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Blow-Up Profiles for a Semilinear Chemotaxis System Arising in Biology

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Abstract. We consider the semilinear coupled system of parabolic-elliptic partial differential equations arising in chemotaxis involving forcing source of exponential growth type and homogeneous Dirichlet boundary conditions. The local existence and uniqueness of nonnegative classical solutions are proved. Also, a lower bound for the blow-up time if the solution blows up in finite time is derived. Moreover, the exponential decay of the associated energies are also studied. The results we obtained here essentially extend some existing results in this area.

Keywords. Chemotaxis system, classical solution, blow up, decay
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1. Introduction and motivation

There is an extensive literature on the theory of coupled systems of partial differential equations, some contributions along this line have been made, among them for instance are [10,14,15,22,23] and the references therein. In the study of the biological relevance of chemotaxis, a coupled system of partial differential equations, which describes the aggregation of certain types of bacteria, was proposed by Keller and Segel [6] (see also [7]), which reads

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v) + g(u), & x\in\Omega,\ t>0,\\ \varepsilon v_t = d_2\Delta v + f(u,v), & x\in\Omega,\ t>0,\end{cases} \tag{1}$$

where Ω, representing the capacity, is an open domain in $\mathbb{R}^n$ $(n\geq 1)$, Δ stands for the Laplacian operator, ∇ is the gradient operator, $u=u(x,t)$ denotes the

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cell density and $v = v(x,t)$ represents the oxygen concentration with (constant) diffusion rates $d_1, d_2 > 0$, χ is a positive constant linked to the sensitivity with respect to chemotaxis, and g models possible production or death of cells.

The cross-diffusive term in the first equation reflects the assumption that individual cells at least partially adapt their motion so as to prefer to migrate toward increasing oxygen concentrations.

Since the work of Keller and Segel, this type of model (1) and its variations have been investigated quite thoroughly during the past three decades. For instance, for the system (1) with $\Omega = \mathbb{R}^n$, $d_1, d_2, \chi = 1$, $g(u) = 0$ and $f(u,v) = u$, Kozono et al. [9] considered the problem whether there does exist a finite-time self-similar solution of the backward type for the case of either $\varepsilon = 0, n \geq 3$ or $\varepsilon = 1, n \geq 2$, and Sugiyama and Yahagi [19] investigated the uniqueness and continuity of weak solutions with respect to the initial data for the Keller-Segel system of degenerate type. For more contributions along this line, we refer reader to [16–18] and the references therein.

In view of the biologically meaningful question whether or not cell populations spontaneously form aggregates, some studies focused on the issue whether solutions remain bounded or blow up (see, e.g., [2,3,24]). Here we sketach some references, but not a list of all references is included. Kozono and Sugiyama in [8] showed global existence of strong solutions to the system (1) with $\Omega = \mathbb{R}^n$ $(n \geq 3)$, $d_1, d_2, \chi, \varepsilon = 1$, $g(u) = 0$ and $f(u,v) = u - \gamma v$ for small initial data in the scaling invariant class. The authors also proved the uniqueness of strong solutions as well as the decay property in $L^p(\mathbb{R}^n)$ as $t \to \infty$. In the case when $d_1, d_2 = 1$, $\varepsilon = 0$ and $f(u,v) = u - v$, the system (1), under homogeneous Neumann boundary conditions, has been studied by Tello and Winkler [21], where the authors proved the existence of global bounded classical solutions and global weak solutions, established the stability of nonzero equilibrium and obtained some multiplicity and bifurcation results. In [5], for the case when $d_1, d_2 = 1$, $\varepsilon = 0$, $g(u) = 0$ and $f(u,v) = u - v$, Jäger and Luckhaus dealt with the blow-up and global existence for the system (1) with homogeneous Dirichlet boundary condition and certain initial conditions

$$u(x,0) = u_0(x) \geq 0 \quad \text{and} \quad \int_\Omega (u_0 - 1)dx = 0, \quad x \in \Omega.$$

Sugiyama and Yahagi [20] studied the extinction, decay and blow-up for the quasi-linear Keller-Segel system of fast diffusion type.

Another interesting aspects connected to a deeper understanding of the system (1) and its variations are the question of lower/upper bounds for the time of blow-up. Here, it is worth mentioning that the chemotaxis system in [5] was investigated by Payne and Song [12], where they derived a lower bound for the time of blow-up and established some explicit criterions ensuring that the solutions remain bounded for all time and the associated energy decay exponentially in time. Moreover, from [13], one can find results on the parabolic-parabolic system (1) with $d_1 = 1,\ \varepsilon = 1,\ g(u) = 0, f(u,v) = k_1u - k_2v$, in which the authors gave a lower bound for the time of blow-up.

Let Ω be a bounded convex domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary. In the present paper, we will extend previous works on chemotaxis systems to the coupled system of elliptic-parabolic equations involving forcing source of exponential growth and homogeneous Dirichlet boundary conditions. More precisely, we are interested in studying the IBVP

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \omega u^q, & x \in \Omega,\ t > 0, \\ 0 = \Delta v + u - v, & x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0, & \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{2}$$

where $\chi > 0,\ \omega \geq 0, q \geq 1$ are constants. It is the theme of this paper to investigate the lower bound for the blow-up time and decay criterions of the associated energies to the system (2). To accomplish these goals, we first prove a local existence and uniqueness result for nonnegative classical solutions. The theorems formulated are extensions of many previous results on the Keller-Segel system of partial differential equations.

When using homogeneous Dirichlet boundary conditions for the chemotaxis system, one prescribes the disappearance of individual cells and oxygen near the boundary.

From biological point of view, solutions to the system (2), representing densities, must satisfy

$$u \geq 0 \quad \text{and} \quad v \geq 0.$$

Thus it is reasonable to require throughout that the initial data $u_0 \in C^0(\overline{\Omega})$ be nonnegative.

2. Local existence of nonnegative solutions

We begin our study of the system (2) with a result of local existence and uniqueness of nonnegative classical solutions.

In view of $\Delta v = v - u$, one has

$$\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \Delta v = \nabla u \cdot \nabla v + u(v - u),$$

which admits us to rewrite the system (2) as equivalent one

$$\begin{cases} u_t - \Delta u + \chi \nabla u \cdot \nabla v + \chi u(v - u) - \omega u^q = 0, & x \in \Omega,\ t > 0, \\ \Delta v + u - v = 0, & x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, & \\ v|_{\partial\Omega} = 0, & \\ u|_{t=0} = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

Note that no cross-diffusive term in the first equation of (3) exists.

The main result in this section is given in the following theorem.

Theorem 2.1. *Let $q \geq 1$ and $0 < \alpha < 1$. Assume that an initial datum $u_0 \in C^{0,\alpha}(\overline{\Omega})$ is given. Then there exists an unique classical solution (u, v) to the system (2) defined on the maximal interval $[0, t_{max})$ of existence, for which one has*

$$u \in C^{2+\alpha, 1+\frac{\alpha}{2}}_{loc}(\overline{\Omega} \times [0, t_{max})), \quad v \in C^{2+\alpha, \frac{\alpha}{2}}_{loc}(\overline{\Omega} \times [0, t_{max})).$$

Moreover, if $t_{max} < +\infty$, then

$$\lim_{t \to t_{max}} \|u(t)\|_{L^\infty(\Omega)} = +\infty. \tag{4}$$

Proof. Fix $u_0 \in C^{0,\alpha}(\overline{\Omega})$. Assume that $[0, T^*)$ is the maximal interval of existence of nonnegative solution h for the IVP in the form

$$\begin{cases} h_t = \chi h^2 + \omega h^q, \quad t > 0, \\ h(0) = \|u_0\|_{L^\infty(\Omega)} \geq 0. \end{cases}$$

Let the set Ω_{u_0} be defined by

$$\Omega_{u_0} = \{u \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T]); 0 \leq u(x,t) \leq h(t), \quad (x,t) \in \Omega \times [0, T]\},$$

where $T = \frac{1}{2} T^*$.

For each $u \in \Omega_{u_0}$, let us consider the following IBVP

$$\begin{cases} w_t - \Delta w + \chi \nabla w \cdot \nabla v + \chi w(v - u) - \omega u^q = 0, \quad (x,t) \in \Omega \times (0, T], \\ w|_{\partial\Omega} = 0, \\ w|_{t=0} = u_0, \end{cases} \tag{5}$$

where v is the unique solution to the BVP of form

$$\begin{cases} -\Delta v + v = u, \quad x \in \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

In view of $u \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$, it follows that $v \in C^{2+\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$ and hence $\nabla v \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$, which together with the observation $u^q \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$ yields, from a basic theory of linear parabolic equations, that there exists an unique classical solution $w \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$ to (5), In particular, one has $w \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$.

According to the strong maximum principle of elliptic equations, we obtain $v \geq 0$ in $\overline{\Omega} \times [0, T]$. Also, since $w \equiv 0$ is a lower solution of (5), the comparison principle of parabolic equations implies that $w \geq 0$ in $\overline{\Omega} \times [0, T]$.

Furthermore, noting $u \in \Omega_{u_0}$, $h \in C^1[0,T]$ and $v \geq 0$ in $\Omega \times [0,T]$, a direct calculation yields

$$\begin{aligned}
&(w-h)_t - \Delta(w-h) + \chi\nabla(w-h)\cdot\nabla v + \chi(w-h)(v-u) \\
&= w_t - h_t - \Delta w + \chi\nabla w\cdot\nabla v + \chi w(v-u) - \chi h(v-u) \\
&= -\chi h^2 - \omega h^q - \chi h(v-u) + \omega u^q \\
&\leq \chi h(-h-v+u) \\
&\leq 0.
\end{aligned}$$

From this together with $w-h\leq 0$ on $\partial\Omega\times(0,T]$ and in $\Omega\times\{0\}$ we see, again by the comparison principle of parabolic equations, that $w\leq h$ in $\overline{\Omega}\times[0,T]$. Therefore, we conclude $w\in\Omega_{u_0}$.

We seek for solutions in Ω_{u_0}. To this end, based on the arguments above we define a mapping as

$$\Gamma : \Omega_{u_0} \to \Omega_{u_0} \quad \text{with } \Gamma(u) = w,$$

where w is the unique solution to (5). It is clear that Γ is well defined. And then following a standard argument (see, e.g., [1]), we see that Γ is continuous. Moreover, since the imbedding $C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[0,T]) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[0,T])$ is compact, Γ is compact. Therefore, applying the Schauder fixed point theorem yields that Γ admits at least one fixed point in Ω_{u_0}, which in fact gives solutions to (2).

Next, we prove the uniqueness of the solutions. Assume that both (u_1,v_1) and (u_2,v_2) are the solutions of (2). As above, it suffices to show $\varphi:=u_1-u_2\equiv 0$. Note that

$$\begin{aligned}
\varphi_t = \Delta\varphi - \chi\nabla\varphi\cdot\nabla v_1 - \chi\nabla u_2\cdot\nabla(v_1-v_2) - \chi\varphi v_1 \\
- \chi u_2(v_1-v_2) + \chi(u_1^2-u_2^2) + \omega(u_1^q-u_2^q).
\end{aligned} \tag{6}$$

Multiplying both sides of (6) by φ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varphi^2 dx + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 dx \leq M\int_\Omega \varphi^2 dx,$$

where M is a computable constant. Here, we have tacitly used the following estimates:

$$\begin{aligned}
|\chi(\nabla\varphi\cdot\nabla v_1)\varphi| &\leq \frac{1}{2}|\nabla\varphi|^2 + \frac{\chi^2}{2}|\nabla v_1|^2\varphi^2, \\
|\chi(\nabla u_2\cdot\nabla(v_1-v_2))\varphi| &\leq \frac{\chi}{2}|\nabla(v_1-v_2)|^2 + \frac{\chi}{2}|\nabla u_2|^2\varphi^2,
\end{aligned}$$

and

$$\int_\Omega |\nabla(v_1-v_2)|^2 dx + \int_\Omega (v_1-v_2)^2 dx \leq \int_\Omega \varphi^2 dx.$$

Hence, an application of Gronwall's inequality shows that $\varphi\equiv 0$, as desired.

Finally, a standard procedure enables us to obtain (4) (cf., e.g., [1,4]). This completes the proof of the theorem. □

3. Lower bound for blow-up time and decay criteria

Let $u_0 \in C^{0,\alpha}(\overline{\Omega})$ and $q \geq 1$. As shown in Theorem 2.1, there exists an unique classical solution (u, v) to the system (2) defined on the maximal interval $[0, t_{max})$ of existence. In the sequel, we let $t_{max} < +\infty$. Then from (4) it follows that there exists $t^* > 0$ and $p \geq 1$ such that

$$\lim_{t \to t^*} \|u(t)\|_{L^{2p}(\Omega)} = +\infty, \tag{7}$$

in which we may assume without loss of generality that $p \geq \frac{3q}{2}$.

Here, our objective is to obtain a lower bound for the blow-up time t^*. For the sake of convenience, we write

$$A_0 = \frac{\chi(p-1)(2p-1)}{p}, \qquad A_1 = \frac{2p\omega|\Omega|^{\frac{2p-3q+3}{2p}}}{3},$$

$$A_2 = \frac{3^{\frac{3}{4}}}{2\rho^{\frac{3}{2}}}\left(\frac{\chi(2p-1)}{p} + \frac{4p\omega}{3}\right), \quad B_1 = \frac{3^{\frac{3}{4}}\chi(2p-1)}{2\rho^{\frac{3}{2}}p}, \quad d = \max_{x \in \overline{\Omega}} |x|^2,$$

where $\rho = \min_{\partial\Omega} x \cdot \nu > 0$. Among that, ν denotes the unit normal vector directed outward on $\partial\Omega$ and $|\Omega|$ denotes the volume of Ω.

Theorem 3.1. *Let $q > 1$ and $\Omega \subset \mathbb{R}^3$. Suppose in addition that (u, v) is the unique nonnegative classical solution of the system* (2) *defined on the maximal interval $[0, t_{max})$ of existence. If $t_{max} < +\infty$, then* (7) *implies that*

$$t^* \geq \int_{\phi(0)}^{+\infty} \frac{d\xi}{A_0\xi + A_1\xi^{\frac{2p-3}{2p}} + A_2\xi^{\frac{3}{2}} + A_3\xi^3}, \tag{8}$$

where A_3 depending on ρ, p, d is a nonnegative constant and

$$\phi(0) = \int_\Omega u_0(x)^{2p} dx.$$

Proof. Let

$$\phi(t) = \int_\Omega u^{2p} dx, \quad t \in [0, t^*). \tag{9}$$

We employ Hölder's inequality and Young's inequality to estimate

$$\int_\Omega u^{2p+1} dx \leq \left(\int_\Omega u^{2p} dx\right)^{\frac{p-1}{p}} \left(\int_\Omega u^{3p} dx\right)^{\frac{1}{p}} \leq \frac{p-1}{p}\phi + \frac{1}{p}\int_\Omega u^{3p} dx, \tag{10}$$

and

$$\int_\Omega u^{2p+q-1} dx \leq \frac{2}{3}\int_\Omega u^{3p} dx + \frac{1}{3}\int_\Omega u^{3q-3} dx \leq \frac{2}{3}\int_\Omega u^{3p} dx + \frac{1}{3}|\Omega|^{\frac{2p-3q+3}{2p}}\phi^{\frac{3q-3}{2p}}. \tag{11}$$

Then differentiating (9) and using the fact $u|_{\partial\Omega} = 0$ we have

$$\begin{aligned}\phi'(t) &= 2p\int_\Omega u^{2p-1}u_t dx\\ &= 2p\int_\Omega u^{2p-1}(\Delta u - \chi\nabla u\cdot\nabla v - \chi u\Delta v + \omega u^q)dx\\ &= -2p(2p-1)\int_\Omega u^{2p-2}|\nabla u|^2dx - \chi(2p-1)\int_\Omega u^{2p}\Delta v dx + 2p\omega\int_\Omega u^{2p+q-1}dx\\ &= -\frac{2(2p-1)}{p}\int_\Omega|\nabla(u^p)|^2dx + \chi(2p-1)\int_\Omega u^{2p+1}dx - \chi(2p-1)\int_\Omega u^{2p}v dx\\ &\quad + 2p\,\omega\int_\Omega u^{2p+q-1}dx.\end{aligned}$$

Therefore, by (10), (11) and the fact $u, v \geq 0$ one has

$$\phi'(t) \leq A_0\phi - \tfrac{2(2p-1)}{p}\textstyle\int_\Omega|\nabla(u^p)|^2dx + C_1\int_\Omega u^{3p}dx + A_1\phi^{\frac{3q-3}{2p}}, \tag{12}$$

where $C_1 = \frac{\chi(2p-1)}{p} + \frac{4p\omega}{3}$.

Next, noticing that Ω is a convex domain, a similar argument with that in [11, Estimate (2.16)] enables to show that the following estimate

$$\int_\Omega u^{3p}dx \leq \frac{1}{3^{\frac{3}{4}}}\left\{\frac{3}{2\rho}\int_\Omega u^{2p}dx + \left(\frac{d^{\frac{1}{2}}}{\rho}+1\right)\left(\int_\Omega u^{2p}dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla(u^p)|^2dx\right)^{\frac{1}{2}}\right\}^{\frac{3}{2}} \tag{13}$$

holds. From this and the inequality

$$(a+b)^{\frac{n+1}{n}} \leq 2^{\frac{1}{n}}\left(a^{\frac{n+1}{n}} + b^{\frac{n+1}{n}}\right) \quad \text{for } a, b > 0, \tag{14}$$

it follows that

$$\begin{aligned}\int_\Omega u^{3p}dx &\leq \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\left\{\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\phi^{\frac{3}{2}} + \left(\frac{d^{\frac{1}{2}}}{\rho}+1\right)^{\frac{3}{2}}\phi^{\frac{3}{4}}\left(\int_\Omega|\nabla(u^p)|^2dx\right)^{\frac{3}{4}}\right\}\\ &\leq \frac{3^{\frac{3}{4}}}{2\rho^{\frac{3}{2}}}\phi^{\frac{3}{2}} + \frac{1}{2^{\frac{3}{2}}3^{\frac{3}{4}}\varepsilon^3}\left(\frac{d^{\frac{1}{2}}}{\rho}+1\right)^{\frac{3}{2}}\phi^3 + \frac{3^{\frac{1}{4}}\varepsilon}{2^{\frac{3}{2}}}\left(\frac{d^{\frac{1}{2}}}{\rho}+1\right)^{\frac{3}{2}}\int_\Omega|\nabla(u^p)|^2dx.\end{aligned} \tag{15}$$

Here we also tacitly used Hölder's inequality and Young's inequality with $\varepsilon > 0$. Inserting this estimate into (12), it follows that

$$\phi'(t) \leq \left(-\frac{2(2p-1)}{p} + C_2\right)\int_\Omega|\nabla(u^p)|^2dx + A_0\phi + A_1\phi^{\frac{2p-3}{2p}} + A_2\phi^{\frac{3}{2}} + A_3\phi^3, \tag{16}$$

where

$$A_3 = \frac{1}{2^{\frac{3}{2}} 3^{\frac{3}{4}} \varepsilon^3} \left(\frac{d^{\frac{1}{2}}}{\rho} + 1 \right)^{\frac{3}{2}} \left(\frac{\chi(2p-1)}{p} + \frac{4p\omega}{3} \right),$$

$$C_2 = \frac{3^{\frac{1}{4}} \varepsilon}{2^{\frac{3}{2}}} \left(\frac{d^{\frac{1}{2}}}{\rho} + 1 \right)^{\frac{3}{2}} \left(\frac{\chi(2p-1)}{p} + \frac{4p\omega}{3} \right).$$

To simplify the right side of (16), we choose an appropriate ε such that $-\frac{2(2p-1)}{p} + C_2 = 0$. Then we can estimate (16) as

$$\phi'(t) \leq A_0 \phi + A_1 \phi^{\frac{2p-3}{2p}} + A_2 \phi^{\frac{3}{2}} + A_3 \phi^3. \tag{17}$$

Now, integrating (17) over $(0, t)$, we have

$$t \geq \int_{\phi(0)}^{\phi(t)} \frac{d\xi}{A_0 \xi + A_1 \xi^{\frac{2p-3}{2p}} + A_2 \xi^{\frac{3}{2}} + A_3 \xi^3},$$

which proves that the assertion (8) remains true. □

Theorem 3.2. *Let the hypotheses in Theorem* 3.1 *hold except for $q > 1$ to be replaced by $q = 1$. If $t_{max} < +\infty$, then* (7) *implies that*

$$t^* \geq \int_{\phi(0)}^{+\infty} \frac{d\xi}{(2p\omega + A_0)\xi + B_1 \xi^{\frac{3}{2}} + B_2 \xi^3}, \tag{18}$$

where B_2 depending on ρ, p, d is a nonnegative constant and

$$\phi(0) = \int_\Omega u_0(x)^{2p} dx.$$

Proof. Assume that the function ϕ is defined the same as in Theorem 3.1. Following from the same idea as in the proof of Theorem 3.1, one obtain the following estimate

$$\phi'(t) \leq -\frac{2(2p-1)}{p} \int_\Omega |\nabla(u^p)|^2 dx + \frac{\chi(2p-1)}{p} \int_\Omega u^{3p} dx + (2p\omega + A_0)\phi,$$

and hence

$$\phi'(t) \leq \left(C_3 - \frac{2(2p-1)}{p} \right) \int_\Omega |\nabla(u^p)|^2 dx + (2p\omega + A_0)\phi + B_1 \phi^{\frac{3}{2}} + B_2 \phi^3$$

by using (13), (14) and (15) with ε replaced by $\delta > 0$, where

$$B_2 = \frac{\chi(2p-1)}{2^{\frac{3}{2}} 3^{\frac{3}{4}} \delta^3 p} \left(\frac{d^{\frac{1}{2}}}{\rho} + 1 \right)^{\frac{3}{2}}, \quad C_3 = \frac{3^{\frac{1}{4}} \chi \delta (2p-1)}{2^{\frac{3}{2}} p} \left(\frac{d^{\frac{1}{2}}}{\rho} + 1 \right)^{\frac{3}{2}}.$$

Now, choosing an appropriate δ such that $C_3 - \frac{2(2p-1)}{p} = 0$, we get

$$\phi'(t) \leq (2p\omega + A_0)\phi + B_1\phi^{\frac{3}{2}} + B_2\phi^3. \tag{19}$$

Integrating (19) over $(0, t)$, we have

$$t \geq \int_{\phi(0)}^{\phi(t)} \frac{d\xi}{(2p\omega + A_0)\xi + B_1\xi^{\frac{3}{2}} + B_2\xi^3},$$

which proves the desired result. □

Remark 3.3. For the case when $\Omega \subset \mathbb{R}^2$, we infer, by a similar argument with that in [13, Estimate (3.7)], that the following estimate

$$\int_\Omega u^3 dx \leq \frac{\sqrt{2}}{2\rho}\left(\int_\Omega u^2 dx\right)^{\frac{3}{2}} + \frac{\sqrt{2}}{2}\left(1 + \frac{d^{\frac{1}{2}}}{\rho}\right)\int_\Omega u^2 dx \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$$

remains true. Then noticing this and following from the same idea as those in Theorems 3.1 and 3.2, we can obtain the characterization of lower bound of the blow-up time for the system (2) when $\Omega \subset \mathbb{R}^2$.

Next, we focus on the exponential decay of the associated energies for the system (2). We only consider the case when $\Omega \subset \mathbb{R}^3$, the case when $\Omega \subset \mathbb{R}^2$ being completely similar.

Write, $p \geq 1$,

$$D_1 = \frac{2(2p-1)}{p}, \quad D_2 = \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\left(\frac{d^{\frac{1}{2}}}{\rho} + 1\right)^{\frac{3}{2}}\left(\frac{\chi(2p-1)}{p} + 2p\omega\right),$$
$$D_3 = \frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\left(\frac{3}{2\rho}\right)^{\frac{3}{2}}\left(\frac{\chi(2p-1)}{p} + 2p\omega\right).$$

To state our main result, we need the following condition:

$$-D_1\lambda_1 + D_2\lambda_1^{\frac{3}{4}}\psi(0)^{\frac{1}{2}} + D_3\psi(0)^{\frac{1}{2}} + A_0 < 0, \tag{20}$$

where $\psi(0) = \int_\Omega u_0(x)^{2p} dx$ and λ_1 is the first eigenvalue for the BVP as follows:

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \varphi|_{\partial\Omega} = 0 & \\ \varphi > 0, & x \in \Omega. \end{cases}$$

Remark 3.4. Let us note that upon an appropriate choices of $\chi, \omega, \psi(0)$, one can obtain condition (20).

We are now ready to state our main result about the decay of energies for the system (2).

Theorem 3.5. *Let* $q \geq 2$ *and* $p = q - 1$. *Suppose in addition that* $\psi(0) > 0$ *and the condition* (20) *is satisfied. Then the solution decays exponentially to zero in* $L^{2p}(\Omega)$.

Proof. Let (u, v) be the unique solution of (2). We put

$$\psi(t) = \int_\Omega u^{2p} dx, \quad \eta(t) = \int_\Omega v^{2p} dx.$$

Following from the same idea as in the proof of Theorem 3.1, we have

$$\begin{aligned}\psi'(t) &\leq -\frac{2(2p-1)}{p}\int_\Omega |\nabla(u^p)|^2 dx + \left(\frac{\chi(2p-1)}{p} + 2p\omega\right)\int_\Omega u^{3p} dx + A_0\psi \\ &\leq \left(\int_\Omega |\nabla(u^p)|^2 dx\right)^{\frac{3}{4}}\left(-D_1\left(\int_\Omega |\nabla(u^p)|^2 dx\right)^{\frac{1}{4}} + D_2\psi^{\frac{3}{4}}\right) + D_3\psi^{\frac{3}{2}} + A_0\psi.\end{aligned}$$

Inserting the result $\int_\Omega |\nabla(u^p)|^2 dx \geq \lambda_1 \int_\Omega u^{2p} dx$, we obtain

$$\psi'(t) \leq \left(\int_\Omega |\nabla(u^p)|^2 dx\right)^{\frac{3}{4}}\left(-D_1\lambda_1^{\frac{1}{4}}\psi(t)^{\frac{1}{4}} + D_2\psi(t)^{\frac{3}{4}}\right) + D_3\psi(t)^{\frac{3}{2}} + A_0\psi(t). \tag{21}$$

We claim that

$$\psi'(t) < 0. \tag{22}$$

Indeed, from (20) and (21) it is easy to see that $\psi'_+(0) < 0$. Hence, there exists $t_1 > 0$ for which $\psi(t)$ is strictly decreasing in $(0, t_1)$. And then, noticing $\psi(t_1) < \psi(0)$ and $-D_1\lambda_1 + D_2\lambda_1^{\frac{3}{4}}\psi(0)^{\frac{1}{2}} + D_3\psi(0)^{\frac{1}{2}} + A_0 < 0$, we have

$$\psi(t_1)\left(-D_1\lambda_1 + D_2\lambda_1^{\frac{3}{4}}\psi(t_1)^{\frac{1}{2}} + D_3\psi(t_1)^{\frac{1}{2}} + A_0\right) < 0,$$

which implies that $\psi'(t_1) < 0$. The above procedure may be repeated indefinitely up to obtain desired result.

Moreover, from the arguments above, we see

$$\psi'(t) \leq \psi(t)\left(-D_1\lambda_1 + D_2\lambda_1^{\frac{3}{4}}\psi(t)^{\frac{1}{2}} + D_3\psi(t)^{\frac{1}{2}} + A_0\right).$$

At the same time, by (22) and (20) there exits a positive constant α such that

$$D_2\lambda_1^{\frac{3}{4}}\psi(t)^{\frac{1}{2}} + D_3\psi(t)^{\frac{1}{2}} < D_2\lambda_1^{\frac{3}{4}}\psi(0)^{\frac{1}{2}} + D_3\psi(0)^{\frac{1}{2}} < D_1\lambda_1 - A_0 - \alpha.$$

Thus, one has $\psi'(t) \leq -\alpha\psi(t)$, which yields

$$\psi(t) \leq \psi(0)\exp(-\alpha t). \tag{23}$$

This proves that u decays exponentially to zero in $L^{2p}(\Omega)$. Next, to study the decay behavior of $\eta(t)$, we multiply both sides of the second equation in (2) by v^{2p-1} and then integrate over Ω to obtain

$$\int_\Omega uv^{2p-1}dx = \int_\Omega v^{2p}dx + \frac{2p-1}{p^2}\int_\Omega |\nabla(v^p)|^2 dx.$$

So, by Hölder's inequality,

$$\frac{1}{2p}\int_\Omega v^{2p}dx + \frac{2p-1}{p^2}\int_\Omega |\nabla(v^p)|^2 dx \le \frac{1}{2p}\int_\Omega u^{2p}dx,$$

which together with (23) yields

$$\eta(t) \le \psi(t) \le \psi(0)\exp(-\alpha t),$$

as desired. This completes the proof. □

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