rectitifable set $A \subseteq Y$ of the integral current T is well approximated, in a strong sense, by a C^1 submanifold of dimension n, $M \subseteq Y$, i.e. $\mathscr{H}^n(M \ominus A) < \varepsilon$, and that M is, locally, the image of a tangent n plane by a C^1 diffeomorphism of

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the ambient space which is as close to the identity as one wishes. In turn, the approximation of A by M relies upon a Lusin type approximation of Lipschitz maps $f : \mathbb{R}^n \to Y$ by C^1 maps, see [7, 3.1.6] or [6, 6.6.1]. The proof of this boils down to the Whitney Extension Theorem (see e.g. [15] or [13, Chapter VI]) and Rademacher Theorem (see e.g. [6, 3.1.2] or [7, 3.1.6]).

In this paper we address the problem of approximating rectifiable G chains in a Banach space Y by polyhedral G chains, i.e. we replace the coefficient group $\mathbb{Z}$ by a general complete normed Abelian group G, and we replace the ambient space ℓ_2^N by a general Banach space Y. We work in the context set up in [4]. Unless Y has the Radon-Nikodym property, a Lipschitz map $f : \mathbb{R}^n \to Y$ need not be differentiable anywhere. In fact, the Radon-Nikodym property of Y is equivalent to the Lusin type approximation property of f by a C^1 map. This means we cannot expect the above approximation to hold at that level of generality.

Here is a classical example illustrating what can go wrong with the differentiability of Lipschitz maps. We let $n = 1$ and $Y = L_1[0,1]$, and we consider $f : [0,1] \to Y : t \to \mathbb{1}_{[0,t]}$. One checks that $\operatorname{Lip} f = 1$ and that f is differentiable nowhere. Yet we seek a polyhedral approximation of the chain $T = f_\# [\![0,1]\!]$. Recalling that $\mathscr{M}(T) = \mathscr{H}^1(f([0,1]))$ is simply the *variation* $V_0^1 f$ of the map f, we readily find a polygonal line P with the same endpoints as T, with $\mathscr{M}(P)$ as close as we please to $\mathscr{M}(T)$, and with $\operatorname{spt} P$ contained in a small tubular neighborhood of $\operatorname{spt} T$. In fact P is simply obtained as $P = \hat{f}_\# [\![0,1]\!]$, considering a fine partition $I_1, \dots, I_\kappa$ of the domain $[0,1]$ and defining a PL map $\hat{f} : [0,1] \to Y$ that coincides with f at the endpoints of the I_k. Our goal is to extend the scope of this observation.

In case $n \geqslant 2$, V is an n dimensional Banach space, $B \subseteq V$ is a bounded Borel set, and $f : B \to Y$ is an injective Lispchitz map with $\max\{\operatorname{Lip} f, \operatorname{Lip} f^{-1}\}$ nearly equal to 1, we ought to approximate the chain $T = f_\#(g \cdot [\![B]\!])$, $g \in G$, by a polyhedral chain P. In case B itself is a polyhedron one can consider simplicial subdivisions K of B and the corresponding PL approximation $\hat{f}$ of f. One classical trap here is that if the simplexes of which K consists are too thin then $\mathscr{M}(\hat{f}_\#(g \cdot [\![B]\!]))$ may by far exceed $\mathscr{M}(T)$ (recall H. A. Schwarz' accordeon [12] and see the definition of the *shape* of a simplex before Lemma 3.1, and Proposition 3.2 for a relevant estimate which is common practice in finite elements method for instance). Now if B is merely a Borel set then one first needs to approximate it by a polyhedron S, possibly from the outside, and hence extend f as well. Such extension $\tilde{f}$ indeed exists (use the usual *Whitney cube* procedure, see e.g. [13, Chapter VI]) but in general $\operatorname{Lip} \tilde{f} \leqslant \mathbf{c}(\dim V) \operatorname{Lip} f$ with $\mathbf{c}(\dim V)$ much larger than 1. Therefore S must not wander too much outside of B: relevant technicalities are taken care of in Theorem 3.3. Use of the homotopy formula then yields a polyhedral approximation $P \in \mathscr{P}_n(Y;G)$ such that: $\mathscr{M}(P) < \varepsilon + \mathscr{M}(T)$; $\operatorname{spt} P \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$ and $\mathscr{F}(P - T) < \varepsilon$. The latter means that P and T are nearly homologous in a "small way", i.e. that the equality $P - T = \partial R$ nearly holds for some R, with $\mathscr{M}(R) < \varepsilon$, specifically that $\mathscr{M}(P - T - \partial R) < \varepsilon$.

Now if $T \in \mathscr{R}_n(Y;G)$ is a general n dimensional rectifiable G chain, then $T = \sum_{i=1}^{\infty} T_i$ where the sum is mass convergent and each T_i is of the type considered in the previous paragraph. Thus T_i is nearly some polyhedral P_i. Choosing a large integer N one can thus approximate T in the sense above by a polyhedral $P = \sum_{i=1}^{N} P_i$. However each P_i contributes, possibly a lot, to ∂P, so that $\mathscr{M}(\partial P)$ will in general be much larger than $\mathscr{M}(\partial T)$. In order to remedy this problem one then wants to fill in the gaps $T - \sum_{i=1}^{N} P_i$. In case Y is finite dimensional, the technique used in [7, 4.2.20] consists in applying the Deformation Theorem (see [7, 4.2.9] and [14]). This result does not hold when Y is infinite dimensional, but according to the principle developed in the Appendix (see Theorem A.1), it very nearly does – in an appropriate sense – when Y has the metric (or bounded) approximation property and $\operatorname{spt} T$ is assumed to be compact. As an illustration of this principle, the Appendix also contains a compactness theorem that applies to showing existence for a Plateau problem in the separable Hilbert space.

Technical variations of this theme applied to both ∂T and a slight modification of T then yield our main

Theorem. *Assume the Banach space Y has the metric approximation property, $T \in \mathscr{R}_n(Y;G)$ is so that $\partial T \in \mathscr{R}_{n-1}(Y;G)$ and $\operatorname{spt} T$ is compact, and let $\varepsilon > 0$. There then exists $P \in \mathscr{P}_n(Y;G)$ such that*

(A) $\mathscr{M}(P) < \varepsilon + \mathscr{M}(T)$
(B) $\mathscr{M}(\partial P) < \varepsilon + \mathscr{M}(\partial T)$
(C) $\mathscr{F}(T - P) < \varepsilon$
(D) $\operatorname{spt} P \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$.

2. Preliminaries

In the remaining part of this paper we let V and Y be Banach spaces, with $n = \dim V < \infty$. Furthermore $(G, |\cdot|)$ denotes a complete normed Abelian group.

2.1. Rectifiable G chains. For the definition of $\mathscr{R}_n(Y;G)$, whose members are called *n dimensional rectifiable G chains in Y*, we refer to [4]. Here we recall (see [4, §3.6]) that such $T \in \mathscr{R}_n(Y;G)$ is characterized as an $\mathscr{H}^n$ equivalence class of a pair $(A, \mathbf{g})$ where $A \subseteq Y$ is a Borel n rectifiable subset of Y and $\mathbf{g}$ is a G valued orientation of A such that $|\mathbf{g}| \in L_1(\mathscr{H}^n \llcorner A)$. We abbreviate this by writing $T = \mathscr{H}^n \llcorner A \wedge \mathbf{g}$. This means that at $\mathscr{H}^n$ almost every $x \in A$ where an n dimensional approximate tangent space W_x of A is defined, $\mathbf{g}(x)$ is a choice of a G valued orientation of W_x, i.e. an equivalence class of a pair $(\mathscr{O}, g)$ where $\mathscr{O}$ is an orientation of W_x and $g \in G \setminus \{0\}$ (the equivalence relation being $(\mathscr{O}, g) \cong (-\mathscr{O}, -g)$). Furthermore $\mathbf{g}(x)$ depends on x in a Borel way (this is conveniently stated in [4] in terms of *almost parametrizations* of A), and $|\mathbf{g}(x)|$ is the norm of the coefficient g in $\mathbf{g}(x) = (\mathscr{O}, g)$. Corresponding to T we define a finite measure $\|T\|$ in Y (denoted μ_T in [4]) by $\|T\| = |\mathbf{g}| \cdot \mathscr{H}^n \llcorner A$. When $\mathbf{g} = g$ is constant and A has a canonical orientation, we also use the notation $g \cdot [\![A]\!]$. This is the case for instance when $A \subseteq \mathbb{R}^n$ is given the orientation of $\mathbb{R}^n$, and when $\sigma \subseteq Y$ is an oriented n-simplex.

2.2. Lipschitz extensions. If $A \subseteq V$ and $f : A \to Y$ is Lipschitz then H. Whitney's construction of an extension of f by means of so-called *Whitney cubes* (see [15] where both V and Y are Euclidean) applies verbatim in the present case, as has been reported in [8]. Thus there exists a Lipschitz map $\tilde{f} : V \to Y$ such that $\tilde{f} \restriction_A = f$ and $\operatorname{Lip} \tilde{f} \leqslant \mathbf{c}_{2.0}(n) \operatorname{Lip} f$.

2.3. Piecewise linearity. Our reference for elementary statements regarding simplicial complexes and PL maps is [11]. A *cell* in V is a bounded set which is the finite intersection of closed half spaces. Each cell is the convex hull of its finitely many extreme points, called its *vertices*. An *n-simplex* has $n+1$ vertices $x_0, x_1, \ldots, x_n$ such that $x_1 - x_0, \ldots, x_n - x_0$ are linearly independent. A *polyhedron* is a finite union of cells. Each polyhedron is the set $|K|$ of a simplicial complex K (see e.g. [11, Theorem 2.11]). The n skeleton of K is denoted $K^{(n)}$. A *PL map* is a map $f : |K| \to Y$ where K is a simplicial complex, subject to the requirements that (a) f is continuous, and (b) each restriction $f \restriction_\sigma$, $\sigma \in K$, is affine.

2.4. Jacobians. Here we recall a particular case of the *area formula*, [1]. If W is a Banach space with $k = \dim W < \infty$, and $f : W \to Y$ is Lipschitz, then f is metrically differentiable at $\mathscr{H}^k$ almost every $x \in W$, [9, Theorem 2]. We denote its metric differential at x by mdf_x. This is a seminorm on W defined by the requirement that

$$\|f(x+h) - f(x)\|_Y = (mdf_x)(h) + o(\|h\|_W),$$

$h \in W$. Its *Jacobian* is defined as

$$(J_k f)(x) = J_k(mdf_x) = \frac{\mathscr{H}^k(W \cap \{\|\cdot\|_W \leqslant 1\})}{\mathscr{H}^k(W \cap \{mdf_x \leqslant 1\})}.$$

Since readily $(mdf_x)(h) \leqslant (\operatorname{Lip} f)\|h\|_W$ it follows that $(J_k f)(x) \leqslant (\operatorname{Lip} f)^k$. Now if $A \subseteq W$ is Borel then

$$\int_Y \operatorname{card}(A \cap f^{-1}\{y\}) d\mathscr{H}^k(y) = \int_A (J_k f)(x) d\mathscr{H}^k(x). \tag{1}$$

2.5. Jacobian of a map of two variables. We consider a map $f: V_1 \times V_2 \to Y$ where V_1, V_2, Y are Banach spaces with $m_1 = \dim V_1 < \infty$ and $m_2 = \dim V_2 < \infty$. We claim that for $\mathscr{H}^{m_1}$ almost every $x_1 \in V_1$ and $\mathscr{H}^{m_2}$ almost every $x_2 \in V_2$, f is metrically differentiable at (x_1, x_2), the map $V_2 \to Y : \xi_2 \mapsto f(x_1, \xi_2)$ is metrically differentiable at x_2, the map $V_1 \to Y : \xi_1 \mapsto f(\xi_1, x_2)$ is metrically differentiable at x_1, and

$$(J_{m_1+m_2} f)(x_1, x_2) \leqslant \mathbf{c}_{2.0}(m_1, m_2)(J_{m_1} f \restriction_{V_1 \times \{x_2\}})(x_1) \cdot (J_{m_2} f \restriction_{\{x_1\} \times V_2})(x_2). \quad (2)$$

The first part of our claim follows from 2.4 and Fubini's Theorem, since $\mathscr{H}^{m_1} \otimes \mathscr{H}^{m_2}$ and $\mathscr{H}^{m_1+m_2}$ are both Haar measures on $V_1 \times V_2$. The specific value of the Jacobian $(J_{m_1+m_2} f)(x_1, x_2)$ depends on the choice of a norm on $V_1 \times V_2$, but since these are all equivalent, the upper bound above doesn't depend on such choice (it is implemented in the constant $\mathbf{c}_{2.0}(m_1, m_2)$). Letting $\|\cdot\|_{V_1}$ and $\|\cdot\|_{V_2}$ be the norms of V_1 and V_2, we henceforth assume that $V_1 \times V_2$ is equipped with the norm $\|(h_1, h_2)\| = \max\{\|h_1\|_{V_1}, \|h_2\|_{V_2}\}$. We next observe that $\operatorname{im} f$ is separable, thus isometrically isomorphic as a metric space to a subset of $\ell_\infty(\mathbb{N})$ (see e.g. [1, end of §2]). Since the metric differential is invariant under such isometry, we may as well assume $Y = \ell_\infty(\mathbb{N})$. The latter being the dual of a separable space, f is weakly* differentiable almost everywhere and $mdf_x(h) = \|wdf_x(h)\|$ according to [1, Theorem 3.5]. It follows that the proof of (2) reduces to the case when $f = L$ is linear. Our choice of a norm $\|\cdot\|$ on $V_1 \times V_2$ readily implies that $\mathscr{H}^{m_1} \otimes \mathscr{H}^{m_2} = \mathscr{H}^{m_1+m_2}$. Furthermore, if we let $L_1 : V_1 \to Y : h_1 \to L(h_1, 0)$ and $L_2 : V_2 \to Y : h_2 \to L(0, h_2)$ then $L(h_1, h_2) = L_1(h_1) + L_2(h_2)$ and therefore $(V_1 \times V_2) \cap \{\|L\| \leqslant 1\} \supseteq (V_1 \cap \{\|L_1\|_{V_1} \leqslant 1/2\}) \times (V_2 \cap \{\|L_2\|_{V_2} \leqslant \frac{1}{2}\})$. It ensues that

$$\begin{aligned}
\frac{\mathscr{H}^{m_1+m_2}\{\|\cdot\| \leqslant 1\}}{\mathscr{H}^{m_1+m_2}\{\|L\| \leqslant 1\}} &= \frac{\mathscr{H}^{m_1}(\|\cdot\|_{V_1} \leqslant 1\}) \cdot \mathscr{H}^{m_2}(\{\|\cdot\|_{V_2} \leqslant 1\})}{\mathscr{H}^{m_1+m_2}\{\|L\| \leqslant 1\}} \\
&\leqslant \frac{\mathscr{H}^{m_1}(\|\cdot\|_{V_1} \leqslant 1\}) \cdot \mathscr{H}^{m_2}(\{\|\cdot\|_{V_2} \leqslant 1\})}{\mathscr{H}^{m_1}(\{\|L_1\|_{V_1} \leqslant \frac{1}{2}\}) \cdot \mathscr{H}^{m_2}(\{\|L_2\|_{V_2} \leqslant \frac{1}{2}\})} \\
&\leqslant 2^{m_1+m_2}(J_{m_1} L_1)(J_{m_2} L_2).
\end{aligned}$$

2.6. Homotopy formula. Here $f_0, f_1 : V \to Y$ are Lipschitz maps. We consider the affine homotopy

$$H : [0,1] \times V \to Y : (t, x) \mapsto f_0(x) + t(f_1(x) - f_0(x)).$$

If $T \in \mathscr{R}_k(V; G)$, $0 \leqslant k \leqslant m$, then

$$f_{1\#}T - f_{0\#}T = \partial H_\#([\![0,1]\!] \times T) + H_\#([\![0,1]\!] \times \partial T) \quad (3)$$

where the Cartesian product of $[\![0,1]\!]$ and a chain in V is a chain in $\mathbb{R} \times V$

defined in the obvious way. In order to prove (3) we compute $\partial H_{\#}([\![0,1]\!]\times T) = H_{\#}\partial([\![0,1]\!]\times T)$ and we obtain the formula $\partial([\![0,1]\!]\times T) = [\![1]\!]\times T - [\![0]\!]\times T - [\![0,1]\!]\times \partial T$ first in the case when T is a Lipschitz chain, reasoning as in [4, §5.1], and in the general case by mass approximation. In order to complete the proof of (3) on then notices that $H_{\#}([\![j]\!]\times T) = f_{j\#}T$ because $H(j,x) = f_j(x)$, $j=0,1$, so that [4, Proposition 5.5.2(1)] applies.

In order to estimate the flat norm of $f_{1\#}T - f_{0\#}T$ we will next refer to 2.5 to find an upper bound for $J_{1+k}H(t,x)$. Given (t,x) we put $H_t(x) = H(t,x) = H_x(t)$. If H, H_t and H_x are metrically differentiable respectively at (t,x), x and t, then

$$(J_1H_t)(x) \leqslant \operatorname{Lip} H_t \leqslant \|f_1(x) - f_0(x)\|$$

and

$$(J_kH_x)(t) \leqslant (\operatorname{Lip} H_x)^k \leqslant (t \operatorname{Lip} f_1 + (1-t)\operatorname{Lip} f_0)^k \leqslant \max\{\operatorname{Lip} f_0, \operatorname{Lip} f_1\}^k.$$

Thus

$$(J_{1+k}H)(t,x) \leqslant \mathbf{c}_{2.5}(1,k)\max\{\operatorname{Lip} f_0, \operatorname{Lip} f_1\}^k \|f_1(x) - f_0(x)\|$$

according to 2.5. We now apply the area formula (1) to find out that

$$\begin{aligned}\mathscr{M}(H_{\#}([\![0,1]\!]\times T)) &\leqslant \int_{\mathbb{R}\times V} (J_{1+k}H)(t,x)d(\mathscr{L}^1 \otimes \|T\|)(t,x)\\ &\leqslant \mathbf{c}_{2.5}(1,k)\max\{\operatorname{Lip} f_0, \operatorname{Lip} f_1\}^k \int_V \|f_1(x) - f_0(x)\| d\|T\|(x).\end{aligned}$$

Applying this formula to both T and ∂T we thus obtain

$$\mathscr{F}(f_{1\#}T - f_{0\#}T) \leqslant \mathbf{c}_{2.0}(k, \operatorname{Lip} f_0, \operatorname{Lip} f_1)\left(\sup_{x\in \operatorname{spt} T} \|f_1(x) - f_0(x)\|\right)\mathscr{N}(T), \quad (4)$$

where

$$\mathbf{c}_{2.0}(k, \operatorname{Lip} f_0, \operatorname{Lip} f_1) = \mathbf{c}_{2.5}(1,k)\max\{(\operatorname{Lip} f_0)^k, (\operatorname{Lip} f_1)^k, (\operatorname{Lip} f_0)^{k-1}, (\operatorname{Lip} f_1)^{k-1}\}.$$

3. Approximating Lipschitz maps by PL maps

If $f : V \to Y$ and σ is an n-dimensional simplex in V we let

$$A(\sigma, f) : \sigma \to Y$$

be the affine map that coincides with f on the vertices of σ. Thus if $x \in \sigma = \operatorname{co}\{x_0, x_1, \dots, x_,\}$ has barycentric coordinates $t_1, \dots, t_n \in \mathbb{R}^+$, $\sum_{i=0}^n t_i = 1$,

$$x = \sum_{i=0}^n t_i x_i$$

then

$$A(\sigma, f)(x) = \sum_{i=0}^{n} t_i f(x_i).$$

We observe that

$$\begin{aligned}\|A(\sigma, f)(x) - f(x)\| &= \left\| \sum_{i=0}^{n} t_i f(x_i) - \sum_{i=0}^{n} t_i f(x) \right\| \\ &\leqslant \sum_{i=0}^{n} t_i \|f(x_i) - f(x)\| \\ &\leqslant \operatorname{diam} f(\sigma).\end{aligned}$$

Therefore if f is continuous and if σ is small then $A(\sigma, f)$ is a good approximation of $f\restriction_\sigma$ in the norm $\|\cdot\|_\infty$. In particular, if f is Lipschitz then

$$\|A(\sigma, f) - f\restriction_\sigma\|_\infty \leqslant (\operatorname{Lip} f)(\operatorname{diam} \sigma). \tag{5}$$

In case K is a simplicial complex in V we put

$$\operatorname{mesh} K = \max\{\operatorname{diam} \sigma : \sigma \in K^{(n)}\}.$$

Now if we define

$$A(K, f) : |K| \to Y$$

to coincide with $A(\sigma, f)$ on each $\sigma \in K^{(n)}$, we notice that $A(K, f)$ is a PL map. We infer from (5) that

$$\|A(K, f) - f\restriction_{|K|}\|_\infty \leqslant (\operatorname{Lip} f)(\operatorname{mesh} K). \tag{6}$$

If $\langle K_j \rangle_j$ is a sequence of successive subdivisions of K such that $\lim_j \operatorname{mesh}(K_j) = 0$ then $\lim_j \|A(K_j, f) - f\restriction_{|K|}\|_\infty = 0$. To find such a sequence we may use, for instance, barycentric subdivisions. If no further restriction is imposed upon K_j, however, it may happen that $\lim_j \operatorname{Lip} A(K_j, f) = \infty$ (recall [12]). In the remaining part of this section, we explain how to avoid this obstacle.

We define the shape of an n-dimensional simplex σ in Euclidean space ℓ_2^n by the formula

$$\operatorname{shape} \sigma = \frac{\mathscr{L}^n(\sigma)}{(\operatorname{diam} \sigma)^n}.$$

Shape appears (under the name of *fullness*) in the following computation taken from H. Whitney's [16, Ch. IV Lemma 15b(2)]. We denote by $|\cdot|$ the Euclidean norm of ℓ_2^n.

Lemma 3.1. *Let σ be an n-simplex in ℓ_2^n, and let $u_1, \ldots, u_n$ be independent unit vectors parallel to the edges of σ. It follows that for every $a_1, \ldots, a_n \in \mathbb{R}$ the following holds:*

$$\left| \sum_{i=1}^{n} a_i u_i \right| \geqslant (n!)(\operatorname{shape}\sigma) \max_{i=1,\ldots,n} |a_i|.$$

Proof. We start by recalling that $\mathscr{L}^n(\operatorname{co}\{0, e_1, \ldots, e_n\}) = (n!)^{-1}$, according to Fubini's Theorem applied inductively on n. Considering $\sigma = \operatorname{co}\{x_0, x_1, \ldots, x_n\}$ as the affine image of $\operatorname{co}\{0, e_1, \ldots, e_n\}$ by a map A such that $A(e_i) = x_0 + v_i$, $v_i = x_i - x_0$, $i = 1, \ldots, n$, we infer that

$$\mathscr{L}^n(\sigma) = \frac{1}{(n!)} \, \| v_1 \wedge \cdots \wedge v_n \| \tag{7}$$

where $\| v_1 \wedge \cdots \wedge v_n \|$ denotes the absolute value of the determinant of the matrix DA whose rows contains the coordinates of the edges $v_1, \ldots, v_n$.

We now show that

$$\left| \sum_{i=1}^{n} a_i u_i \right| \geqslant \| u_1 \wedge \cdots \wedge u_n \| \max_{i=1,\ldots,n} |a_i|. \tag{8}$$

This will establish the lemma since

$$\| u_1 \wedge \cdots \wedge u_n \| = \frac{\| v_1 \wedge \cdots \wedge v_n \|}{\| v_1 \| \cdots \| v_n \|} \geqslant \frac{(n!)\mathscr{L}^n(\sigma)}{(\operatorname{diam}\sigma)^n} \geqslant (n!)(\operatorname{shape}\sigma)$$

in view of (7). We prove (8) by reductio ad absurdum. Assuming if possible that the reverse inequality holds, we write $a = |a_n| = \max_{i=1,\ldots,n} |a_i|$ (renumbering the a_i's if necessary) and we define

$$w = b_1 u_1 + \cdots + b_{n-1} u_{n-1} + a_n u_n$$

where the coefficients $b_1, \ldots, b_{n-1}$ are chosen so as to minimize $\| w \|$. Then,

$$\begin{aligned} \| w \| &\leqslant \left| \sum_{i=1}^{n} a_i u_i \right| \\ &< a \, \| u_1 \wedge \cdots \wedge u_n \| \\ &= \| u_1 \wedge \cdots \wedge u_{n-1} \wedge w \| \\ &\leqslant \| u_1 \| \cdots \| u_{n-1} \| \,.\, \| w \| = \| w \|, \end{aligned}$$

a contradiction. □

In the finite dimensional Banach space V we consider an Auerbach system $e_1,\dots,e_n,e_1^*,\dots,e_n^*$, chosen once for all. This means that $e_1,\dots,e_n$ are unit vectors in V, that $e_1^*,\dots,e_n^*$ are unit covectors in V^* and that $e_j^*(e_i)=\delta_{i,j}$, the Kronecker symbol, $i,j=1,\dots,n$. We then consider a Euclidean structure on V defined in order that $e_1,\dots,e_n$ is a Euclidean basis, and we denote by $|\cdot|$ the corresponding Euclidean norm. One easily checks that

$$\frac{1}{\sqrt{n}}\|x\| \leqslant |x| \leqslant \sqrt{n}\|x\| \tag{9}$$

for every $x\in V$. Both inequalities are based on the relation $e_i^*(x)=\langle x,e_i\rangle$, $i=1,\dots,n$. The first one follows from the triangle inequality applied to $\|x\|=\|\sum_i e_i^*(x)e_i\|$, and comparing the norms of ℓ_1^n and ℓ_2^n. The second one follows from evaluating $x^*=\sum_j e_j^*(x)e_j^*$ at x, and the inequality $\|x^*\|\leqslant n\|x\|$.

In the next result, $\operatorname{shape}\sigma$ refers to the shape of the simplex with respect to the Euclidean structure of V, associated with the chosen Auerbach basis $e_1,\dots,e_n$. Furthermore $|||L|||$ denotes the operator norm of a linear map $L: V\to Y$. Finally we recall that the oscillation of Dg on a subset $S\subseteq V$ is defined by

$$\operatorname{osc}(Dg;S)=\sup\{|||Dg(x)-Dg(x')|||: x,x'\in S\}.$$

Proposition 3.2. *Assume $g:V\to Y$ is continuously differentiable, σ is an n-dimensional simplex in V, and x_0 is a vertex of σ. It follows that*

$$|||DA(\sigma,g)-Dg(x_0)|||\leqslant \mathbf{c}_{3.0}(n)\left(\frac{\operatorname{osc}(Dg;\sigma)}{\operatorname{shape}\sigma}\right).$$

Proof. We let $x_0,x_1,\dots,x_n$ be a numbering of the vertices of σ (the first one of which being that appearing in the statement of the proposition). We also let $v_i=x_i-x_0$ be the edges of σ, and $u_i=|v_i|^{-1}v_i$ be the corresponding Euclidean unit vectors, $i=1,\dots,n$. Writing $x\in\sigma$ in barycentric coordinates, $x=\sum_{i=0}^n t_ix_i$, we infer that

$$A(\sigma,g)(x)=g(x_0)+\sum_{i=1}^n t_i(g(x_i)-g(x_0))=g(x_0)+L(x-x_0)$$

where the second equality defines the linear part $L:V\to Y$ of $A(\sigma;g)$, i.e. $L=DA(\sigma;g)$. We note that

$$\begin{aligned}\|L(v_i)-Dg(x_0)(v_i)\| &= \|g(x_i)-g(x_0)-Dg(x_0)(v_i)\|\\ &= \left\|\int_0^1 (Dg(x_0+t(x_i-x_0))(v_i)-Dg(x_0)(v_i))\,d\mathscr{L}^1(t)\right\|\\ &\leqslant \operatorname{osc}(Dg;\sigma)\|v_i\|\end{aligned}$$

Now if $u = \sum_{i=1}^n a_i u_i$ then

$$\begin{aligned}\|L(u) - Dg(x_0)(u)\| &= \left\| \sum_{i=1}^n a_i \,|\!|\!|\, v_i \,|\!|\!|^{-1} \Big(L(v_i) - Dg(x_0)(v_i) \Big) \right\| \\ &\leqslant \sum_{i=1}^n |a_i| \,|\!|\!|\, v_i \,|\!|\!|^{-1} \|v_i\| \operatorname{osc}(Dg;\sigma) \\ &\leqslant n \left(\max_{i=1,\dots,n} |a_i| \right) \sqrt{n} \operatorname{osc}(Dg;\sigma) \\ &\leqslant \frac{n^{\frac{3}{2}} \,|\!|\!|\, u \,|\!|\!|\, \operatorname{osc}(Dg;\sigma)}{(n!)(\operatorname{shape}\sigma)} \\ &\leqslant \left(\frac{n^2}{n!} \right) \left(\frac{\operatorname{osc}(Dg;\sigma)}{\operatorname{shape}\sigma} \right) \|u\|,\end{aligned}$$

according to (9) and Lemma 3.1. □

Theorem 3.3. *Assume that*

(A) $f : V \to Y$ *is Lipschitz;*
(B) $B \subseteq V$ *is a nonnegligible bounded Borel set;*
(C) $U \supseteq B$ *is open and bounded;*
(D) $\varepsilon > 0$.

There then exists a polyhedron S *in* V *such that*

(E) $S \subseteq U$ *and* $\mathscr{H}^n(S \ominus B) < \epsilon$*;*

and there exists $\eta_0 > 0$ *with the following property. For every* $0 < \eta \leqslant \eta_0$ *there exist a simplicial complex* K_η *and a PL map* $\hat{f}_\eta : |K_\eta| \to Y$ *such that*

(F) $|K_\eta| = S$*;*
(G) $\|\hat{f}_\eta - f \restriction_S \|_\infty \leqslant \eta$*;*
(H) $K_\eta^{(n)} = K_g \cup K_b$ *where*
   (i) *For every* $\sigma \in K_g$ *one has* $\operatorname{Lip}(\hat{f}_\eta \restriction_\sigma) < \varepsilon + \operatorname{Lip}(f \restriction_B)$*;*
   (ii) $\mathscr{H}^n(\cup K_b) < \varepsilon$ *and for every* $\sigma \in K_b$ *one has* $\operatorname{Lip}(\hat{f}_\eta \restriction_\sigma) < \varepsilon + \operatorname{Lip} f$.

Proof. We consider *dyadic cubes* in V relative to the Auerbach system $e_1, \dots, e_n$ chosen before Proposition 3.2. There exists a set S which is a finite union of dyadic cubes and has the following properties: $S \subseteq U$ and

$$\mathscr{H}^n(S \ominus B) < \frac{\varepsilon}{2}. \tag{10}$$

This S is the polyhedron in conclusion (E).

We choose a mollifier function $\varphi : V \to \mathbb{R}$ of class C^1 such that $\varphi \geqslant 0$, $\operatorname{supp}\varphi \subseteq V \cap \{x : \|x\| \leqslant 1\}$, and $\int_V \varphi d\mathscr{H}^n = 1$. Given $r > 0$ we then let $\varphi_r(x) = r^{-n}\varphi(r^{-1}x)$, $x \in V$. We use Bochner integration to define the convolution product

$$f_r(x) := \int_V \varphi_r(x-\xi)f(\xi)d\mathscr{H}^m(\xi) = \int_V \varphi_r(\xi)f(x-\xi)d\mathscr{H}^m(\xi),$$

upon noticing that the integrand is indeed *strongly* measurable (i.e. the limit $\mathscr{H}^n$ a.e. of a sequence of simple maps), see [5, Chapter 2]. It is easy to see that $\|f - f_r\|_\infty \leqslant r \operatorname{Lip} f$, that f_r is of class C^1, and that $\operatorname{Lip} f_r \leqslant \operatorname{Lip} f$.

We let

$$B_1 = B \cap \left\{ x : \lim_{r\to 0^+} \frac{\mathscr{H}^n(B \cap \mathbf{B}(x,r))}{\mathscr{H}^n(\mathbf{B}(x,r))} = 1 \right\},$$

so that $\mathscr{H}^n(B \backslash B_1) = 0$, according to the Lebesgue Density Theorem. For $\beta > 0$ to be determined momentarily, Egoroff's Theorem guarantees the existence of a compact subset $C \subseteq B_1$ such that $\mathscr{H}^n(B_1 \setminus C) < \beta$, and the existence of $r_0 > 0$ such that

$$\mathscr{H}^n(B \cap \mathbf{B}(x,r)) \geqslant (1-\beta)\mathscr{H}^n(\mathbf{B}(x,r)) \tag{11}$$

for every $x \in C$ and every $0 < r \leqslant r_0$. For the remaining part of this proof we assume $0 < r \leqslant r_0$. Given $x \in V$ we put $G_x = \mathbf{B}(0,r) \cap \{\xi : x - \xi \in B\}$. Since readily $x - G_x = B \cap \mathbf{B}(x,r)$, we infer from (11) that

$$\mathscr{H}^n(\mathbf{B}(0,r) \setminus (G_x \cap G_{x'})) < 2\beta\mathscr{H}^n(\mathbf{B}(0,r))$$

whenever $x, x' \in C$. In that case,

$$\begin{aligned}
&\|f_r(x) - f_r(x')\| \\
&= \left\| \int_V \varphi_r(\xi)\big(f(x-\xi) - f(x'-\xi)\big)d\mathscr{H}^n(\xi) \right\| \\
&\leqslant \left\| \int_{G_x \cap G_{x'}} \varphi_r(\xi)\big(f(x-\xi) - f(x'-\xi)\big)d\mathscr{H}^n(\xi) \right\| \\
&\quad + \left\| \int_{\mathbf{B}(0,r)\setminus(G_x\cap G_{x'})} \varphi_r(\xi)\big(f(x-\xi) - f(x'-\xi)\big)d\mathscr{H}^n(\xi) \right\| \\
&\leqslant \|x - x'\|(\operatorname{Lip} f\restriction_B) + \|x-x'\|(\operatorname{Lip} f)r^{-n}(\sup\varphi)2\beta\mathscr{H}^n(\mathbf{B}(0,r)) \\
&\leqslant \|x - x'\|\Big((\operatorname{Lip} f\restriction_B) + 2\beta(\operatorname{Lip} f)(\sup\varphi)\mathscr{H}^n(\mathbf{B}(0,1))\Big).
\end{aligned}$$

It is now clear how to choose β small enough so that C and f_r have the following properties.

$$\mathscr{H}^n(B \setminus C) < \frac{\varepsilon}{2} \tag{12}$$

and

$$\operatorname{Lip}(f_r\restriction_C) < \frac{\varepsilon}{3} + \operatorname{Lip}(f\restriction_B) \tag{13}$$

for every $0 < r \leqslant r_0$. In order to apply the above inequality later, we note that if x is a Lebesgue point of C then $|||Df_r(x)||| \leqslant \mathrm{Lip}(f_r \restriction_C)$.

Given $0 < r \leqslant r_0$ we define

$$\omega_r(\delta) = \sup \{|||Df_r(x) - Df_r(x')||| : x, x' \in \operatorname{Clos} U \text{ and } \|x - x'\| \leqslant \delta\} .$$

We next choose $\delta_r > 0$ small enough for

$$\omega_r(\delta_r) < \frac{\varepsilon}{3} \min \left\{ 1, \frac{s(n)}{\mathbf{c}_{3.2}(n)} \right\}$$

where $s(n) > 0$ will be determined momentarily.

Now we choose a decomposition $S = \cup_j Q_j$ into finitely many dyadic cubes Q_j, all of a same generation, so that $\operatorname{diam} Q_j < \min\{\delta_r, r\}$. We let K_r denote the simplicial complex obtained from the *complete barycentric subdivision* of each of these cubes Q_j. The simplexes used are clearly all homothetic to those belonging to the complete barycentric subdivision of the unit cube, and therefore there exists $s(n) > 0$ such that $\mathrm{shape}(\sigma) \geqslant s(n)$ for all $\sigma \in K_r^{(n)}$ (the shape of a simplex is defined relative to the Euclidean structure of V for which the Auerbach system $e_1, \ldots, e_n$ is an orthonormal basis).

We consider the PL map $\hat{f}_r = A(K_r, f_r)$. Recall that $\operatorname{mesh} K_r < r$, therefore

$$\|\hat{f}_r - f \restriction_S \|_\infty \leqslant \|A(K_r, f_r) - f_r \restriction_S \|_\infty + \|f_r - f\|_\infty \leqslant 2r \operatorname{Lip} f,$$

according to (6). Up to a change of parameter $\eta = 2r \operatorname{Lip} f$ we note that conclusions (F) and (G) are satisfied.

Furthermore, for each $\sigma \in K_r^{(n)}$, if x_0 is a vertex of σ, we infer from Proposition 3.2 that

$$|||DA(\sigma, f_r) - Df_r(x_0)||| \leqslant \mathbf{c}_{3.2}(n) \left(\frac{\operatorname{osc}(Df_r; \sigma)}{\operatorname{shape} \sigma} \right) \leqslant \frac{\mathbf{c}_{3.2}(n) \omega_r(\delta_r)}{s(n)} < \frac{\varepsilon}{3}. \quad (14)$$

We let $C_1 = C \cap \{x : \Theta^n(\mathscr{H}^n \llcorner C, x) = 1\}$ and we decompose $K_r^{(n)} = K_g \cup K_b$ where

$$K_g = K_r^{(n)} \cap \{\sigma : \sigma \cap C_1 \neq \emptyset\}$$

and

$$K_b = K_r^{(n)} \setminus K_g.$$

If $\sigma \in K_g$ there exists $x_0' \in C_1$ such that $\|x_0 - x_0'\| < \delta_r$. Thus, referring to (13) and (14), we obtain $\operatorname{Lip} A(\sigma, f_r) = |||DA(\sigma, f_r)||| < \frac{\varepsilon}{3} + |||Df_r(x_0)||| \leqslant \frac{\varepsilon}{3} + \omega_r(\delta_r) + |||Df_r(x_0')||| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \mathrm{Lip}(f_r \restriction_C) \leqslant \varepsilon + \mathrm{Lip}(f \restriction_B)$. Conclusion (H)(i) is now established. Furthermore,

$$\mathscr{H}^n(\cup K_b) \leqslant \mathscr{H}^n(S \setminus C_1) \leqslant \mathscr{H}^n(S \setminus B) + \mathscr{H}^n(B \setminus C) < \varepsilon$$

according to (12) and (10). If $\sigma \in K^{(n)}$ and x_0 is a vertex of σ then

$$\operatorname{Lip} A(\sigma, f_r) = |||DA(\sigma, f_r)||| < \frac{\varepsilon}{3} + |||Df_r(x_0)||| \leqslant \frac{\varepsilon}{3} + \operatorname{Lip} f_r \leqslant \frac{\varepsilon}{3} + \operatorname{Lip} f.$$

This completes the proof. □

One technical point of Theorem 3.3 is that the Lipschitz constant of the approximating PL map $\hat{f}$ *outside* of B (but close to B) is nearly not larger than the Lipschitz constant of f in B (in general $|K|$ will not be contained in B). If one relaxes this request then the proof simplifies. The following version may be of independent interest.

Theorem 3.4. *Assume that*

(A) *$f : V \to Y$ is Lipschitz;*
(B) *K is a simplicial complex in V;*
(C) *$\varepsilon > 0$.*

There then exists a simplicial map $\hat{f} : |K| \to Y$ such that

(D) *$\|\hat{f} - f\restriction_{|K|}\|_\infty < \varepsilon$;*
(E) *$\operatorname{Lip} \hat{f} < \varepsilon + \operatorname{Lip} f$.*

Since we will not use this result in the present paper, we merely sketch its proof. We start by replacing the polyhedron $|K|$ by its convex hull S_0. It is itself a polyhedron, and admits a simplicial decomposition K_0. The point is that there exists a sequence $\langle K_j \rangle_j$ of simplicial decompositions of S_0, all being refinements of both K and K_0, and so that $\lim_j \operatorname{mesh}(K_j) = 0$ as well as $\inf_j \operatorname{shape}(K_j) > 0$, see [16, Appendix 2 §4]. Thus $\lim_j \|A(K_j, f_{r_j}) - f\restriction_{|K_0|}\|_\infty = 0$, and an application of Proposition 3.2 shows that $\operatorname{Lip} A(\sigma, f_{r_j}) < \varepsilon_j + \operatorname{Lip} f$ for all $\sigma \in K_j^{(n)}$. Since S_0 is convex one can replace σ by K_j in the last inequality.

4. Approximating rectifiable chains by polyhedral chains

Theorem 4.1. *Assume that $\varepsilon > 0$ and*

(A) *$B \subseteq V$ is a bounded Borel subset;*
(B) *$f : B \to Y$ is Lipshitz;*
(C) *$\beta > 0$ and $(1+\beta)^{-1}\|x - x'\| \leqslant \|f(x) - f(x')\| \leqslant (1+\beta)\|x - x'\|$ whenever $x, x' \in B$;*
(D) *$g \in G$.*

There then exists a simplicial complex K in V and a PL map $\hat{f} : |K| \to Y$ such that on letting $T = f_\#(g \cdot [\![B]\!])$ and $P = \hat{f}_\#(g \cdot [\![|K|]\!])$ the following hold:

(E) *$\mathscr{M}(P) < \varepsilon + (1 + 2\beta)^{2m} \mathscr{M}(T)$;*
(F) *$\mathscr{F}(P - T) < \varepsilon$;*
(G) *$\operatorname{spt} P \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$.*

Proof. We start by extending f to the whole of V, using the same symbol f for such extension (see 2.2). We apply Theorem 3.3 with $U = \mathbf{U}(B, \frac{\varepsilon}{2}(\operatorname{Lip} f)^{-1})$ and some $\hat{\varepsilon}$ to be determined in the course of the present proof. We thus obtain a polyhedron $S \subseteq \mathbf{U}(B, \frac{\varepsilon}{2}(\operatorname{Lip} f)^{-1})$ such that $\mathscr{H}^n(S \ominus B) < \hat{\varepsilon}$. We define $P_0 = g \cdot [\![S]\!] \in \mathscr{P}_n(V;G)$ and $T_0 = g \cdot [\![B]\!] \in \mathscr{R}_n(V;G)$. The simplicial complex K and the PL map $\hat{f}$ of our conclusion will be the K_η and $\hat{f}_\eta$ given by Theorem 3.3, corresponding to some η sufficiently small.

We start by observing that

$$\mathscr{M}(T) = |g|\mathscr{H}^n(f(B)) \geqslant |g|(1+\beta)^{-n}\mathscr{H}^n(B). \tag{15}$$

Furthermore, since $|K_\eta| = S$ we have $\hat{f}_{\eta\#}P_0 = \sum_{\sigma \in K^{(n)}} \hat{f}_{\eta\#}(g \cdot [\![\sigma]\!])$. It therefore ensues from conclusions (E) and (H) of Theorem 3.3, and from (15), that

$$\begin{aligned}
\mathscr{M}\left(\hat{f}_{\eta\#}P_0\right) &\leqslant \sum_{\sigma \in K_g} \mathscr{M}\left(\hat{f}_{\eta\#}(g \cdot [\![\sigma]\!])\right) + \sum_{\sigma \in K_b} \mathscr{M}\left(\hat{f}_{\eta\#}(g \cdot [\![\sigma]\!])\right) \\
&< (\hat{\varepsilon} + \operatorname{Lip}(f \restriction_B))^n\, |g| \sum_{\sigma \in K_g} \mathscr{H}^n(\sigma) + (\hat{\varepsilon} + \operatorname{Lip} f)^n\, |g| \sum_{\sigma \in K_b} \mathscr{H}^n(\sigma) \\
&\leqslant (\hat{\varepsilon} + 1 + \beta)^n\, |g|\mathscr{H}^n(S) + (\hat{\varepsilon} + \operatorname{Lip} f)^n\, |g|\mathscr{H}^n(K_b) \\
&\leqslant (\hat{\varepsilon} + 1 + \beta)^n\, |g|\, (\hat{\varepsilon} + \mathscr{H}^n(B)) + (\hat{\varepsilon} + \operatorname{Lip} f)^n\, |g|\hat{\varepsilon} \\
&\leqslant (\hat{\varepsilon} + 1 + \beta)^n\, (\hat{\varepsilon}|g| + (1+\beta)^n \mathscr{M}(T)) + (\hat{\varepsilon} + \operatorname{Lip} f)^n\, |g|\hat{\varepsilon}.
\end{aligned}$$

It should now be obvious how to choose $\hat{\varepsilon}$ sufficiently small – depending upon β, $|g|$, $\operatorname{Lip} f$, $\mathscr{H}^n(B)$ and ε –, in order that

$$\mathscr{M}\left(\hat{f}_{\eta\#}P_0\right) < \varepsilon + (1+2\beta)^{2n}\mathscr{M}(T).$$

In other words, conclusion (E) will be verified for each $0 < \eta \leqslant \eta_0$.

We notice that conclusion (G) is verified for every $0 < \eta \leqslant \frac{\varepsilon}{2}$. Indeed, it follows from Theorem 3.3(G) that

$$\operatorname{spt} P \subseteq \hat{f}_\eta(S) \subseteq \mathbf{B}[f(S), \eta] \subseteq \mathbf{B}\left[f(B), \eta + \frac{\varepsilon}{2}\right] \subseteq \mathbf{B}[\operatorname{spt} T, \varepsilon].$$

Finally we show that conclusion (F) is verified if η is sufficiently small. We consider the affine homotopy $H_\eta(t,x) = f(x) + t(\hat{f}_\eta(x) - f(x))$ and we apply (4) together with Theorem 3.3(G) to obtain

$$\mathscr{F}\left(\hat{f}_{\eta\#}P_0 - f_\# P_0\right) \leqslant \mathbf{c}_{2.6}(n, \operatorname{Lip} \hat{f}_\eta, \operatorname{Lip} f)\mathscr{N}(P_0)\eta.$$

Since also

$$\mathscr{M}(f_\# P_0 - f_\# T_0) \leqslant (\operatorname{Lip} f)^n \mathscr{H}^n(S \ominus B) \leqslant (\operatorname{Lip} f)^n \hat{\varepsilon},$$

it should now be obvious how to choose $\hat{\varepsilon}$ and η so that conclusion (F) holds. □

From now on, let $n \geqslant 2$.

Theorem 4.2. *Assume that $T \in \mathscr{R}_n(Y;G)$ and $\varepsilon > 0$. There then exists $P \in \mathscr{P}_n(Y;G)$ subject to the following requirements.*

(A) $\mathscr{M}(P) < \varepsilon + \mathscr{M}(T)$;

(B) $\mathscr{F}(P - T) < \varepsilon$;

(C) $\operatorname{spt} P \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$;

(D) *If $\partial T \in \mathscr{P}_{n-1}(Y;G)$, $\operatorname{spt} T$ is compact and Y is finite dimensional, then P can be chosen so that $\partial P = \partial T$, and conclusion (B) can be strengthen to $P - T = \partial Z$ for some $Z \in \mathscr{R}_{n+1}(Y;G)$ such that*

  (i) $\mathscr{M}(Z) < \varepsilon$;

  (ii) $\operatorname{spt} Z \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$;

(E) *If $\partial T \in \mathscr{P}_{n-1}(Y;G)$, $\operatorname{spt} T$ is compact and Y has the metric approximation property, then P can be chosen so that $\mathscr{M}(\partial P) \leqslant \mathscr{M}(\partial T)$, and in case $\partial T = 0$ conclusion (B) can be strengthen to $P - T = \partial Z$ for some $Z \in \mathscr{R}_{n+1}(Y;G)$ such that*

  (i) $\mathscr{M}(Z) < \varepsilon$;

  (ii) $\operatorname{spt} Z \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$;

Proof. We choose $\beta > 0$ sufficiently small for

$$(1 + 2\beta)^{2n}\mathscr{M}(T) < \hat{\varepsilon} + \mathscr{M}(T).$$

Recalling 2.1 we associate with T a Borel n rectifiable set $A \subseteq Y$ and a Borel G valued orientation $\mathbf{g}$ of A such that $T = \mathscr{H}^n \llcorner A \wedge \mathbf{g}$ and $\|T\| = |\mathbf{g}| \cdot \mathscr{H}^n \llcorner A$. We represent T as an n dimensional *parametrized G chain* $[\gamma, E_i, g]$, using an almost bilipschitz parametrization of A, as in [4, §3.2]. We apply Lusin's Theorem [7, 2.3.5] to finitely many of the $\gamma \restriction_{E_i}$ to find a closed set $C \subseteq \cup_i E_i$ such that $g \restriction_C$ is continuous and $\|T\|(A \setminus \gamma(C)) < \hat{\varepsilon}$ where $\hat{\varepsilon}$ will be determined in the course of this proof.

We further decompose $\gamma(C) = N \cup (\cup_{i=1}^{\infty} A_i)$, where each A_i is Borel, the oscillation of g on each $B_i = \gamma^{-1}(A_i)$ doesn't exceed $\hat{\varepsilon} \inf_{B_i} |g|$, $\mathscr{H}^n(N) = 0$, and for each $i = 1, 2, \ldots$ there exists an n dimensional Banach space $(V_i, \|\cdot\|_i)$ and a Lipschitz map $f_i : B_i \to Y$ such that $A_i = f_i(B_i)$ and $(1+\beta)^{-1}\|x - x'\|_i \leqslant \|f_i(x) - f_i(x')\| \leqslant (1+\beta)\|x - x'\|_i$. That such a decomposition be possible follows from the definition of n rectifiability and [1, Lemma 5.2 and Theorem 8.2]. We also choose $x_i \in A_i$ and we let $g_i \in G$ be so that $f_{i\#}(g_i \cdot [\![B_i]\!]) = \mathscr{H}^n \llcorner A_i \wedge \mathbf{g}(x_i)$. Therefore

$$\mathscr{M}\left(f_{i\#}(g_i \cdot [\![B_i]\!]) - T \llcorner A_i\right) \leqslant \hat{\varepsilon}\mathscr{M}(T \llcorner A_i). \tag{16}$$

We now apply Theorem 4.1 to $2^{-i}\hat{\varepsilon}, f_i, V_i, B_i, \beta$ and g_i. We obtain a polyhedron $S_i \subseteq V_i$, a simplicial complex K_i such that $|K_i| = S_i$, a PL map $\hat{f}_i : S_i \to Y$

with the following properties. Letting $P_i = \hat{f}_{i\#}(g_i \cdot [\![S_i]\!]) \in \mathscr{P}_n(Y;G)$,

$$\begin{aligned} \mathscr{M}(P_i) &< 2^{-i}\hat{\varepsilon} + \big((1+2\beta)^{2n} + \hat{\varepsilon}\big)\,\mathscr{M}(T \llcorner A_i) \\ \mathscr{F}(P_i - T \llcorner A_i) &< 2^{-i}\hat{\varepsilon} + \hat{\varepsilon}\mathscr{M}(T \llcorner A_i) \\ \operatorname{spt} P_i &\subseteq \mathbf{B}(\operatorname{spt}(T \llcorner A_i), 2^{-i}\hat{\varepsilon}), \end{aligned}$$

according to Theorem 4.1(E)–(G), and (16).

We choose an integer N large enough for

$$\mathscr{M}\left(T - \sum_{i=1}^{N} T \llcorner A_i\right) < \hat{\varepsilon} + \|T\|(A \setminus C) < 2\hat{\varepsilon}. \tag{17}$$

We claim that $P = \sum_{i=1}^N P_i$ satisfies our conclusions (A)–(C). Indeed, $\mathscr{M}(P) \leqslant \sum_{i=1}^N \mathscr{M}(P_i) \leqslant \sum_{i=1}^N \big[2^{-i}\hat{\varepsilon} + ((1+2\beta)^{2n} + \hat{\varepsilon})\,\mathscr{M}(T \llcorner A_i)\big] \leqslant \hat{\varepsilon} + \big(\hat{\varepsilon} + \mathscr{M}(T)\big) + \hat{\varepsilon}\mathscr{M}(T) \leqslant \varepsilon + \mathscr{M}(T)$, provided $\hat{\varepsilon}$ is chosen small enough according to ε and $\mathscr{M}(T)$. Furthermore,

$$\begin{aligned} \mathscr{F}(P - T) &\leqslant \mathscr{F}\left(\sum_{i=1}^N P_i - T \llcorner A_i\right) + \mathscr{M}\left(T - \sum_{i=1}^N T \llcorner A_i\right) \\ &\leqslant \sum_{i=1}^N \big[2^{-i}\hat{\varepsilon} + \hat{\varepsilon}\mathscr{M}(T \llcorner A_i)\big] + 2\hat{\varepsilon} \\ &\leqslant \hat{\varepsilon} + \hat{\varepsilon}\mathscr{M}(T) + 2\hat{\varepsilon} \\ &< \varepsilon \end{aligned}$$

provided $\hat{\varepsilon}$ is chosen small enough. Finally,

$$\operatorname{spt} P \subseteq \cup_{i=1}^N \operatorname{spt} P_i \subseteq \cup_{i=1}^N \mathbf{B}(\operatorname{spt}(T \llcorner A_i), 2^{-i}\hat{\varepsilon}) \subseteq \mathbf{B}(\operatorname{spt} T, \hat{\varepsilon}).$$

We now turn to proving conclusion (D). The proof will consist in *modifying* the polyhedral G chain P obtained above, according to the Deformation Theorem [14]. To this end we suppose that the P obtained so far verifies conclusions (A)–(C) with some $\tilde{\varepsilon}$ instead of ε, which will be chosen small according to various quantities including $\dim Y$. Since $\mathscr{F}(P - T) < \tilde{\varepsilon}$ and $d = \dim Y < \infty$ we infer from the definition of flat norm and the fact that Y is an absolute $\mathbf{c}(d)$-Lipschitz retract that there are $Q \in \mathscr{R}_n(Y;G)$ and $R \in \mathscr{R}_{n+1}(Y;G)$ such that $P - T = Q + \partial R$ and $\mathscr{M}(Q) + \mathscr{M}(R) < \mathbf{c}(d)\tilde{\varepsilon}$. We now show how to modify Q and R, not increasing their mass too much and making sure their supports remain close to that of T. We let $u(y) = \operatorname{dist}(y, \operatorname{spt} T)$, $y \in Y$, and we notice that P has been defined so that $\operatorname{spt} P \subseteq \{u < t\}$ whenever $t > \tilde{\varepsilon}$. Thus, for such t, $P - T = (P - T) \llcorner \{u < t\} = Q \llcorner \{u < t\} + (\partial R) \llcorner \{u < t\} = Q \llcorner \{u < t\} + \partial(R \llcorner \{u < t\}) - \langle R, u, t\rangle$. Furthermore,

$$\int_{\sqrt{\tilde{\varepsilon}}}^{2\sqrt{\tilde{\varepsilon}}} \mathscr{M}(\langle R, u, t\rangle)\, d\mathscr{L}^1(t) \leqslant 2(n+1)\mathscr{M}(R),$$

according to [4, 3.7.1(9)]. There thus exists $\sqrt{\tilde{\varepsilon}} < t < 2\sqrt{\tilde{\varepsilon}}$ such that

$$\mathscr{M}(\langle R, u, t\rangle) \leqslant \frac{4(n+1)}{\sqrt{\tilde{\varepsilon}}}\mathscr{M}(R) \leqslant 4(n+1)\mathbf{c}(d)\sqrt{\tilde{\varepsilon}}.$$

We now define $Q' = Q \llcorner \{u < t\} - \langle R, u, t\rangle$ and $R' = R \llcorner \{u < t\}$. It follows that $P - T = Q' + \partial R'$ and

$$\begin{aligned}
\mathscr{M}(Q') < \tilde{\varepsilon} + 4(n+1)\mathbf{c}(d)\sqrt{\tilde{\varepsilon}} &\leqslant \mathbf{c}'(d,n)\sqrt{\tilde{\varepsilon}}, \\
\mathscr{M}(R') &< \tilde{\varepsilon}, \\
(\operatorname{spt} Q') \cup (\operatorname{spt} R') &\subseteq \mathbf{B}(\operatorname{spt} T, 2\sqrt{\tilde{\varepsilon}}),
\end{aligned}$$

because there is no restriction to assume that $\tilde{\varepsilon} < 1$.

Applying the Deformation Theorem in Y to the chain Q' with some ε' (which will be determined momentarily), we obtain $Q' - P' = Q'' + \partial R''$ for some $P' \in \mathscr{P}_n(Y;G)$, $Q'' \in \mathscr{P}_n(Y;G)$ (Q'' is polyhedral because so is $\partial Q' = \partial P - \partial T$), $R'' \in \mathscr{R}_{n+1}(Y;G)$ such that

$$\begin{aligned}
\mathscr{M}(P') &\leqslant \mathbf{c}''(d)\mathscr{M}(Q'), \\
\mathscr{M}(Q'') \leqslant \varepsilon'\mathbf{c}''(d)\mathscr{M}(\partial Q') &= \varepsilon'\mathbf{c}''(d)\mathscr{M}(\partial P - \partial T), \\
\mathscr{M}(R'') &\leqslant \varepsilon'\mathbf{c}''(d)\mathscr{M}(Q'), \\
(\operatorname{spt} P') \cup (\operatorname{spt} Q'') \cup (\operatorname{spt} R'') &\subseteq \mathbf{B}(\operatorname{spt} Q', \varepsilon'\mathbf{c}''(d)).
\end{aligned}$$

We claim that the polyhedral G chain $P - P' - Q''$ (replacing P) verifies all four conclusions of the theorem, with $Z = R' + R''$. We start by observing that

$$P - P' - Q'' - T = (P - T) - P' - Q'' = Q' + \partial R' - P' - Q'' = \partial R'' + \partial R' = \partial Z.$$

This immediately shows that $\partial P = \partial T$. Furthermore

$$\mathscr{M}(Z) \leqslant \mathscr{M}(R') + \mathscr{M}(R'') < \tilde{\varepsilon} + \varepsilon'\mathbf{c}''(d)\mathscr{M}(Q') \leqslant \tilde{\varepsilon} + \varepsilon'\mathbf{c}''(d)\mathbf{c}'(d,n)\sqrt{\tilde{\varepsilon}}$$

so that conclusion (D)(i) (and hence also conclusion (B)) is verified provided $\tilde{\varepsilon}$ is chosen small enough and $\varepsilon' \leqslant 1$. Furthermore,

$$\operatorname{spt} Z \subseteq (\operatorname{spt} R') \cup (\operatorname{spt} R'') \subseteq \mathbf{B}\left(\operatorname{spt} T, 2\sqrt{\tilde{\varepsilon}} + \varepsilon'\mathbf{c}''(d)\right),$$

so that conclusion (D)(ii) is verified as well provided $\tilde{\varepsilon}$ and ε' are both chosen small enough. Regarding conclusion (A) we observe that

$$\begin{aligned}
\mathscr{M}(P - P' - Q'') &\leqslant \mathscr{M}(P) + \mathscr{M}(P') + \mathscr{M}(Q'') \\
&< \tilde{\varepsilon} + \mathscr{M}(T) + \mathbf{c}''(d)\mathscr{M}(Q') + \varepsilon'\mathbf{c}''(d)\mathscr{M}(\partial P - \partial T) \\
&\leqslant \tilde{\varepsilon} + \mathscr{M}(T) + \mathbf{c}''(d)\mathbf{c}'(d,n)\sqrt{\tilde{\varepsilon}} + \varepsilon'\mathbf{c}''(d)\mathscr{M}(\partial P - \partial T) \\
&\leqslant \varepsilon + \mathscr{M}(T)
\end{aligned}$$

provided $\tilde{\varepsilon}$ is chosen small enough according to $d = \dim Y$ and n, and ε' is chosen small enough (in applying the Deformation Theorem) according to $\mathscr{M}(\partial P - \partial T)$ and $d = \dim Y$. Finally conclusion (C) holds as well because

$$\begin{aligned}(\operatorname{spt} P) \cup (\operatorname{spt} P') \cup (\operatorname{spt} Q'') &\subseteq \mathbf{B}(\operatorname{spt} T, \tilde{\varepsilon}) \cup \mathbf{B}\big[\operatorname{spt} Q', \varepsilon' \mathbf{c}''(d)\big] \\ &\subseteq \mathbf{B}(\operatorname{spt} T, \tilde{\varepsilon}) \cup \mathbf{B}\big[\operatorname{spt} T, 2\sqrt{\tilde{\varepsilon}} + \varepsilon' \mathbf{c}''(d)\big].\end{aligned}$$

It remains only to prove conclusion (E). Given $\hat{\varepsilon}$ we associate with the compact set $\operatorname{spt} T$ a finite dimensional subspace $Y' \subseteq Y$ and a linear map π according to the definition of M.A.P. (see the Appendix). We notice that $\pi_\# T \in \mathscr{R}_n(Y'; G)$ and that $\partial \pi_\# T = \pi_\# \partial T$ is polyhedral because π is linear. According to the previous case there exists $P \in \mathscr{P}_n(Y'; G)$ and $Z \in \mathscr{P}_{n+1}(Y'; G)$ such that

$$\begin{aligned}\mathscr{M}(P) &< \hat{\varepsilon} + \mathscr{M}(\pi_\# T) \\ P - \pi_\# T = \partial Z &\text{ and } \mathscr{M}(Z) < \hat{\varepsilon} \\ (\operatorname{spt} P) \cup (\operatorname{spt} Z) &\subseteq \mathbf{B}(\operatorname{spt} \pi_\# T, \hat{\varepsilon}).\end{aligned}$$

Referring to Theorem A.1 it is now clear that

$$\begin{aligned}\mathscr{M}(P) &< \hat{\varepsilon} + \mathscr{M}(T) \\ \mathscr{F}(T - P) &< \hat{\varepsilon}\,(1 + \mathbf{c}_{2.6}(n, 1, 1))\, \mathscr{N}(T) \\ \operatorname{spt} P &\subseteq \mathbf{B}(\operatorname{spt} T, 2\hat{\varepsilon}) \\ \mathscr{M}(\partial P) &= \mathscr{M}(\pi_\# \partial T) \leqslant \mathscr{M}(\partial T).\end{aligned}$$

Choosing $\hat{\varepsilon}$ small enough according to ε, n and $\mathscr{N}(T)$ completes the proof of the the first part of conclusion (E). If we assume that $\partial T = 0$ then the homotopy formula yields

$$\pi_\# T - T = \partial H_\#([\![0,1]\!] \times T)$$

where H is the affine homotopy between π and id_Y. Since

$$\begin{aligned}\mathscr{M}(H_\#([\![0,1]\!] \times T)) &\leqslant \mathbf{c}_{2.6}(1, n+1)\hat{\varepsilon}\mathscr{M}(T) \\ \operatorname{spt} H_\#([\![0,1]\!] \times T) &\subseteq \mathbf{B}(\operatorname{spt} T, \hat{\varepsilon}),\end{aligned}$$

conclusions (E)(i) and (ii) follow upon choosing $\hat{\varepsilon}$ small enough according to ε, n and $\mathscr{M}(T)$. □

Remark 4.3. It follows in particular from conclusion (E) that P is a cycle whenever T is a cycle. In fact one can further strengthen conclusion (E) by requesting that $\partial P = \partial T$ even when T is not a cycle, as in the case when Y is finite dimensional. This can be seen from the proof above by replacing P with $P' = P - S$ where $S = H_\#([\![0,1]\!] \times \partial T)$ and $H(t, y) = \pi(y) + t(y - \pi(y))$.

It indeed ensues from the homotopy formula that $\partial P' = \partial T$, and that $\mathscr{M}(S)$ is bounded by a multiple of $\hat{\varepsilon}$. However, a *standard mistake* would consist in claiming that P' is polyhedral because S is polyhedral. In fact S need not be polyhedral. Here one ought to approximate the Lipschitz affine homotopy H by a PL map $\hat{H}$ such that $\hat{H}(0,\cdot) = H(0,\cdot)$, $\hat{H}(1,\cdot) = H(1,\cdot)$, and the relevant n dimensional Jacobians of $\hat{H}$ are not much larger than those of H. It is possible to modify the proof of Theorem 3.3 in order to obtain such an approximation, but we will not do it here since it is not needed for our next result.

Theorem 4.4. *Assume Y has the metric approximation property, $T \in \mathscr{R}_n(Y;G)$ is so that $\partial T \in \mathscr{R}_{n-1}(Y;G)$ and $\operatorname{spt} T$ is compact, and let $\varepsilon > 0$. There then exists $P \in \mathscr{P}_n(Y;G)$ such that*
(A) $\mathscr{M}(P) < \varepsilon + \mathscr{M}(T)$
(B) $\mathscr{M}(\partial P) < \varepsilon + \mathscr{M}(\partial T)$;
(C) $\mathscr{F}(T - P) < \varepsilon$;
(D) $\operatorname{spt} P \subseteq \mathbf{B}(\operatorname{spt} T, \varepsilon)$.

Proof. The proof consists in applying twice Theorem 4.2(E) to two different chains. We first apply it to the chain ∂T, with $\hat{\varepsilon} = \frac{\varepsilon}{2}$. We obtain $P_0 \in \mathscr{P}_{n_1}(Y;G)$ such that

$$\begin{aligned}
\mathscr{M}(P_0) &< \hat{\varepsilon} + \mathscr{M}(\partial T)\\
\operatorname{spt} P_0 &\subseteq \mathbf{B}(\operatorname{spt} \partial T, \hat{\varepsilon})\\
P_0 - \partial T &= \partial Z \text{ for some } Z \in \mathscr{R}_n(Y;G)\\
\mathscr{M}(Z) &< \hat{\varepsilon}\\
\operatorname{spt} Z &\subseteq \mathbf{B}(\operatorname{spt} \partial T, \hat{\varepsilon}).
\end{aligned}$$

We next define $T' = T + Z \in \mathscr{R}_n(Y;G)$. Notice that $\partial T' = P_0$ is polyhedral. It therefore ensues from Theorem 4.2(E) again that there exists $P \in \mathscr{P}_n(Y;G)$ such that

$$\begin{aligned}
\mathscr{M}(P) &< \hat{\varepsilon} + \mathscr{M}(T') \leqslant 2\hat{\varepsilon} + \mathscr{M}(T)\\
\mathscr{M}(\partial P) &\leqslant \mathscr{M}(\partial T') = \mathscr{M}(P_0) < \hat{\varepsilon} + \mathscr{M}(\partial T)\\
\mathscr{F}(P - T) &\leqslant \mathscr{F}(P - T') + \mathscr{M}(T' - T) < 2\hat{\varepsilon}\\
\operatorname{spt} P &\subseteq \mathbf{B}(\operatorname{spt} T', \hat{\varepsilon}) \subseteq \mathbf{B}(\operatorname{spt} T, 2\hat{\varepsilon}).
\end{aligned}$$

□

A. Appendix: The bounded approximation property of Banach spaces

We recall that a Banach space Y has the *bounded approximation property* (abbreviated B.A.P.) if the following holds. There exists $1 \leqslant \lambda < \infty$ such that for every compact set $C \subseteq Y$ and every $\varepsilon > 0$ there exist a finite dimensional subspace $Y' \subseteq Y$ and a bounded linear map $\pi : Y \to Y'$ with $\operatorname{Lip} \pi \leqslant \lambda$ and $\|y - \pi(y)\| \leqslant \varepsilon$ for every $y \in C$. In case one can choose $\lambda = 1$ we say that Y has the *metric approximation property* (abbreviated M.A.P.). It is useful to notice that for Y to have the B.A.P. it suffices that the definition be satisfied for *finite* sets C.

The following approximation principle is rather useful.

Theorem A.1 (Approximation principle is spaces having the B.A.P.). *Assume Y is a Banach space having the bounded approximation property. Let $C \subseteq Y$ be a compact set and $\varepsilon > 0$. Let λ, Y' and π be associated with C and ε in the definition. If $T \in \mathscr{F}_n(Y; G)$ then*

(A) $\pi_\# T \in \mathscr{F}_n(Y'; G)$;
(B) $\mathscr{M}(\pi_\# T) \leqslant \lambda^n \mathscr{M}(T)$ *and* $\mathscr{M}(\partial \pi_\# T) \leqslant \lambda^{n-1} \mathscr{M}(\partial T)$;
(C) $\mathscr{F}(T - \pi_\# T) \leqslant \varepsilon \mathbf{c}_{2.6}(n, \lambda, 1) \mathscr{N}(T)$;
(D) $\operatorname{spt} \pi_\# T \subseteq Y' \cap \mathbf{B}(\operatorname{spt} T, \varepsilon) \subseteq Y' \cap \mathbf{B}(\pi(C), \varepsilon)$.

Proof. Conclusions (A), (B) and (D) are obvious, whereas (C) follows from the homotopy formula as in (4) (applied with $f_1 = \pi$ and $f_0 = \mathrm{id}_Y$). □

Theorem A.1 says that if one is concerned about properties of chains T in $X = \mathscr{F}_n(Y; G) \cap \{T : \operatorname{spt} T \subseteq C\}$ that are not sensitive to small perturbations of T relative to the *localized topology* of X, then the analysis is the same as if T were supported in some finite dimensional space. The localized topology of X is a sequential locally convex topology characterized by the condition $T_j \to 0$ if and only if $\mathscr{F}(T_j) \to 0$ and $\sup_j \mathscr{N}(T_j) < \infty$. See [3], [10, §§10.2–10.4] or the forthcoming [2] for more on localized topologies.

This is illustrated in the last step of the proof of Theorem 4.2 of the present paper, for instance. Here we give another application, showing how the Deformation Theorem almost applies in X. Here $\hat{\mathscr{N}}$ denotes the *slicing normal mass* defined in [4].

Theorem A.2. *Let C be a compact metric space, and $M > 0$. Assume that $G \cap \{g : |g| \leqslant m\}$ is compact for every $m > 0$. It follows that*

$$\mathscr{F}_n(C; G) \cap \{T : \hat{\mathscr{N}}(T) \leqslant M\}$$

is $\mathscr{F}$-compact.

Proof. Since the slicing normal mass is lower semicontinuous with respect to $\mathscr{F}$ convergence, it suffices to establish that the set above is totally bounded. We first recall how this is a consequence of the Deformation Theorem [14] in case $C \subseteq Y'$ and Y' is *finite dimensional*. Let $N = \dim Y'$ and choose a basis $e_1, \ldots, e_N$ of Y'. There exists a constant κ_N with the following property. Given $\varepsilon > 0$ we denote by $\mathfrak{F}_{n,\varepsilon}$ the collection of all *oriented* n-faces of the ε-cubical decomposition of Y' according to the basis $e_1, \ldots, e_N$. In other words we consider the cubes $C_{k_1,\ldots,k_N} = Y' \cap \{y : \varepsilon k_i \leqslant e_i^*(y) \leqslant \varepsilon(1 + k_i) \text{ for every } i = 1, \ldots, N\}$, corresponding to integers $k_1, \ldots, k_N \in \mathbb{Z}$, and $\mathfrak{F}_{n,\varepsilon}$ consists of all m-dimensional faces of the cubes $C_{k_1,\ldots,k_N}$ together with a choice of an orientation (relative to that of the basis $e_1, \ldots, e_N$). The Deformation Theorem implies that for every $T \in \mathscr{F}_n(Y'; G)$ there exists $P \in \mathscr{P}_n(Y'; G)$ such that the following hold:

(1) There are $g_F \in G$ corresponding to each $F \in \mathfrak{F}_{n,\varepsilon}$ such that

$$P = \sum_{F \in \mathfrak{F}_{m,\varepsilon}} g_F \cdot [\![F]\!];$$

(2) $\mathscr{N}(P) \leqslant \kappa_N \mathscr{N}(T)$;

(3) $\mathscr{F}(P - T) \leqslant \varepsilon \kappa_N \mathscr{N}(T)$;

(4) $\operatorname{spt} P \subseteq \mathbf{B}(\operatorname{spt} T, \kappa_N \varepsilon)$.

We now further assume that $\operatorname{spt} T \subseteq C$ and $\mathscr{N}(T) \leqslant M$. Since $g_F = 0$ if $(\operatorname{Clos} F) \cap \mathbf{B}(\operatorname{spt} T, \kappa_N \varepsilon) = \emptyset$, according to (4), it follows that $g_F \neq 0$ only for a finite collection $\mathfrak{F}_{n,\varepsilon,C}$ of F's depending only on C and ε. We also notice that $\inf\{\mathscr{H}^m(F) : F \in \mathfrak{F}_{n,\varepsilon}\} =: \alpha > 0$. SInce $\alpha \max_F |g_F| \leqslant \mathscr{M}(P) \leqslant \kappa_N M$ we immediately infer that $\max_F |g_F| \leqslant \kappa_N M \alpha^{-1}$. Now the assumption on the coefficient group $(G, |\cdot|)$ implies that

$$\mathscr{P}_n(Y'; G) \cap \left\{ P : P = \sum_{F \in \mathfrak{F}_{n,\varepsilon,C}} g_F \cdot [\![F]\!] \text{ and } \max_F |g_F| \leqslant \kappa_N M \alpha^{-1} \right\}$$

is $\mathscr{M}$-compact, and hence also $\mathscr{F}$-compact. The theorem now easily follows in case Y' is finite dimensional.

We now consider the general case. Recall that C is isometric to a compact subset (still denoted C) of some Banach space Y having the M.A.P., for instance $Y = \ell_\infty(\mathbb{N})$ or $Y = C[0,1]$. Given $\varepsilon > 0$ we associate Y' and π with C and ε as in the definition of M.A.P. Each $T \in \mathscr{F}_n(Y; G)$ with $\operatorname{spt} T \subseteq C$ and $\mathscr{N}(T) \leqslant M$ is $\varepsilon \mathbf{c}_{2.6}(n,1,1)M$-close in the $\mathscr{F}$ norm to a member of $\mathscr{F}_n(Y'; G) \cap \{T' : \operatorname{spt} T' \subseteq \pi(C) \text{ and } \mathscr{N}(T') \leqslant M\}$, according to Theorem A.1. Since the latter is $\mathscr{F}$ compact according to the previous paragraph, the proof is complete. $\square$

Remark A.3. It is maybe worth pointing out the following consequence of the Compactness Theorem. We let ℓ_2 denote the separable Hilbert space and G a group verifying the hypothesis of Theorem A.2. Given $T_0 \in \mathscr{F}_n(\ell_2; G)$ such that $\operatorname{spt} \partial T_0$ is compact, the following Plateau problem

$$(\mathscr{P}) \begin{cases} \text{minimize } \mathscr{M}(T) \\ \text{among } T \in \mathscr{F}_n(\ell_2; G) \text{ with } \partial T = \partial T_0 \end{cases}$$

admits a minimizer. There indeed exists a minimizing sequence supported in the convex hull C of $\operatorname{spt} \partial T_0$, because C is a 1-Lipschitz retract of ℓ_2. Since C is compact, such sequence is relatively compact with respect to the $\mathscr{F}$ norm, according to Theorem A.2. The limit of a converging subsequence minimizes, because $\mathscr{M}$ is lower semicontinuous in ℓ_2.

For the reader's convenience we now give examples of Banach spaces having the M.A.P., together with elementary proofs. None of the following is new.

Proposition A.4. *Hilbert spaces have the M.A.P.*

Proof. Let Y be a Hilbert space, let $C \subseteq Y$ be compact and $\varepsilon > 0$. Choose a maximal subset $F \subseteq C$ such that $\|y - y'\| \geqslant \frac{\varepsilon}{2}$ whenever $y, y' \in F$ are distinct. Since C is compact, F is finite. We let $Y' = \operatorname{span} F$ and we let π be the orthogonal projector on Y'. One readily checks that Y' and π have the sought for properties. □

Proposition A.5. *$C[0,1]$ has the M.A.P.*

Proof. If $C \subseteq C[0,1]$ is compact and $\varepsilon > 0$ then there exists $\delta > 0$ such that $|u(t) - u(t')| < \varepsilon$ whenever $u \in C$ and $t, t' \in [0,1]$ are so that $|t - t'| < \delta$. Let $I_1, \dots, I_\kappa$ be a finite cover of $[0,1]$ by open intervals of length less than δ, and $\varphi_1, \dots, \varphi_\kappa$ a partition of unity associated with it. Choose arbitrarily $t_k \in I_k$. Let $Y' = \operatorname{span}\{\varphi_1, \dots, \varphi_\kappa\}$ and given $u \in C[0,1]$ define $\pi(u) = \sum_{k=1}^{\kappa} u(t_k)\varphi_k$. One readily checks that π is linear, that $\operatorname{Lip} \pi \leqslant 1$, and that $\|\pi(u) - u\|_\infty < \varepsilon$ when $u \in C$. □

Notice that π obtained in Proposition A.4 is a linear *retract* on Y', a property not shared by π obtained in Proposition A.5. The proof of Proposition A.5 generalizes to the case of the Banach space $C(K)$ where K is a Hausdorff compact topological space. As $\ell_\infty(\mathbb{N}) \cong C(K)$ isometrically for some K, this in particular implies the following, for which we give an elementary proof instead.

Proposition A.6. *$\ell_\infty(\mathbb{N})$ has the M.A.P.*

Proof. We start with a construction. Given $u \in \ell_\infty(\mathbb{N})$ and $\varepsilon > 0$ we find finitely many disjoint intervals I_j, $j \in J_{u,\varepsilon}$, in the real line such that

$$[-\|u\|_\infty, \|u\|_\infty] = \cup_{j \in J_{u,\varepsilon}} I_j$$

and each I_j has length less that ε. We then define

$$A_{u,j} := \mathbb{N} \cap \{\xi : u(\xi) \in I_j\} = u^{-1}(I_j)$$

and we notice that $A_{u,j}$, $j \in J_{u,\varepsilon}$, is a finite partition of $\mathbb{N}$ with the property that if $\xi_1, \xi_2 \in A_{u,j}$ for some $j \in J_{u,\varepsilon}$ then $|u(\xi_1) - u(\xi_2)| \leqslant \varepsilon$.

Next, given a subset $A \subseteq \mathbb{N}$, we define a linear map $Q_A : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ by the formula

$$Q_A(v)(\zeta) := \begin{cases} v(\min A) & \text{if } \zeta \in A \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to prove the proposition. We first observe that there is no restriction to assume the given compact set C is finite. Given a finite collection $u_1, \dots, u_\kappa$ and $\varepsilon > 0$, we apply the construction of the first paragraph to each u_k and we obtain κ finite partitions of $\mathbb{N}$, $A_{u_k,j}$, $j \in J_{u_k,\varepsilon}$, $k = 1, \dots, \kappa$. We then choose a new finite partition $A_1, \dots, A_N$ of $\mathbb{N}$ with the property that for every $k = 1, \dots, \kappa$, each A_n is contained in $A_{u_k,j}$ for some $j \in J_{u_k,\varepsilon}$. We then define $P := \sum_{n=1}^N Q_{A_n}$. This is readily a linear operator on $\ell_\infty(\mathbb{N})$. Since $A_1, \dots, A_N$ is a partition, the definition of the Q_{A_n} implies that each $\zeta \in \mathbb{N}$ belongs to exactly one $A_{n(\zeta)}$ and hence, for each $v \in \ell_\infty(\mathbb{N})$

$$P(v)(\zeta) = \sum_{n=1}^N Q_{A_n}(v)(\zeta) = Q_{A_{n(\zeta)}}(v)(\zeta) = v(\min A_{n(\zeta)}).$$

This readily implies that $\operatorname{Lip} P \leqslant 1$. It also readily shows that the range of P is spanned by e_{A_n}, $n = 1, \dots, N$ where

$$e_{A_n}(\zeta) = \begin{cases} 1 & \text{if } \zeta \in A_n \\ 0 & \text{otherwise.} \end{cases}$$

Finally if $v = u_k$ for some $k = 1, \dots, \kappa$, and $\zeta \in \mathbb{N}$ is given then $A_{n(\zeta)}$ is contained in some $A_{u_k,j}$. The definition of $A_{u_k,j}$ and the formula above for $P(u_k)(\zeta)$ then imply that $|u_k(\zeta) - P(u_k)(\zeta)| \leqslant \varepsilon$ because $\zeta, \min A_{n(\zeta)} \in A_{n(\zeta)} \subseteq A_{u_k,j}$. Since ζ is arbitrary we infer that $\|u_k - P(u_k)\|_\infty \leqslant \varepsilon$ and the proof is complete. □

The last two propositions are interesting (as far as we are concerned about applications of Theorem A.1) because any separable metric space admits an isometric embedding into $\ell_\infty(\mathbb{N})$ and an isometric embedding into $C[0,1]$.

Finally we recall that if a (separable) Banach Y has a Schauder basis $e_1, e_2, \dots$ then it has the B.A.P. Letting $Y_n = \operatorname{span}\{e_1, \dots, e_n\}$ and $\pi_n : Y \to Y_n$ be defined by $\pi_n = \sum_{k=1}^n e_k^* e_k$, it follows indeed from the Open Mapping Theorem that $\sup_n \operatorname{Lip} \pi_n = \lambda < \infty$.

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