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Kuratowski's Measure of Noncompactness with Respect to Thompson's Metric

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Abstract. It is known that the interior of a normal cone K in a Banach space is a complete metric space with respect to Thompson's metric d. We prove that Kuratowski's measure of noncompactness τ in (K°, d) has the Mazur-Darbo property and that, as a consequence, an analog of Darbo-Sadovskii's fixed point theorem is valid in (K°, d). We show that the properties of τ partly differ to the classical case. Among others τ is nicely compatible with the multiplication in ordered Banach algebras.

Keywords. Ordered Banach spaces, Thompson metric, measure of noncompactness, fixed points

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1. Introduction

Let $(E, \|\cdot\|)$ be a real Banach space ordered by a normal cone K, that is K is a closed convex subset of E such that $\lambda K \subseteq K$ $(\lambda \geq 0)$, $K \cap (-K) = \{0\}$, inducing an ordering by $x \leq y :\iff y - x \in K$, and

$$\exists c \geq 1 :\ 0 \leq x \leq y \ \Rightarrow\ \|x\| \leq c\|y\|.$$

Moreover we assume that K is solid, that is K has nonempty interior K°, and we set $x \ll y :\iff y - x \in K^\circ$. In this situation K° endowed with the Thompson metric [12]

$$d(x, y) := \log(\min\{\alpha \geq 1 : x \leq \alpha y,\ y \leq \alpha x\})$$

is a complete metric space. Let $\mathcal{B}$ denote the set of all bounded sets in (K°, d). In this paper we investigate the corresponding Kuratowski measure of noncompactness $\tau : \mathcal{B} \to [0, \infty)$ defined by

$$\tau(A) = \inf \left\{ \lambda \geq 0 : \exists n \in \mathbb{N}, A_1, \dots, A_n \in \mathcal{B} : \operatorname{diam}(A_k) \leq \lambda,\ A \subseteq \bigcup_{k=1}^{n} A_k \right\}.$$

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Of course, τ shares the general properties of Kuratowski's measure of noncompactness on a complete metric space, see Proposition 2.1. But in addition we will see that τ has the Mazur-Darbo property, that is

$$\tau(A) = \tau(\mathrm{conv}(A)) \quad (A \in \mathcal{B}).$$

This fact allows a version of Darbo-Sadovskii's fixed point theorem with respect to Thompson's metric. Moreover τ is nicely compatible with the multiplication in ordered Banach algebras. We will see that then

$$\tau(A \cdot B) \leq \tau(A) + \tau(B) \quad (A, B \in \mathcal{B}).$$

In Section 5 we prove a fixed point theorem in Banach algebras, which can be applied to certain functional-integral equations, for example.

2. Properties of d and τ

For $A \in \mathcal{B}$ and $r > 0$ we set

$$A_r := \bigcup_{a \in A} \{x \in K^\circ : d(x, a) \leq r\}.$$

For the following basic properties of Kuratowski's measure of noncompactness (which hold in general complete metric spaces) see [2].

Proposition 2.1. *For all* $A, B \in \mathcal{B}$:

1. $\tau(A) \leq \mathrm{diam}(A)$;
2. $A \subseteq B \ \Rightarrow \ \tau(A) \leq \tau(B)$;
3. $\tau(A \cup B) = \max\{\tau(A), \tau(B)\}$;
4. $\tau(\overline{A}) = \tau(A)$;
5. A *is compact* $\iff$ $A = \overline{A}$, $\tau(A) = 0$;
6. $\tau(A_r) \leq \tau(A) + 2r$ $(r > 0)$.

Next, let $p \in K^\circ$ be fixed and let $\|\cdot\|_p$ denote the Minkowski functional

$$\|x\|_p = \min\{\alpha \geq 0 : -\alpha p \leq x \leq \alpha p\} \quad (x \in E).$$

Then $\|\cdot\|_p$ is an equivalent norm on E [5, Prop.19.9].

Example 2.2. Consider $E = C[0,1]$ with maximum norm $\|\cdot\|_\infty$ and

$$K = \{x \in E : x(t) \geq 0 \ (t \in [0,1])\}.$$

Then $\|\cdot\|_\infty = \|\cdot\|_p$ for $p(\cdot) = 1 \in K^\circ$ and, for $x, y \in K^\circ$ one has

$$d(x,y) = \max_{t \in [0,1]} \left| \log \frac{x(t)}{y(t)} \right| = \left\| \log\left(\frac{x}{y}\right) \right\|_\infty .$$

It would be interesting to study also Hausdorff's measure of noncompactness in this setting and to obtain representations in the spirit of, e.g. the results in [2].

We write

$$\mathrm{dist}_p(x, \partial K) := \inf\{\|x - y\|_p : y \in \partial K\}$$

for the distance to the boundary of K with respect to this norm. We have the following

Lemma 2.3. *For any* $x \in K$:

$$x - \mathrm{dist}_p(x, \partial K)p \geq 0.$$

Proof. The assertion is clear for $x \in \partial K$. So assume that $x \in K^\circ$. Since K is convex and closed, the set $\{s \geq 0 : x - sp \in K\}$ is then a compact interval $[0, t]$ where $t > 0$. Let $x_0 := x - tp \in \partial K$. For $s \in [0, t)$ we have

$$x - sp = x_0 + (t - s)p \geq (t - s)p,$$

so $x - sp \in K^\circ$.

Clearly, $\mathrm{dist}_p(x, \partial K) \leq \|x - x_0\|_p = t$. If, on the other hand, $\mathrm{dist}_p(x, \partial K) < t$, we would find $\alpha \in (\mathrm{dist}_p(x, \partial K), t)$ and $y \in \partial K$ with

$$\|x - y\|_p \leq \alpha \iff -\alpha p \leq x - y \leq \alpha p.$$

But then $x - \alpha p \leq y$, which would imply $y \in K^\circ$, a contradiction. □

The following proposition lists some properties of d and its connection to $\|\cdot\|_p$, and inequalities concerning convex combinations. For not exactly these, but quite related inequalities, see [11, Section 2.2.].

Proposition 2.4. *For all* $x, y, x_1, \ldots, x_n, y_1, \ldots, y_m \in K^\circ$:

1. $\|x - y\|_p \leq (\exp(d(x, y)) - 1) \exp(\max\{d(x, p), d(y, p)\})$;
2. $d(x, y) \leq \dfrac{\|x - y\|_p}{\min\{\mathrm{dist}_p(x, \partial K), \mathrm{dist}_p(y, \partial K)\}}$;
3. $-\log(\mathrm{dist}_p(x, \partial K)) \leq d(x, p)$;
4. $x \in \mathrm{conv}\{x_1, \ldots, x_n\}$, $y \in \mathrm{conv}\{y_1, \ldots, y_m\} \Rightarrow$
$$d(x, y) \leq \max\{d(x_j, y_k) : j = 1, \ldots, n,\ k = 1, \ldots, m\};$$
5. *if* $x = \sum_{k=1}^n \alpha_k x_k$, $y = \sum_{k=1}^n \beta_k x_k$ *are convex combinations, then*
$$d(x, y) \leq \exp(\max\{d(x_j, x_k) : j, k = 1, \ldots, n\}) \sum_{k=1}^n |\alpha_k - \beta_k|.$$

Proof. 1.) We have

$$\begin{aligned} x - y &\leq (\exp(d(x, y)) - 1)y \\ &\leq (\exp(d(x, y)) - 1) \exp(d(y, p))p \\ &\leq (\exp(d(x, y)) - 1) \exp(\max\{d(x, p), d(y, p)\})p. \end{aligned}$$

Analogously $y - x \le (\exp(d(x,y)) - 1)\exp(\max\{d(x,p), d(y,p)\})p$, thus

$$\|x-y\|_p \le (\exp(d(x,y)) - 1)\exp(\max\{d(x,p), d(y,p)\}).$$

2.) From $y - \operatorname{dist}_p(y, \partial K)p \ge 0$ (see Lemma 2.3) we obtain

$$x = x - y + y \le \|x-y\|_p p + y \le \left(1 + \frac{\|x-y\|_p}{\operatorname{dist}_p(y,\partial K)}\right) y,$$

consequently

$$x \le \left(1 + \frac{\|x-y\|_p}{\min\{\operatorname{dist}_p(x,\partial K), \operatorname{dist}_p(y,\partial K)\}}\right) y,$$

and analogously

$$y \le \left(1 + \frac{\|x-y\|_p}{\min\{\operatorname{dist}_p(x,\partial K), \operatorname{dist}_p(y,\partial K)\}}\right) x.$$

Hence

$$\begin{aligned} d(x,y) &\le \log\left(1 + \frac{\|x-y\|_p}{\min\{\operatorname{dist}_p(x,\partial K), \operatorname{dist}_p(y,\partial K)\}}\right) \\ &\le \frac{\|x-y\|_p}{\min\{\operatorname{dist}_p(x,\partial K), \operatorname{dist}_p(y,\partial K)\}}. \end{aligned}$$

3.) For each $z \in \partial K$ we have

$$x - z \le \|x-z\|_p p \le \|x-z\|_p \exp(d(x,p))x \;\Rightarrow\; (1 - \|x-z\|_p \exp(d(x,p)))x \le z.$$

Since $x \in K^\circ$ we conclude $1 \le \|x-z\|_p \exp(d(x,p))$, and by taking the infimum over all $z \in \partial K$ we obtain $1 \le \operatorname{dist}_p(x,\partial K)\exp(d(x,p))$.

4.) Let

$$x = \sum_{j=1}^n \alpha_j x_j, \quad y = \sum_{k=1}^m \beta_k y_k$$

be convex combinations, and set

$$\gamma := \max\{d(x_j, y_k) : j = 1,\dots,n,\ k = 1,\dots,m\}.$$

Now

$$\begin{aligned} x &= \sum_{j=1}^n \sum_{k=1}^m \beta_k \alpha_j x_j \\ &\le \sum_{j=1}^n \sum_{k=1}^m \beta_k \alpha_j \exp(d(x_j, y_k)) y_k \\ &\le \exp(\gamma) \sum_{k=1}^m \sum_{j=1}^n \alpha_j \beta_k y_k \\ &= \exp(\gamma) \sum_{k=1}^m \beta_k y_k \\ &= \exp(\gamma) y. \end{aligned}$$

Analogously $y \le \exp(\gamma)x$ and therefore $d(x,y) \le \gamma$.

5.) Set $\delta := \max\{d(x_j,x_k) : j,k = 1,\dots,n\}$. For each $j \in \{1,\dots,n\}$ we have

$$x-y \le \sum_{k=1}^{n} |\alpha_k - \beta_k| x_k \le \sum_{k=1}^{n} |\alpha_k - \beta_k| \exp(d(x_k,x_j)) x_j \le \left(\exp(\delta)\sum_{k=1}^{n}|\alpha_k-\beta_k|\right) x_j.$$

Thus $x-y = \sum_{j=1}^{n} \beta_j (x-y) \le (\exp(\delta)\sum_{k=1}^{n}|\alpha_k-\beta_k|)\sum_{j=1}^{n}\beta_j x_j$ which implies

$$x \le \left(1+\exp(\delta)\sum_{k=1}^{n}|\alpha_k-\beta_k|\right) y.$$

Analogously

$$y \le \left(1+\exp(\delta)\sum_{k=1}^{n}|\alpha_k-\beta_k|\right) x,$$

and therefore

$$d(x,y) \le \log\left(1+\exp(\delta)\sum_{k=1}^{n}|\alpha_k-\beta_k|\right) \le \exp(\delta)\sum_{k=1}^{n}|\alpha_k-\beta_k|. \qquad \square$$

The next proposition delineates some compatibility of τ and the algebraic operations on E. Let us arrange that we call E an ordered Banach algebra if it is Banach algebra with unit $\mathbf{1}$ and if K satisfies in addition

$$\mathbf{1} \in K, \quad K \cdot K \subseteq K.$$

Proposition 2.5. *Let $A, B \in \mathcal{B}$ and $\alpha > 0$. Then*

1. $\tau(A+B) \le \max\{\tau(A),\tau(B)\}$;
2. $\tau(\alpha A) = \tau(A)$;
3. *if in addition E is an ordered Banach algebra, then*

$$\tau(A\cdot B) \le \tau(A) + \tau(B).$$

Proof. Let $\varepsilon > 0$ and let $A_1,\dots,A_n$ and $B_1,\dots,B_m$ be finite covers of A and B, respectively, with

$$\operatorname{diam}(A_j) \le \tau(A)+\varepsilon \quad (j=1,\dots,n), \quad \operatorname{diam}(B_k) \le \tau(B)+\varepsilon \quad (k=1,\dots,m).$$

1.) We have

$$A+B \subseteq \bigcup_{j=1,k=1}^{n,m} (A_j + B_k).$$

Now, let $a_1, a_2 \in A_j$, $b_1, b_2 \in B_k$. Then

$$\begin{aligned} a_1 + b_1 &\leq \exp(d(a_1,a_2))a_2 + \exp(d(b_1,b_2))b_2 \\ &\leq \exp(\max\{d(a_1,a_2), d(b_1,b_2)\})(a_2+b_2) \\ &\leq \exp(\max\{\tau(A),\tau(B)\}+\varepsilon)(a_2+b_2), \end{aligned}$$

and analogously

$$a_2 + b_2 \leq \exp(\max\{\tau(A),\tau(B)\}+\varepsilon)(a_1+b_1).$$

Thus $d(a_1+b_1, a_2+b_2) \leq \max\{\tau(A),\tau(B)\}+\varepsilon$, and therefore

$$\operatorname{diam}(A_j + B_k) \leq \max\{\tau(A),\tau(B)\}+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get $\tau(A+B) \leq \max\{\tau(A),\tau(B)\}$.

2.) First note that $d(x,y) = d(\alpha x, \alpha y)$ $(x, y \in K^\circ)$. We have

$$\alpha A \subseteq \bigcup_{j=1}^{n} \alpha A_j, \quad \operatorname{diam}(\alpha A_j) = \operatorname{diam}(A_j) \quad (j = 1, \ldots, n).$$

Thus $\tau(\alpha A) = \tau(A)$.

3.) We first prove that $A \cdot B \subseteq K^\circ$: If $a \in A$, $b \in B$ then $a \in K^\circ$ implies

$$\exists \lambda > 0 : \ \mathbf{1} \leq \lambda a \ \Rightarrow \ b \leq \lambda ab \ \Rightarrow \ \frac{1}{\lambda} b \leq ab,$$

and since $\frac{b}{\lambda} \in K^\circ$ we have $ab \in K^\circ$. Next,

$$A \cdot B \subseteq \bigcup_{j=1,k=1}^{n,m} A_j \cdot B_k.$$

Let $a_1, a_2 \in A_j$, $b_1, b_2 \in B_k$. Then

$$a_1 b_1 \leq \exp(d(a_1,a_2) + d(b_1,b_2)) a_2 b_2 \leq \exp(\tau(A) + \tau(B) + 2\varepsilon) a_2 b_2,$$

and analogously

$$a_2 b_2 \leq \exp(\tau(A) + \tau(B) + 2\varepsilon) a_1 b_1.$$

Thus $d(a_1 b_1, a_2 b_2) \leq \tau(A) + \tau(B) + 2\varepsilon$, and so

$$\operatorname{diam}(A_j \cdot B_k) \leq \tau(A) + \tau(B) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get $\tau(A \cdot B) \leq \tau(A) + \tau(B)$. □

3. Convex hulls

We now prove that τ has the Mazur-Darbo property:

Theorem 3.1. *For all $A \in \mathcal{B}$*

1. $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{conv}(A))$;
2. $\tau(A) = \tau(\operatorname{conv}(A))$.

Proof. 1.) Let $A \in \mathcal{B}$ and let

$$x = \sum_{j=1}^{n} \alpha_j x_j, \quad y = \sum_{k=1}^{m} \beta_k y_k$$

be convex combinations of elements of A. According to Proposition 2.4 we have

$$d(x, y) \le \max\{d(x_j, y_k) : j = 1, \dots, n,\ k = 1, \dots, m\} \le \operatorname{diam}(A).$$

Thus $\operatorname{conv}(A) \in \mathcal{B}$ and

$$\operatorname{diam}(\operatorname{conv}(A)) \le \operatorname{diam}(A).$$

Since clearly $\operatorname{diam}(A) \le \operatorname{diam}(\operatorname{conv}(A))$ the assertion follows.

2.) Again let $A \in \mathcal{B}$. According to Proposition 2.1, we have $\tau(A) \le \tau(\operatorname{conv}(A))$. Let $\varepsilon > 0$ and let $A_1, \dots, A_n \in \mathcal{B}$ be such that

$$A \subseteq \bigcup_{k=1}^{n} A_k, \quad \operatorname{diam}(A_k) \le \tau(A) + \varepsilon \quad (k = 1, \dots, n).$$

Without loss of generality we may assume $A \cap A_k \neq \emptyset$ and that A_k is convex $(k = 1, \dots, n)$. For

$$\lambda \in \Lambda := \left\{ \lambda = (\lambda_1, \dots, \lambda_n) \in [0, \infty)^n : \sum_{k=1}^{n} \lambda_k = 1 \right\}$$

we set

$$M(\lambda) := \sum_{k=1}^{n} \lambda_k A_k, \quad M := \sum_{\lambda \in \Lambda} M(\lambda).$$

Since all A_k are convex, a short calculation shows that M is convex too. Since $A \subseteq M$ we obtain $\operatorname{conv}(A) \subseteq M$. Since Λ is compact we can find a finite subset S of Λ such that $\Lambda \subseteq \bigcup_{\lambda \in S} B_\varepsilon(\lambda)$, with $B_\varepsilon(\lambda)$ the open ball with center λ and radius ε in $\mathbb{R}^n$, endowed with the l^1-norm $\|\cdot\|_1$. Now let $x \in M$, thus $x \in M(\lambda)$ for some $\lambda \in \Lambda$, that is

$$\exists (a_1, \dots, a_n) \in A_1 \times \cdots \times A_n : \ x = \sum_{k=1}^{n} \lambda_k a_k.$$

Let $\mu \in S$ with $\|\lambda - \mu\|_1 < \varepsilon$. Set

$$\delta := \max\{d(a_j, a_k) : j, k = 1, \dots, n\}.$$

First note that $A \cap A_k \neq \emptyset$ $(k = 1, \dots, n)$ implies $\delta \leq \operatorname{diam}(A) + 2(\tau(A) + \varepsilon)$. By means of Proposition 2.4, we get

$$\begin{aligned} d\left(x, \sum_{k=1}^{n} \mu_k a_k\right) &= d\left(\sum_{k=1}^{n} \lambda_k a_k, \sum_{k=1}^{n} \mu_k a_k\right) \\ &\leq \exp(\delta)\|\mu - \lambda\|_1 \\ &\leq \varepsilon \exp(\operatorname{diam}(A) + 2(\tau(A) + \varepsilon)) =: r(\varepsilon). \end{aligned}$$

This means $x \in (M(\mu))_{r(\varepsilon)}$, therefore $\operatorname{conv}(A) \subseteq M \subseteq \bigcup_{\mu \in S}(M(\mu))_{r(\varepsilon)}$. By means of Propositions 2.1, 2.4, and 2.5, we conclude

$$\begin{aligned} \tau(\operatorname{conv}(A)) &\leq \tau\left(\bigcup_{\mu \in S}(M(\mu))_{r(\varepsilon)}\right) \\ &= \max_{\mu \in S} \tau\left((M(\mu))_{r(\varepsilon)}\right) \\ &\leq 2r(\varepsilon) + \max_{\mu \in S} \tau(M(\mu)) \\ &= 2r(\varepsilon) + \max_{\mu \in S} \tau\left(\sum_{k=1}^{n} \mu_k A_k\right) \\ &\leq 2r(\varepsilon) + \max_{k=1,\dots,n} \tau(A_k) \\ &\leq 2r(\varepsilon) + \tau(A) + \varepsilon. \end{aligned}$$

Since $r(\varepsilon) \to 0$ $(\varepsilon \to 0+)$ we have $\tau(\operatorname{conv}(A)) \leq \tau(A)$. □

4. Fixed points of condensing mappings

Let $C \subseteq K^\circ$ and let $f : C \to K^\circ$ be a function. As usual we call f *condensing* (with respect to τ), if

$$A \in \mathcal{B},\ A \subseteq C,\ \tau(A) > 0 \ \Rightarrow\ f(A) \in \mathcal{B},\ \tau(f(A)) < \tau(A).$$

Since Darbo's result on the measure of noncompactness of convex sets is the key to Darbo-Sadovskii's fixed point theorem [5, Theorem 9.1], [13, Chapter 11.5], one expects by means of Theorem 3.1 an analog result with respect to τ. Note that all topological properties in the following theorem are meant with respect to the topology in (K°, d).

Theorem 4.1. *Let $\emptyset \neq C \subseteq K^\circ$ be closed, convex and bounded, and let $f : C \to C$ be continuous and condensing with respect to τ. Then f has a fixed point.*

Remark 4.2. According to Proposition 2.4, a set $M \subseteq K^\circ$ is closed and bounded in (K°, d) if and only if M is norm-closed, norm-bounded and

$$\inf_{x \in M} \operatorname{dist}_p(x, \partial K) > 0.$$

Moreover, if (x_n) is a sequence in K° and $x_0 \in K^\circ$, then

$$d(x_n, x_0) \to 0 \ (n \to \infty) \quad \Longleftrightarrow \quad \|x_n - x_0\| \to 0 \ (n \to \infty).$$

This can also be seen from Proposition 2.4, or compare [9, Chapter 2.3]. In particular, for any set $M \subseteq K^\circ$ a function $f : M \to K^\circ$ is d-continuous if and only if f is norm-continuous.

The proof of Theorem 4.1 follows the classical proof of the Darbo-Sadovskii theorem and is repeated here for convenience of the reader.

Proof. We fix $x_0 \in C$ and set

$$\mathcal{C} = \{M : x_0 \in M \subseteq C,\ M = \overline{\operatorname{conv}}(M),\ f(M) \subseteq M\}.$$

Clearly $\mathcal{C} \subseteq \mathcal{B}$, $C \in \mathcal{C}$ and

$$C_1 := \bigcap_{M \in \mathcal{C}} M \in \mathcal{C}.$$

Let

$$C_2 := \overline{\operatorname{conv}}(\{x_0\} \cup f(C_1)).$$

Then $x_0 \in C_2$ and C_2 is closed and convex. Since $x_0 \in C_1$ and $f(C_1) \subseteq C_1$ we get $\{x_0\} \cup f(C_1) \subseteq C_1$, thus also $C_2 \subseteq C_1$. Therefore

$$f(C_2) \subseteq f(C_1) \subseteq \{x_0\} \cup f(C_1) \subseteq C_2,$$

hence $C_2 \in \mathcal{C}$. We conclude $C_1 = C_2$. Assume by contradiction that $\tau(C_1) > 0$. Then, according to Theorem 3.1,

$$\tau(C_1) = \tau(\overline{\operatorname{conv}}(\{x_0\} \cup f(C_1))) = \tau(\{x_0\} \cup f(C_1)) = \tau(f(C_1)) < \tau(C_1).$$

Thus $\tau(C_1) = 0$. Summing up $C_1 \neq \emptyset$ is convex, compact in (K°, d), and $f(C_1) \subseteq C_1$. Proposition 2.4 shows that C_1 is also norm-compact, and since f is continuous, Schauder's fixed point theorem proves the existence of a fixed point of f in C_1. $\square$

5. Example

A recurrent type of condensing mappings are certain perturbations of contractions [1], [5, Chapter 9], [13, Chapter 11.6]. There are several results on contractions with respect to the Thompson metric, in particular for mixed monotone mappings, see [6–8, 11], [9, Chapter 2.3] and the references given there. Using a condition given by Guo [6] we have

Lemma 5.1. *Let* $l \geq 0$ *and let* $g : K^\circ \times K^\circ \to K^\circ$ *satisfy*

1. $x \mapsto g(x, y)$ *is monotone increasing* ($y \in K^\circ$);
2. $y \mapsto g(x, y)$ *is monotone decreasing* ($x \in K^\circ$);
3. $g(\alpha x, \frac{y}{\alpha}) \leq \alpha^l g(x, y)$ ($\alpha \geq 1,\ x, y \in K^\circ$).

Then $x \mapsto g(x, x)$ *is d-Lipschitz continuous with Lipschitz constant* l.

Proof. Let $x, y \in K^\circ$. Then

$$g(x, x) \leq g(\exp(d(x, y))y, \exp(-d(x, y))y) \leq \exp(ld(x, y))g(y, y),$$

and

$$g(y, y) \leq g(\exp(d(x, y))x, \exp(-d(x, y))x) \leq \exp(ld(x, y))g(x, x).$$

Thus $d(g(x, x), g(y, y)) \leq ld(x, y)$. □

In the following theorem let E be an ordered Banach algebra, and for $x \leq y$ let $[x, y]$ denote the order interval $\{z \in E : x \leq z \leq y\}$. For applications of the classical measures of noncompactness in Banach algebras see [3, 4].

Theorem 5.2. *Let* $g : K^\circ \times K^\circ \to K^\circ$ *satisfy the assumptions in Lemma* 5.1 *with* $l < 1$, *let* $0 \ll q_1 \leq q_2$ *and let* $h : K^\circ \to [q_1, q_2]$ *be continuous and norm-compact (i.e.* $h(K^\circ)$ *is relativly compact in* $(E, \|\cdot\|)$*). Then* $f : K^\circ \to K^\circ$, $f(x) = g(x, x) \cdot h(x)$ *has a fixed point.*

Proof. First note that each order interval in K° is bounded and closed in norm and with respect to d, so $h(K^\circ)$ is relatively compact in (K°, d). Next, we fix $w \in K^\circ$. Then, for $\alpha \geq 1$

$$\frac{g(\alpha w, \frac{w}{\alpha})q_2}{\alpha} \leq \alpha^{l-1} g(w, w)q_2 \to 0 \quad (\alpha \to \infty).$$

Moreover, for $\alpha \geq 1$ we have $g(w, w) = g(\alpha \frac{w}{\alpha}, \frac{\alpha w}{\alpha}) \leq \alpha^l g(\frac{w}{\alpha}, \alpha w)$. Thus

$$\alpha g\left(\frac{w}{\alpha}, \alpha w\right) q_1 \geq \alpha^{1-l} g(w, w)q_1 \quad (\alpha \geq 1).$$

Hence, there exists $\alpha_0 \geq 1$ such that

$$g\left(\alpha_0 w, \frac{w}{\alpha_0}\right) q_2 \leq \alpha_0 w, \quad g\left(\frac{w}{\alpha_0}, \alpha_0 w\right) q_1 \geq \frac{w}{\alpha_0}.$$

For $C := [\frac{w}{\alpha_0}, \alpha_0 w]$ we get $f(C) \subseteq C$: Let $\frac{w}{\alpha_0} \leq x \leq \alpha_0 w$. Then

$$\frac{w}{\alpha_0} \leq g\left(\frac{w}{\alpha_0}, \alpha_0 w\right) q_1 \leq f(x) \leq g\left(\alpha_0, \frac{w}{\alpha_0}\right) q_2 \leq \alpha_0 w.$$

By Lemma 5.1, $x \mapsto g(x,x)$ is continuous, hence $f_{|C} : C \to C$ is continuous, and C is a convex closed and bounded subset of (K°, d). Finally we show that $f_{|C}$ is a τ-set contraction, hence condensing: By setting $\widetilde{g}(x) = g(x,x)$ we get from Lemma 5.1 and Proposition 2.5 that for each $A \subseteq C$

$$\tau(f(A)) \leq \tau(\widetilde{g}(A) \cdot h(A)) \leq \tau(\widetilde{g}(A)) + \tau(h(A)) \leq l\tau(A) + 0 = l\tau(A).$$

Now, application of Theorem 4.1 proves the existence of a fixed point of f in C. □

The authors are grateful to the referee for the observation that Theorem 5.2 can also be proved by a suitable modification of the proof of a fixed point theorem by Krasnoselskii [10] (cf., e.g. [5, p. 71, paragraph after Theorem 1]).

Example 5.3. Take for example $E = C([0,1], \mathbb{R})$, endowed with the maximum norm and ordered by the natural cone

$$K := \{x \in C([0,1], \mathbb{R}) : x(t) \geq 0 \ (t \in [0,1])\},$$

with

$$K^\circ = \{x \in C([0,1], \mathbb{R}) : x(t) > 0 \ (t \in [0,1])\}.$$

By setting

$$g(x,y)(t) = \sqrt{x(1-t)} + \frac{1}{\sqrt{y(t^2)}}, \quad h(x) = \int_0^1 \exp(\lambda t s \sin x(s))\, ds,$$

$$q_1(t) = \exp(-|\lambda|), \quad q_2(t) = \exp(|\lambda|) \quad (t \in [0,1]),$$

Theorem 5.2 proves the existence of a positive solution of the functional-integral equation

$$x(t) = \left(\sqrt{x(1-t)} + \frac{1}{\sqrt{x(t^2)}}\right) \int_0^1 \exp(\lambda t s \sin x(s))\, ds$$

for each $\lambda \in \mathbb{R}$.

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