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On $p_s(x)$-Laplacian Parabolic Problems with Non-Globally Lipschitz Forcing Term

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Abstract. In this work we prove continuity of solutions with respect to initial conditions and exponent parameters and we prove upper semicontinuity of a family of global attractors for problems of the form

$$\frac{\partial u_s}{\partial t} - \operatorname{div}(|\nabla u_s|^{p_s(x)-2}\nabla u_s) + f(x, u_s) = g,$$

where $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a non-globally Lispchitz Carathéodory mapping, $g \in L^2(\Omega)$, Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 1$ and $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ ($p > 2$ constant) as s goes to infinity.

Keywords. Variable exponents, electrorheological fluids, $p_s(x)$-Laplacian, parabolic problems, global attractors, upper semicontinuity.

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1. Introduction

In the last seven years various researchers have spent efforts to obtain results on existence, uniqueness, blow-up, vanishing, local boundedness and localization of solutions for parabolic problems with variable exponents (see, for example, [3,6–9,14–16,21,27,33]). However, until the moment few works have been appeared in the literature about global attractors, see [25] and references therein. The theory of problems with variable exponents has application in electrorheological fluids (fluids characterized by the ability to drastically change the mechanical properties under the influence of exterior electromagnetic field) (see [12,23,24]), image processing (see [1,11,17]) and the models of porous medium equations

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with variable exponents of nonlinearity were considered in [6, 7, 31]. We also refer the reader to [19] for an overview of differential equations with variable exponents.

In [2], G. Akagi and K. Matsuura studied the limiting behavior of solutions for nonlinear diffusion equations driven by the $p(x)$-Laplacian as $p(\cdot)$ diverges to the infinity.

In [18] P. Harjulehto, P. Hästö and M. Koskenoja considered Dirichlet energy integral minimizers in variable exponent Sobolev spaces. In the paper [4], B. Amaziane, L. Pankratov and V. Prytula studied homogenization of $p_\epsilon(x)$-Laplacian elliptic equations and in [5], B. Amaziane, L. Pankratov and A. Piatnitski studied nonlinear flow through double porosity media in variable exponent Sobolev spaces, and the authors considered the following initial boundary value problem

$$\begin{cases} \omega^\epsilon(x)\dfrac{\partial u^\epsilon}{\partial t}(t) - \operatorname{div}(k^\epsilon(x)\nabla u^\epsilon|\nabla u^\epsilon|^{p_\epsilon(x)-2}) = g(t,x) & \text{in } Q \\ u^\epsilon = 0 & \text{on }]0,t[\times\partial\Omega, \\ u^\epsilon(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain, Q denotes the cylinder $]0,T[\times\Omega$, $T > 0$ is given, $g \in C([0,T]; L^2(\Omega))$ and $u_0 \in H^2(\Omega)$ are given functions. They studied the minimization problem for functionals in the limit of small ϵ and obtained the homogenized functional.

In [30] we considered the following one dimensional nonlinear PDE problem

$$\begin{cases} \dfrac{\partial u_s}{\partial t}(t) - \dfrac{\partial}{\partial x}\left(|\dfrac{\partial u_s}{\partial x}(t)|^{p_s(x)-2}\dfrac{\partial u_s}{\partial x}(t)\right) = B(u_s(t)), \quad t > 0 \\ u_s(0) = u_{0s}, \end{cases}$$

under Dirichlet homogeneous boundary conditions, where $u_{0s} \in H := L^2(I)$, $I := (c,d)$, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, $p_s(x) \in C^1(\bar{I})$, $p_s^- := \inf_{x\in I} p_s(x) > 2$ for all $s \in \mathbb{N}$, and $p_s(\cdot) \to p$ in $L^\infty(I)$ $(p > 2$ constant) as $s \to \infty$ and proved the continuity of the flows and the upper semicontinuity of the family of global attractors $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ as s goes to infinity.

Let us consider the following nonlinear PDE problem

$$\begin{cases} \dfrac{\partial u_s}{\partial t}(t) - \operatorname{div}\left(|\nabla u_s(t)|^{p_s(x)-2}\nabla u_s(t)\right) + f(x,u_s(t)) = g, \qquad t > 0 \\ u_s(0) = u_{0s}, \end{cases} \tag{1}$$

under Dirichlet homogeneous boundary conditions, where $u_{0s} \in H := L^2(\Omega)$, Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 1$, $g \in L^2(\Omega)$, $p_s(x) \in C^1(\overline{\Omega})$ for all $s \in \mathbb{N}$, $2 < p \leq p_s(x) \leq a$, for all $x \in \Omega$ and for all $s \in \mathbb{N}$. and $p_s(\cdot) \to p$ in

$L^\infty(\Omega)$ (p constant) as $s \to \infty$. We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a non-globally Lipschitz Carathéodory mapping satisfying the following conditions: there exist positive constants ℓ, k, c_1 and $c_2 \geq 1$ such that

$$(f(x,s_1) - f(x,s_2))(s_1 - s_2) \geq -\ell|s_1 - s_2|^2, \quad \forall\, x \in \Omega \text{ and } s_1, s_2 \in \mathbb{R}, \tag{2}$$

$$c_2|s|^{q(x)} - k \leq f(x,s)s \leq c_1|s|^{q(x)} + k, \quad \forall\, x \in \Omega \text{ and } \quad s \in \mathbb{R}, \tag{3}$$

where $q \in C(\overline{\Omega})$ with $2 < q^- := \inf_{x\in\Omega} q(x) \leq q^+ := \sup_{x\in\Omega} q(x)$. For example, if $\alpha_1 > 1$ and $r > 2$, we observe that the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ given by $f(x,u) = \alpha_1|u|^{r-2}u - u$ is not globally Lipschitz and satisfies the condition (2) with $\ell = 1$ and the condition (3) with $c_2 = 1$, $c_1 = \alpha_1$ and $q(x) = r$ for all $x \in I$.

In this work we investigate in what way the parameter $p_s(x)$ affects the dynamic of problem (1), analyzing the continuity properties of the flows and the global attractors with respect to the parameter $p_s(x)$, when f is a locally Lipschitz function.

The paper is organized as follows. In Section 2 we give some preliminary results. In Section 3 we obtain uniform estimates for solutions of (1). In Section 4 we prove that the solutions $\{u_s\}$ of (1) go to the solution u of the limit problem and, after that, we obtain the upper semicontinuity of the global attractors for the problem (1).

2. Preliminaries

We denote

$$\begin{aligned}\tilde{X}_s := \big\{ u : u \in L^{p^-}\left(0,T; W_0^{1,p_s(x)}(\Omega)\right) \cap L^\infty\left(0,T; L^2(\Omega)\right) \cap L^{q(x)}(\Omega \times (0,T)) \\ \text{with } \nabla u \in L^{p_s(x)}(\Omega \times (0,T)) \big\}.\end{aligned}$$

Definition 2.1 ([22, Definition 2.1])**.** A solution of problem (1) is a function $u_s \in \tilde{X}_s$ such that

$$\int_0^t \int_\Omega \left(-u_s \frac{d\varphi}{dt} + |\nabla u_s|^{p_s(x)-2}\nabla u_s \nabla\varphi + f(x,u_s)\varphi \right) dx d\tau = \int_0^t \int_\Omega g\varphi dx d\tau - \int_\Omega u_s \varphi dx \Big|_0^t$$

holds for any $t \leq T$ and all $\varphi \in \tilde{X}_s$ with $\dfrac{d\varphi}{dt} \in \tilde{X}_s^*$, where $\tilde{X}_s^*$ is the dual space of $\tilde{X}_s$.

Theorem 2.2 ([22, Theorem 2.1])**.** *If* $q^+ < \infty$ *then the problem* (1) *admits a unique solution* $u_s \in C([0,T]; L^2(\Omega))$*. Moreover, the mapping* $u_{0s} \mapsto u_s(t)$ *is continuous in* $L^2(\Omega)$*.*

Theorem 2.3 ([22, Theorem 2.2])**.** *The semigroup $\{S_s(t)\}_{t\geq 0}$ associated with problem* (1) *admits an absorbing set in $W_0^{1,p_s(x)}(\Omega)\cap L^{q(x)}(\Omega)$, i.e., there exists a bounded set $B_{0s}\subset W_0^{1,p_s(x)}(\Omega)\cap L^{q(x)}(\Omega)$ such that for any bounded set B in $L^2(\Omega)$ there exists $T_{0s}>0$ such that $S_s(t)B\subset B_{0s}$ for any $t\geq T_{0s}$, where T_{0s} depends on B.*

Corollary 2.4 ([22, Corollary 2.1])**.** *The semigroup associated with problem* (1) *possesses a global attractor $\mathcal{A}_s$ in $L^2(\Omega)$.*

Observe that by Theorem 2.3 we have

$$\|u_s(t)\|_{W_0^{1,p_s(x)}(\Omega)}\leq C_s,\quad \forall\, t\geq T_{0s}.$$

In this work we will prove a uniform estimate on the parameter s (see Theorem 3.4).

We denote $-\Delta_{p(x)}(u):=-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$. We observe (see [29]) that the operator $-\Delta_{p(x)}$ is the subdifferential $\partial\varphi_{p(x)}$ of the convex, proper and lower semicontinuous map $\varphi_{p(x)}:L^2(\Omega)\to\mathbb{R}\cup\{+\infty\}$ given by

$$\varphi_{p(x)}(u):=\begin{cases}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx & \text{if } u\in W_0^{1,p(x)}(\Omega)\\ +\infty & \text{otherwise.}\end{cases}$$

Moreover, we have (see [28, Lemma 1])

Lemma 2.5. *Let $p(x)\in C(\overline{\Omega})$ with $p(x)>2$ in Ω. We have:*

(i) *If $\|v\|_{W_0^{1,p(x)}(\Omega)}\leq 1$, then $\langle -\Delta_{p(x)}(v),v\rangle\geq \|v\|^{p^+}_{W_0^{1,p(x)}(\Omega)}$;*

(ii) *If $\|v\|_{W_0^{1,p(x)}(\Omega)}\geq 1$, then $\langle -\Delta_{p(x)}(v),v\rangle\geq \|v\|^{p^-}_{W_0^{1,p(x)}(\Omega)}$.*

In [13, 20], the authors obtained results on the density of continuous functions in variable exponent Sobolev spaces. Here, in order to prove the continuity of the flows (in Section 4) for problem (1) we need the following

Theorem 2.6. *If $p\in C^1(\Omega)$, then $C_0^\infty(\Omega)\subset D(-\Delta_{p(x)})$.*

Proof. Let $u\in C_0^\infty(\Omega)$. We consider the sets $W_1(u):=\{x\in \mathrm{supp}(u);|\nabla u|>0\}$ and $W_2(u):=\{x\in\mathrm{supp}(u);|\nabla u|=0\}$. Thus $\mathrm{supp}(u)=W_1(u)\cup W_2(u)$ and

$W_1(u) \cap W_2(u) = \emptyset$. We want to show that $-\Delta_{p(x)}(u) \in L^2(\Omega)$. We have,

$$
\begin{aligned}
&\int_\Omega |-\Delta_{p(x)}(u)|^2 dx \\
&= \int_\Omega \left|\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)\right|^2 dx \\
&= \int_{\mathrm{supp}(u)} \left|\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)\right|^2 dx \\
&= \int_{W_1(u)} \left|\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)\right|^2 dx \\
&= \int_{W_1(u)} \Bigg| |\nabla u|^{p(x)-2} \sum_{i=1}^n \left(\frac{\partial p}{\partial x_i}\frac{\partial u}{\partial x_i}\ln|\nabla u|\right) \\
&\quad + \frac{(p(x)-2)}{2}|\nabla u|^{p(x)-4}\sum_{i=1}^n\left(\frac{\partial}{\partial x_i}|\nabla u|^2\frac{\partial u}{\partial x_i}\right) + |\nabla u|^{p(x)-2}\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}\Bigg|^2 dx \\
&\le \int_{W_1(u)} \Bigg(\left| |\nabla u|^{p(x)-2} \sum_{i=1}^n \left(\frac{\partial p}{\partial x_i}\frac{\partial u}{\partial x_i}\ln|\nabla u|\right)\right| \\
&\quad + \left|\frac{(p(x)-2)}{2}|\nabla u|^{p(x)-4}\sum_{i=1}^n\frac{\partial}{\partial x_i}|\nabla u|^2\frac{\partial u}{\partial x_i}\right| + \left|\nabla u|^{p(x)-2}\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}\right|\Bigg)^2 dx. \\
&\le \int_{W_1(u)} \Bigg(\left| |\nabla u|^{p(x)-2} \sum_{i=1}^n \left(\frac{\partial p}{\partial x_i}\frac{\partial u}{\partial x_i}\ln|\nabla u|\right)\right| \\
&\quad + (p(x)-2)|\nabla u|^{p(x)-2}\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}\nabla u\right| + \left|\nabla u|^{p(x)-2}\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}\right|\Bigg)^2 dx.
\end{aligned}
$$

Now, consider the continuous functions $g_i : \mathbb{R}^n \to \mathbb{R}$ for $1 \le i \le n$ given by

$$
g_i(x) = \begin{cases} x_i \ln(|x|) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}.
$$

As $p(\cdot) > 2$ in Ω, we can return from $W_1(u)$ to $\mathrm{supp}(u)$ and we have

$$
\begin{aligned}
&\int_\Omega |-\Delta_{p(x)}(u)|^2 dx \\
&\le \int_{\mathrm{supp}(u)} \Bigg(\left| |\nabla u|^{p(x)-2} \sum_{i=1}^n \frac{\partial p}{\partial x_i} g_i(\nabla u)\right| \\
&\quad + (p(x)-2)\,|\nabla u|^{p(x)-2}\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}\nabla u\right| + \left||\nabla u|^{p(x)-2}\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}\right|\Bigg)^2 dx \\
&< \infty
\end{aligned}
$$

since the function

$$h(x) := \left(\left| |\nabla u|^{p(x)-2} \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} g_i(\nabla u) \right| \right.$$
$$\left. + (p(x)-2)\, |\nabla u|^{p(x)-2} \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} \nabla u \right| + \left| |\nabla u|^{p(x)-2} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \right| \right)^2$$

is continuous in Ω. □

We denote $X_s := W_0^{1,p_s(x)}(\Omega)$, $X_a := W_0^{1,a}(\Omega)$ and $X := W_0^{1,p}(\Omega)$. It is a known result that $X_s \subset H$ with continuous and dense embeddings (see [27]). Moreover,

Lemma 2.7. *There exists a constant $K = K(|\Omega|, n, p) > 0$, independent of s, such that if $u_s \in X_s$, $s \in \mathbb{N}$, then $\|u_s\|_H \leq K\|u_s\|_{X_s}$, for all $s \in \mathbb{N}$.*

Proof. We know that if $p(x) > q(x)$ then $L^{p(x)}(\Omega) \subset L^{q(x)}(\Omega)$ with $\|u\|_{q(x)} \leq 2(|\Omega|+1)\|u\|_{p(x)}$ for all $u \in L^{p(x)}(\Omega)$ (see [12]). Thus if $u_s \in X_s \subset X \subset H$ we have

$$\begin{aligned} \|u_s\|_H &\leq 2(|\Omega|+1)\|u_s\|_p \\ &\leq 2(|\Omega|+1)C_0(|\Omega|, n, p)\|\nabla u_s\|_p \\ &\leq 4(|\Omega|+1)^2 C_0(|\Omega|, n, p)\|\nabla u_s\|_{p_s(x)} = K\|u_s\|_{X_s}, \end{aligned}$$

where $C_0(|\Omega|, n, p)$ is the positive constant in the Poincaré inequality and $K = K(|\Omega|, n, p) := 4(|\Omega|+1)^2 C_0(|\Omega|, n, p)$. □

3. Uniform estimates

We have the following uniform estimates on the solutions of (1):

Lemma 3.1. *Let u_s be a solution of* (1) *with $u_s(0) = u_{0s} \in H$.*

a) *Given $T_0 > 0$, there exists a positive number r_0 such that $\|u_s(t)\|_H \leq r_0$, for each $t \geq T_0$ and $s \in \mathbb{N}$.*
b) *Given a bounded set $B \subset H$, there exists $D_1 > 0$ such that $\|u_s(t)\|_H \leq D_1$ for all $t \geq 0$ and $s \in \mathbb{N}$ such that $u_{0s} \in B$.*

Proof. a) It is enough to consider $u_{0s} \in D(-\Delta_{p_s(x)})$. Let $\tau > 0$, multiplying the equation on (1) by $u_s(\tau)$ we have

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 + \langle -\Delta_{p_s(x)}(u_s(\tau)), u_s(\tau)\rangle + \int_\Omega f(x, u_s(\tau))u_s(\tau)dx = \int_\Omega g u_s(\tau)dx.$$

By (3) we have

$$\int_\Omega f(x,u_s(\tau))u_s(\tau)dx \geq c_2\int_\Omega |u_s(\tau)|^{q(x)}dx - k|\Omega|. \tag{4}$$

Given $T_0>0$, if $\|u_s(\tau)\|_{X_s}>1$ then by Lemma 2.5 and (4) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 &\leq -\|u_s(\tau)\|_{X_s}^{p_s^-} - c_2\int_\Omega |u_s(\tau)|^{q(x)}dx + k|\Omega| + \int_\Omega g u_s(\tau)dx\\ &\leq -\|u_s(\tau)\|_{X_s}^{p} + k|\Omega| + \frac{1}{2}\|g\|_H^2 + \frac{1}{2}\|u_s(\tau)\|_H^2.\end{aligned}$$

Using Lemma 2.7 we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 \leq -\frac{1}{K^p}\|u_s(\tau)\|_H^p + \frac{1}{2}\|u_s(\tau)\|_H^2 + C_1,$$

where $C_1=C_1(|\Omega|,\|g\|_H)$ and K is the constant independent of s of Lemma 2.7.

Now, we consider $\epsilon>0$ arbitrary and $\theta:=\frac{p}{2}$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Then using Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 \leq \left(\frac{2}{p}\epsilon^{\frac{p}{2}} - \frac{1}{K^p}\right)\|u_s(\tau)\|_H^p + C_2,$$

where $C_2=C_2(|\Omega|,\|g\|_H,p,\epsilon)$. Choosing $\epsilon_0>0$ sufficiently small such that $\frac{2}{p}\epsilon_0^{\frac{p}{2}} < \frac{1}{2K^p}$ we obtain $\frac{1}{2}\frac{d}{dt}\|u_s(\tau)\|_H^2 \leq -\frac{1}{2K^p}\|u_s(\tau)\|_H^p + C_3$, where $C_3 = C_3(|\Omega|,\|g\|_H,p,\epsilon_0)$. So,

$$\frac{d}{dt}\|u_s(\tau)\|_H^2 + \frac{1}{K^p}\|u_s(\tau)\|_H^p \leq 2C_3.$$

Let $I_s:=\{\tau\in(0,\infty);\|u_s(\tau)\|_{X_s}>1\}$ and $y_s:I_s\to\mathbb{R}$, $y_s(\tau):=\|u_s(\tau)\|_H^2$ satisfies the differential inequality

$$y_s'(\tau)+K^{-p}[y_s(\tau)]^{\frac{p}{2}} \leq 2C_3.$$

Therefore, from [32, Lemma 5.1, p. 163], we obtain

$$\|u_s(\tau)\|_H^2 \leq \Big(2C_3K^p\Big)^{\frac{2}{p}} + \left[\frac{1}{2K^p}(p-2)T_0\right]^{\frac{-2}{p-2}}, \quad \forall\, \tau\geq T_0.$$

If $\|u_s(\tau)\|_{X_s}\leq 1$, then $\|u_s(\tau)\|_H \leq K\|u_s(\tau)\|_{X_s}\leq K$. So, taking

$$K_0 := \left\{\Big(2C_3K^p\Big)^{\frac{2}{p}} + \left[\frac{1}{2K^p}(p-2)T_0\right]^{\frac{-2}{(p-2)}}\right\}^{\frac{1}{2}}$$

and $r_0 := \max\{K, K_0\}$ we obtain $\|u_s(\tau)\|_H \leq r_0$, for all $\tau \geq T_0$, $s \in \mathbb{N}$, and a) of the lemma is proved.

b) Now, take $T_0 > 0$ and consider u_s a solution of (1) with $u_s(0) = u_{0s} \in B \subset H$. Then, multiplying the equation on (1) by $u_s(t)$ we have

$$\left\langle \frac{d}{dt}u_s(t), u_s(t) \right\rangle + \langle -\Delta_{p_s(x)}(u_s(t)), u_s(t)\rangle + \langle f(x, u_s(t)), \quad u_s(t)\rangle = \langle g, u_s(t)\rangle.$$

Since $\langle -\Delta_{p_s(x)}(u_s(t)), u_s(t)\rangle \geq 0$ and $\langle f(x, u_s(t)), u_s(t)\rangle \geq -k|\Omega|$ (see (4)) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_s(t)\|_H^2 \leq \frac{1}{2}\|u_s(t)\|_H^2 + k|\Omega| + \frac{1}{2}\|g\|_H^2. \tag{5}$$

Integrating (5) from 0 to $t \leq T_0$ and using the Gronwall-Bellman Lemma we obtain

$$\|u_s(t)\|_H^2 \leq \left[\|u_{0s}\|_H^2 + \left(2k|\Omega| + \|g\|_H^2\right) T_0\right] e^{T_0}, \quad \forall\, t \in [0, T_0],$$

and the second part of the lemma is proved. □

Remark 3.2. The constants r_0 and D_1 in Lemma 3.1 depend neither on the initial data nor on s.

Corollary 3.3. *There exists a bounded set B_0 in H such that $\mathcal{A}_s \subset B_0$ for all $s \in \mathbb{N}$.*

Theorem 3.4. *Let u_s be a solution of (1). Given $T_1 > 1$, there exists a positive constant $r_1 > 0$, independent of s, such that*

$$\|u_s(t)\|_{X_s} < r_1,$$

for every $t \geq T_1$ and $s \in \mathbb{N}$.

Proof. Multiplying problem (1) by u_s and using (3) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_s|^2 dx + \min\{1, c_2\}\left[\int_\Omega |\nabla u_s|^{p_s(x)}dx + \int_\Omega |u_s|^{q(x)}dx\right]\\
&\leq \frac{1}{2}\frac{d}{dt}\int_\Omega |u_s|^2 dx + \int_\Omega |\nabla u_s|^{p_s(x)}dx + c_2\int_\Omega |u_s|^{q(x)}dx\\
&\leq \int_\Omega \frac{g\epsilon^{\frac{1}{2}}u_s}{\epsilon^{\frac{1}{2}}}dx + k|\Omega|\\
&\leq \frac{1}{2\epsilon}\int_\Omega |g|^2 dx + \frac{1}{2}\epsilon\int_\Omega |u_s|^2 dx + k|\Omega|.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{dt}\|u_s\|_H^2 + 2\min\{1, c_2\}\int_\Omega \left(|\nabla u_s|^{p_s(x)} + |u_s|^{q(x)}\right) dx\\
&\leq \frac{1}{\epsilon}\int_\Omega |g|^2 dx + \epsilon\int_\Omega |u_s|^2 dx + 2k|\Omega|.
\end{aligned} \tag{6}$$

Now, if $\theta(x) := \dfrac{q(x)}{2}$ we have

$$\begin{aligned}\int_\Omega |u_s|^2 dx &\le \int_\Omega \left(\frac{1}{\theta(x)}|u_s|^{q(x)} + \frac{1}{\theta'(x)}\right) dx\\ &= \int_\Omega \frac{2}{q(x)}|u_s|^{q(x)}dx + \int_\Omega \frac{q(x)-2}{q(x)}dx \\ &\le \int_\Omega |u_s|^{q(x)}dx + C_4|\Omega|.\end{aligned} \tag{7}$$

Using (7) in (6) we obtain $\frac{d}{dt}\|u_s\|_H^2 + 2\min\{1, c_2\}\int_\Omega \left(|\nabla u_s|^{p_s(x)} + |u_s|^{q(x)}\right) dx \le \frac{1}{\epsilon}\|g\|_H^2 + \epsilon\int_\Omega |u_s|^{q(x)}dx + (2k + \epsilon C_4)|\Omega|$. Since $c_2 \ge 1$ we have

$$\frac{d}{dt}\|u_s\|_H^2 + 2\int_\Omega |\nabla u_s|^{p_s(x)}dx + (2-\epsilon)\int_\Omega |u_s|^{q(x)}dx \le \frac{1}{\epsilon}\|g\|_H^2 + (2k + \epsilon C_4)|\Omega|,$$

and taking $\epsilon_0 > 0$ such that $2 - \epsilon_0 \ge 1$ we obtain

$$\begin{aligned}\frac{d}{dt}\|u_s\|_H^2 + \int_\Omega \left(|\nabla u_s|^{p_s(x)} + |u_s|^{q(x)}\right) dx &\le \frac{1}{\epsilon_0}\|g\|_H^2 + (2k + \epsilon_0 C_4)|\Omega| \\ &\le C_5\|g\|_H^2 + C_5|\Omega|,\end{aligned} \tag{8}$$

where C_5 is a positive constant independent of $s \in \mathbb{N}$. Integrating (8) over $[t, t+1]$, $t \ge T_0$, using Lemma 3.1 we obtain

$$\begin{aligned}\int_t^{t+1}\int_\Omega \left(|\nabla u_s(\tau)|^{p_s(x)} + |u_s(\tau)|^{q(x)}\right) dx d\tau &\le C_5\|g\|_H^2 + C_5|\Omega| + \|u_s(t)\|_H^2 \\ &\le C_5\|g\|_H^2 + C_5|\Omega| + r_0^2 := C_6\end{aligned} \tag{9}$$

for all $t \ge T_0$, where $C_6 = C_6(\|g\|_H, |\Omega|)$.

As for each $s \in \mathbb{N}$ the function $[0,T] \ni t \mapsto \tilde{f}_s(t)(\cdot) := g(\cdot) - f(\cdot, u_s(t,\cdot))$ is in $L^2(0,T;H)$ it follows from [10, Theorem 3.6] that for all $\tau \ge T_0$

$$\begin{aligned}&\int_\Omega \left|\frac{du_s}{dt}(\tau)\right|^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\tau)|^{p_s(x)}dx + \int_\Omega \frac{d}{dt}F(x, u_s(\tau))dx \\ &= \int_\Omega \left|\frac{du_s}{dt}(\tau)\right|^2 dx + \langle \partial\varphi_{p_s(x)}(u_s(\tau)), \frac{du_s}{dt}(\tau)\rangle + \int_\Omega f(x, u_s(\tau))\frac{du_s}{dt}(\tau)dx \\ &= \int_\Omega g\frac{du_s}{dt}(\tau)dx \\ &\le \frac{1}{2}\|g\|_H^2 + \frac{1}{2}\left\|\frac{du_s}{dt}(\tau)\right\|_H^2,\end{aligned}$$

where $F(x,s) = \int_0^s f(x,\tau)d\tau$. Thus,

$$\frac{1}{2}\int_\Omega \left|\frac{du_s}{dt}(\tau)\right|^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\tau)|^{p_s(x)}dx + \frac{d}{dt}\int_\Omega F(x, u_s(\tau))dx \le \frac{1}{2}\|g\|_H^2. \tag{10}$$

From assumption (3), there exist positive constants $\tilde{c}_1, \tilde{c}_2, C_7$ such that

$$\tilde{c}_2|s|^{q(x)} - C_7 \leq F(x,s) \leq \tilde{c}_1|s|^{q(x)} + C_7. \tag{11}$$

Integrating (10) over $[\eta, t+1]$, $T_0 \leq t < \eta < t+1$, yields

$$\begin{aligned}
&\int_\Omega \frac{1}{p_s(x)}|\nabla u_s(t+1)|^{p_s(x)}dx + \int_\Omega F(x, u_s(t+1))dx \\
&\leq \frac{1}{2}\int_\eta^{t+1}\int_\Omega \left|\frac{du_s}{dt}(\tau)\right|^2 dx\; d\tau + \int_\Omega \frac{1}{p_s(x)}|\nabla u_s(t+1)|^{p_s(x)}dx + \int_\Omega F(x, u_s(t+1))dx \\
&\leq \frac{1}{2}\|g\|_H^2 + \int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\eta)|^{p_s(x)}dx + \int_\Omega F(x, u_s(\eta))dx.
\end{aligned}$$

Integrating the above inequality with respect to η between t and $t+1$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{p_s(x)}|\nabla u_s(t+1)|^{p_s(x)}dx + \int_\Omega F(x, u_s(t+1))dx \\
&\leq \frac{1}{2}\|g\|_H^2 + \int_t^{t+1}\int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\eta)|^{p_s(x)}dxd\eta + \int_t^{t+1}\int_\Omega F(x, u_s(\eta))dxd\eta.
\end{aligned}$$

Using the above inequality, the assumptions on $p_s(x)$, (11) and (9), we get

$$\begin{aligned}
&\frac{1}{a}\int_\Omega |\nabla u_s(t+1)|^{p_s(x)}dx + \tilde{c}_2\int_\Omega |u_s(t+1)|^{q(x)}dx - C_7|\Omega| \\
&\leq \frac{1}{2}\|g\|_H^2 + \int_t^{t+1}\int_\Omega \frac{1}{p_s(x)}|\nabla u_s(\eta)|^{p_s(x)}dxd\eta + \int_t^{t+1}\int_\Omega F(x, u_s(\eta))dxd\eta \\
&\leq \frac{1}{2}\|g\|_H^2 + \frac{1}{p}\int_t^{t+1}\int_\Omega |\nabla u_s(\eta)|^{p_s(x)}dxd\eta + \tilde{c}_1\int_t^{t+1}\int_\Omega |u_s(\eta)|^{q(x)}dxd\eta + C_7|\Omega| \\
&\leq \frac{1}{2}\|g\|_H^2 + \max\left\{\frac{1}{p}, \tilde{c}_1\right\}\int_t^{t+1}\int_\Omega \left(|\nabla u_s(\eta)|^{p_s(x)} + |u_s(\eta)|^{q(x)}\right)dxd\eta + C_7|\Omega| \\
&\leq \frac{1}{2}\|g\|_H^2 + \max\left\{\frac{1}{p}, \tilde{c}_1\right\}C_6 + C_7|\Omega|.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_s(t)|^{p_s(x)} + |u_s(t)|^{q(x)}\right)dx \\
&\leq \frac{1}{\min\left\{\frac{1}{a}, \tilde{c}_2\right\}}\left[\frac{1}{2}\|g\|_H^2 + \max\left\{\frac{1}{p}, \tilde{c}_1\right\}C_6 + 2C_7|\Omega|\right] =: C_8
\end{aligned} \tag{12}$$

for all $t \geq T_0 + 1$, where $C_8 = C_8(a, \|g\|_H, p, |\Omega|)$.

Now, if $t \geq T_0+1$ and $\|\nabla u_s(t)\|_{p_s(x)} \geq 1$, we obtain $\|u_s(t)\|_{X_s}^p \leq \|u_s(t)\|_{X_s}^{p_s^-} \leq \int_\Omega |\nabla u_s(t)|^{p_s(x)}dx \leq C_8$ and so

$$\|u_s(t)\|_{X_s} \leq \max\left\{C_8^{\frac{1}{p}}, 1\right\} =: r_1 \quad \forall\, t \geq T_0 + 1,\; s \in \mathbb{N}.$$

The proof is completed. □

Remark 3.5. Using (12) we also can conclude that

$$\|u_s(t)\|_{q(x)} \leq \max\{C_8^{\frac{1}{q^-}}, 1\} \quad \forall\, t \geq T_0 + 1,\ s \in \mathbb{N}.$$

As a consequence of Theorem 3.4 we have

Corollary 3.6. a) *There exists a bounded set B_1^s in X_s such that $\mathcal{A}_s \subset B_1^s$.*

b) *Let u_s be a solution of problem* (1). *Given $T_1 > 1$ there exists a positive constant r_2, independent of s, such that*

$$\|u_s(t)\|_X < r_2$$

for all $t \geq T_1$ and $s \in \mathbb{N}$.

c) $\mathcal{A} := \overline{\bigcup_{s\in\mathbb{N}} \mathcal{A}_s}$ *is a compact subset of H.*

Proposition 3.7. *Let u_s be a solution of (1) with initial value u_{0s}. If there is $C > 0$ such that $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$, then there exists a positive constant R_1 such that $\|u_s(t)\|_{X_s} \leq R_1$, for all $t \geq 0$ and $s \in \mathbb{N}$.*

Proof. As $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$, we obtain

$$\|u_{0s}\|_H \leq K\|u_{0s}\|_{X_s} \leq KC, \quad \forall\ s \in \mathbb{N}.$$

By Lemma 3.1, it follows that $\|u_s(\tau)\|_H \leq D_1$ for all $\tau \geq 0$. Thus the result follows using the same arguments as in Theorem 3.4. □

As we have that $\|u_s(\tau)\|_X \leq 2(|\Omega| + 1)\|u_s(\tau)\|_{X_s}$ for all $s \in \mathbb{N}$ it follows from Proposition 3.7 the following

Corollary 3.8. *Let u_s be a solution of* (1) *with initial value u_{0s}. If there is $C > 0$ such that $\|u_{0s}\|_{X_s} \leq C$ for all $s \in \mathbb{N}$, then there exists a positive constant $\widetilde{R_1}$ such that*

$$\|u_s(t)\|_X \leq \widetilde{R_1},$$

for all $t \geq 0$ and $s \in \mathbb{N}$.

4. Continuity with respect to the initial values and upper semicontinuity of attractors

In this section we will prove that, given $T > 0$, the solutions u_s of (1) go to the solution u of

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(x,u) = g, \quad t > 0 \\ \hfill u(0) = u_0 \in H, \end{cases} \tag{13}$$

in $C([0,T];H)$ when $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ as $s \to \infty$. After that, we will obtain the upper semicontinuity on s in H of the family of global attractors

$$\{\mathcal{A}_s \subset H; s \in \mathbb{N}\}$$

of (1) at p. In this section we additionally suppose that f satisfies

$$\|f(\cdot,u(\cdot)) - f(\cdot,v(\cdot))\|_H \leq L(B)\|u-v\|_H. \tag{14}$$

for all $u, v \in B$, where B is a bounded set of H.

Lemma 4.1. *Given $T > 0$, $M := \{u_s : s \in \mathbb{N}, u_s$ is a solution of (1) with $u_s(0) = u_{0s}, \|u_{0s}\|_{X_s} \leq C \ \forall\ s \in \mathbb{N}$ and $u_{0s} \to u_0$ in H as $s \to +\infty\}$ is relatively compact in $C([0,T];H)$.*

Proof. We observe that for each $s \in \mathbb{N}$ the function $[0,T] \ni t \mapsto \tilde{f}_s(t)(\cdot) = g(\cdot) - f(\cdot, u_s(t,\cdot)) \in H$ is in $L^1(0,T;H)$. Moreover, $\{\tilde{f}_s(t)\}_{s\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T;H)$ and consequently uniformly integrable in $L^1(0,T;H)$. Indeed, using (14) and Lemma 3.1 we obtain

$$\begin{aligned}
\int_0^T \|f(\cdot,u_s(\tau,\cdot))\|_H d\tau &\leq \int_0^T \|f(\cdot,u_s(\tau,\cdot)) - f(\cdot,0)\|_H d\tau + \int_0^T \|f(\cdot,0)\|_H d\tau \\
&\leq \int_0^T L(B)\|u_s(\tau,\cdot)\|_H d\tau + CT \\
&\leq (C + L(B)D_1)T,
\end{aligned}$$

for all $s \in \mathbb{N}$. So, $\{\tilde{f}_s(t)\}_{s\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T;H)$. Also we have $\overline{\cap_s D(\varphi_{p_s(x)})} = H$ since $X_a \subset X_s \subset X$, for all $s \in \mathbb{N}$.

With some analogous computations to the proof of Lemma 7 in [30] we can show that:

- For each $u \in \cap_s D(\varphi_{p_s(x)})$ there exists a constant $k(u) > 0$ such that $\varphi_{p_s(x)}(u) \leq k(u)$, for all $s \in \mathbb{N}$;
- Let $M(t) := \{u_s(t); u_s \in M\}$ and let $\{S^s(t)\}$ be the semigroup generated by $-\Delta_{p_s(x)}$ in H. For each $t \in (0,T]$ and $h > 0$ such that $t-h \in (0,T]$, the operator $T_h : M(t) \to H$ defined by $T_h u_s(t) = S^s(h)u_s(t-h)$ is compact.

Thus, by Theorem 3.2 in [26], we obtain that M is relatively compact in $C([0,T];H)$. □

Theorem 4.2. *For each $s \in \mathbb{N}$ let u_s be a solution of (1) with $u_s(0) = u_{0s}$. Suppose that there exists $C > 0$, independent of s, such that $\|u_{0s}\|_{X_s} \leq C$ for every $s \in \mathbb{N}$ and $u_{0s} \to u_0$ in H as $s \to \infty$. Then, for each $T > 0$, $u_s \to u$ in $C([0,T];H)$ as $s \to \infty$, where u is a solution of (13) with $u(0) = u_0 \in H$.*

Proof. By Lemma 4.1 M is relatively compact in $C([0,T];H)$. So, $\{u_s\}$ converges in $C([0,T];H)$ to a function $u:[0,T]\to H$. There exists a bounded set $B\subset H$ such that $u_{0,s},u_0\in B$. So, by Lemma 3.1, $u_s(t')\in B_1=B_H(0,D_1)$ for all $s\in\mathbb{N}$ and $t'\geq 0$. Thus, by (14) we obtain

$$\begin{aligned}\|g(\cdot)-f(\cdot,u_s(t'))\|_H &\leq \|g\|_H+\|f(\cdot,u_s(t'))\|_H\\ &\leq \|g\|_H+\|f(\cdot,u_s(t'))-f(\cdot,0)\|_H+\|f(\cdot,0)\|_H\\ &\leq \|g\|_H+L(B_1)\|u_s(t')\|_H+\|f(\cdot,0)\|_H\\ &\leq \tilde{L},\end{aligned}$$

for all $s\in\mathbb{N}$ and $t'\geq 0$, where $\tilde{L}:=\|g\|_H+L(B_1)D_1+\|f(\cdot,0)\|_H$. Considering $B:\Omega\times H\to H$ given by $B(x,u)=g(x)-f(x,u)$ we have

$$\|B\circ u_s\|_{L^2(0,T;H)}\leq \tilde{L}\sqrt{T},\quad \forall\ s\in\mathbb{N}.$$

As $L^2(0,T;H)$ is a reflexive Banach space there is $w\in L^2(0,T;H)$ such that $B\circ u_s\rightharpoonup w$ in $L^2(0,T;H)$. Since $B(u_s(t))\to B(u(t))$ in H it follows that $w=B\circ u$. Thus, $B\circ u_s\rightharpoonup B\circ u$ in $L^2(0,T;H)$.

Now observe that since $B\circ u_s\rightharpoonup B\circ u$ in $L^2(0,T;H)$ implies that

$$B\circ u_s\rightharpoonup B\circ u\quad\text{in } L^2(\tau,t;H),\quad \forall\ 0\leq\tau\leq t\leq T;$$

and $u_s\to u$ in $C([0,T];H)$ implies that $u_s\to u$ in $C([\tau,t];H)$ and consequently

$$u_s\to u\quad\text{in } L^2(\tau,t;H),\quad \forall\ 0\leq\tau\leq t\leq T;$$

then

$$\langle B\circ u_s-h,u_s-\theta\rangle_{L^2(\tau,t;H)}\to\langle B\circ u-h,u-\theta\rangle_{L^2(\tau,t;H)}$$

for all $\theta,h\in H$.

[10, Proposition 3.6] implies that

$$\frac{1}{2}\|u_s(t)-\phi\|_H^2\leq\frac{1}{2}\|u_s(\tau)-\phi\|_H^2+\int_\tau^t\langle B(u_s(t'))+\Delta_{p_s(x)}(\phi),u_s(t')-\phi\rangle dt'\tag{15}$$

for every $\phi\in D(-\Delta_{p_s(x)})$ and $0\leq\tau\leq t\leq T$.

Now, the idea is take the limit as $s\to\infty$ ($p_s\to p$) on the last inequality. Consider $\overline{\theta}\in C_0^\infty(\Omega)\subset D(-\Delta_{p_s(x)})\subset H$ (see Theorem 2.6) and let be $\overline{h}:=-\Delta_p(\overline{\theta})\in H$. We already knows from (15) that holds

$$\begin{aligned}\frac{1}{2}\parallel u_s(t)-\overline{\theta}\parallel_H^2 &\leq\frac{1}{2}\parallel u_s(\tau)-\overline{\theta}\parallel_H^2\\ &\quad+\int_\tau^t\langle B(u_s(t'))+\Delta_{p_s(x)}(\overline{\theta}),u_s(t')-\overline{\theta}\rangle dt'\\ &=\frac{1}{2}\parallel u_s(\tau)-\overline{\theta}\parallel_H^2+\int_\tau^t\langle B(u_s(t'))-\overline{h},u_s(t')-\overline{\theta}\rangle dt'\\ &\quad+\int_\tau^t\langle\overline{h}+\Delta_{p_s(x)}(\overline{\theta}),u_s(t')-\overline{\theta}\rangle dt'.\end{aligned}\tag{16}$$

Using Dominated Convergence Theorem, Mean Value Theorem and Proposition 3.7 we can show with some computations that

$$\int_\tau^t \langle \overline{h} + \Delta_{p_s(x)}(\overline{\theta}), u_s(t') - \overline{\theta}\rangle dt' \to 0 \quad \text{as } s \to +\infty.$$

So, taking the limit in (16) as $s \to \infty$, we obtain

$$\frac{1}{2}\|u(t) - \overline{\theta}\|_H^2 \leq \frac{1}{2}\|u(\tau) - \overline{\theta}\|_H^2 + \int_\tau^t \langle B(u(t')) + \Delta_p(\overline{\theta}), u(t') - \overline{\theta}\rangle dt'$$

for every $\overline{\theta} \in C_0^\infty(\Omega)$ and $0 \leq \tau \leq t \leq T$.

As $\mathcal{D}(-\Delta_p) \subset W_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p}(\Omega)}$ we can use a density argument to conclude that

$$\frac{1}{2}\|u(t) - \overline{\theta}\|_H^2 \leq \frac{1}{2}\|u(\tau) - \overline{\theta}\|_H^2 + \int_\tau^t \langle B(u(t')) + \Delta_p(\overline{\theta}), u(t') - \overline{\theta}\rangle dt'$$

for every $\overline{\theta} \in \mathcal{D}(-\Delta_p)$ and $0 \leq \tau \leq t \leq T$. Thus, Proposition 3.6 in [10] implies that u is a solution of (13). □

Using the uniform estimates and the continuity of the flows we get

Theorem 4.3. *The family of global attractors $\{\mathcal{A}_s;\ s \in \mathbb{N}\}$ associated with problem* (1) *is upper semicontinuous on s at infinity, in the topology of H.*

Remark 4.4. For the one-dimensional case ($\Omega = I = (c, d)$), if we suppose that f satisfies

$$|f(x, r_1) - f(x, r_2)| \leq L(B)|r_1 - r_2|, \quad \forall\, r_1, r_2 \in B,$$

where B is a bounded set of $\mathbb{R}$ we obtain that f satisfies (14). Indeed, by Corollary 3.8, $\left\|\frac{du_s}{dx}(t)\right\|_p = \|u_s(t)\|_X \leq \widetilde{R_1}$ for all $t \geq 0$ and then for $d \in I$ fixed and $x \in I$

$$|u_s(t, x) - u_s(t, d)| = \left|\int_d^x \frac{du_s}{dy}(t, y)dy\right| \leq \int_I \left|\frac{du_s}{dy}(t, y)\right| dy \leq \widetilde{R_1}|I|^{\frac{1}{p'}}.$$

Remark 4.5. Using Theorem 2.6 and Lemma 2.7, the results in [30, Section 3, 4] can be extended to higher dimension.

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