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Asymptotic Behavior of Solutions for the Time-Delayed Kuramoto-Sivashinsky Equation

Chaosheng Zhu

Abstract. In this paper, we investigate the asymptotic behavior of the solutions for the Kuramoto-Sivashinsky equation with a time delay. We prove the global existence of solutions and energy decay. By using the Liapunov function method, we shall show that the solution is exponentially decay if the delay parameter τ is sufficiently small.

Keywords. Kuramoto-Sivashinsky equation, time-delay, exponential decay, Liapunov function

Mathematics Subject Classification (2010). Primary 35R10, secondary 35B35, 35Q53

1. Introduction

In this paper, we will investigate the asymptotic behavior of the solutions of the time-delayed Kuramoto-Sivashinsky equation

$$u_t(x,t) + u_{xx}(x,t) + u_{xxxx}(x,t) + u(x,t-\tau)u_x(x,t) + u(x,t) = 0, \quad (1)$$

$$u(x,t) = u(x+1,t), \quad (2)$$

$$u(x,s) = u_0(x,s), \quad (3)$$

where $x \in \mathbb{R}^1$, $t > 0$, $-\tau \le s \le 0$.

The Kuramoto-Sivashinsky equation has been independently derived in the context of several extended physical systems driven far from equilibrium by intrinsic instabilities, including instabilities of dissipative trapped ion modes in plasmas [4, 9], instabilities in laminar flame fronts [16], phase dynamics in reaction-diffusion systems [8], and fluctuations in fluid films on inclines [17]. Indeed, equation (1) generically describes the dynamics near long-wave-length primary instabilities in the presence of appropriate (translational, parity and

C. S. Zhu: School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China; zcs@swu.edu.cn

Galilean) symmetries [13]. It is worth noting that there has been the existing literature about delay reaction-diffusion equations [6,12,14], on which our work is based.

There is an extensive literature on the study of equation (1) without delay (see, e.g., [3,5,7,18,20] and the references therein). But few of the equations involving delay. Recently, in [2] authors discuss novel methods of classification and prediction of spatio-temporal dynamics in extended systems. They tested these methods on simulated data for the delay Kuramoto-Sivashinsky equation that describes unstable flame front propagation in uniform mixtures. To explain our motivation of introducing a time delay into Kuramoto-Sivashinsky equation, we consider the rate of change of u, which we denote by $\frac{Du}{Dt}$, is

$$\begin{aligned}\frac{Du}{Dt} &= \frac{d}{dt}u[x(t),t] \\ &= \frac{\partial}{\partial t}u(x,t) + \frac{dx(t)}{dt}\frac{\partial}{\partial x}u(x,t) \\ &= \frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t),\end{aligned}$$

where $x(t)$ is understood to change with time at $u = \frac{dx}{dt}$. However, we might have a delay τ to u. In this case the rate of change of u with the delay τ should be

$$\begin{aligned}\frac{Du}{Dt} &= \frac{d}{dt}u[x(t-\tau),t] \\ &= \frac{\partial}{\partial t}u(x,t) + \frac{dx(t-\tau)}{dt}\frac{\partial}{\partial x}u(x,t) \\ &= \frac{\partial}{\partial t}u(x,t) + u(x,t-\tau)\frac{\partial}{\partial x}u(x,t).\end{aligned}$$

This clearly shows how we obtain the time-delay term $u(x,t-\tau)u_x(x,t)$ in Kuramoto-Sivashinsky equation (1). Here, by using the Lyapunov function method, we shall show that the solution of problems (1)–(3) is exponentially decay if the delay parameter τ is sufficiently small.

We now introduce notation used throughout the paper. $H^s(0,1)$ denotes the usual Sobolev space (see [1,10]) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$. The norm on $L^2(0,1)$ is denoted by $\|\cdot\|$. Let X be a Banach space and $a < b$. We denote by $C^n([a,b];X)$ the space of n times continuously differentiable functions defined on $[a,b]$ with values in X with the supremum norm and we write $C([a,b];X)$ for $C^0([a,b];X)$, that is $u(x,t) \in C^n([a,b];X)$ if and only if $\max_{t\in[a,b]} \|\frac{\partial^n u(x,t)}{\partial t^n}\|_X < +\infty$.

The main result of this paper is stated as follows.

Theorem 1.1. *For any initial condition* $u_0 = u_0(x,s) \in C([-\tau,0], H_0^2(0,1))$, *problem* (1)–(3) *has a unique global mild solution* u *on* $[-\tau,\infty)$ *with*

$$u \in C([-\tau,+\infty), H_0^2(0,1)).$$

Theorem 1.2. *For any initial condition* $u_0 = u_0(x,s) \in C([-\tau,0], H_0^2(0,1))$, *there are* τ_0, ω, $K > 0$ *such that, for* $\tau < \tau_0$, *the solution of* (1)–(3) *satisfies*

$$\|u_{xx}(t)\|^2 \le \frac{K^2}{4}\exp\{-\omega t\}, \quad t \ge 0. \tag{4}$$

Remark 1.3. By Theorem 1.1–1.2, we find that $\|u_{xx}(t)\|^2$ is bounded when $t \in [-\tau,0]$.

This paper is organized as follows. In next section, we prove the existence of the solution. In Section 3, we show that the solution is exponentially decay by using the Lyapunov function method.

2. Existence of the solutions

We now briefly show that problem (1)–(3) is well posed.

Proof of Theorem 1.1. By standard methods as [15, Chapter 6: Theorem 1.4], it is easy to prove that for every initial value $u_0 = u_0(x,s) \in C([-\tau,0], H_0^2(0,1))$, there exists a $T = T(u_0) > 0$ such that problem (1)–(3) has a unique mild solution u on $[-\tau,T]$ with

$$u(x,t) \in C([-\tau,T], H_0^2(0,1)).$$

Furthermore, for any $\tau > 0$, the solution of (1)–(3) does not blow up in finite time. Indeed, integrating by parts, we obtain for $0 \le t \le \tau$,

$$\begin{aligned}
\frac{d}{dt}\|u_{xx}(t)\|^2 &= 2\int_0^1 u_{xxxx}(t)u_t(t)dx \\
&= 2\int_0^1 u_{xxxx}(t)\Big(-u_{xx}(t) - u_{xxxx}(t) - u(t-\tau)u_x(t) - u(t)\Big)dx \\
&= -2\int_0^1 u_{xxxx}(t)u_{xx}dx - 2\|u_{xxxx}(t)\|^2 \\
&\quad - 2\int_0^1 u_{xxxx}(t)u(t-\tau)u_x(t)dx - 2\int_0^1 u_{xxxx}(t)u(t)dx.
\end{aligned}$$

By Young inequality and

$$|u_x(x,t)| \le \|u_{xx}(x,t)\|, \quad 0 \le x \le 1,$$

we get

$$\begin{aligned}
&\frac{d}{dt}\|u_{xx}(t)\|^2 \\
&\leq 2\int_0^1 |u_{xxxx}(t)||u_{xx}|dx - 2\|u_{xxxx}(t)\|^2 \\
&\quad + 2\|u_0\|_{C([-\tau,0],H_0^2(0,1))}\int_0^1 |u_{xxxx}(t)||u_x(t)|dx + 2\int_0^1 |u_{xxxx}(t)||u(t)|dx \\
&\leq \frac{1}{2}\|u_{xxxx}(t)\|^2 + 2\|u_{xx}(t)\|^2 - 2\|u_{xxxx}(t)\|^2 \\
&\quad + \|u_0\|^2_{C([-\tau,0],H_0^2(0,1))}\|u_x(t)\|^2 + \|u_{xxxx}(t)\|^2 + \frac{1}{2}\|u_{xxxx}(t)\|^2 + 2\|u(t)\|^2 \\
&= 2\|u_{xx}(t)\|^2 + \|u_0\|^2_{C([-\tau,0],H_0^2(0,1))}\|u_x(t)\|^2 + 2\|u(t)\|^2 \\
&\leq \left(4 + \|u_0\|^2_{C([-\tau,0],H_0^2(0,1))}\right)\|u_{xx}(t)\|^2,
\end{aligned}$$

which implies that

$$\|u_{xx}(t)\|^2 \leq M\left(\|u_0\|_{C([-\tau,0],H_0^2(0,1))}\right),$$

where $M(\cdot)$ is a positive constant depending on $\|u_0\|_{C([-\tau,0],H_0^2(0,1))}$. Repeating the above procedure, we can prove that for $n\tau \leq t \leq (n+1)\tau$, $(n = 1, 2, \ldots)$

$$\|u_{xx}(t)\|^2 \leq M(n, \|u_0\|_{C([-\tau,0],H_0^2(0,1))})$$

In summary, we have proved the Theorem 1.1. □

3. Exponential decay estimates

To prove that our main result about the exponential stability, we introduce the following notations. For a given initial condition
$u_0 = u_0(x,s) \in C([-\tau,0], H_0^2(0,1))$, denote

$$K = \sup_{-\tau\leq s\leq 0} \|u_{0xx}(s)\| + \sqrt{8\Sigma},$$

where

$$\Sigma = \left(2\|u_0(x,0)\|^2 + \|u_{0xx}(x,0)\|^2\right)\exp\left\{e^{\omega t}\int_{-\tau}^0 \|u_{0x}(s)\|^2 ds + \frac{2e^{\omega\tau}}{\omega}\|u_0(x,0)\|^2\right\}.$$

Set

$$\sigma = \sup\left\{\delta > 0:\ \Sigma \leq \frac{K^2}{4},\ 0 \leq \tau \leq \delta\right\},$$

and let τ_0 small enough, such that for any τ, $0 \leq \tau < \tau_0 \leq \sigma$,

$$\omega = 1 - 2\sqrt{\tau(K^2 + 8K^2\tau + 2K^4\tau)} > 0. \tag{5}$$

Lemma 3.1 ([11]). *Let g, h and y be three positive and integrable functions on (t_0, T) such that y' is integrable on (t_0, T). Assume that*

$$\frac{dy}{dt} \leq gy + h, \quad \text{for } t_0 \leq t \leq T,$$
$$\int_{t_0}^{T} g(s)ds \leq C_1, \quad \int_{t_0}^{T} e^{\delta s} h(s)ds \leq C_2, \quad \int_{t_0}^{T} e^{\delta s} y(s)ds \leq C_3,$$

where δ, C_1, C_2 and C_3 are positive constants. Then

$$y(t) \leq (C_2 + \delta C_3 + y(t_0))e^{C_1} e^{\delta(t_0 - t)}, \quad \text{for } t_0 \leq t \leq T.$$

Proof of Theorem 1.2. Let

$$T_0 = \sup \left\{ \delta : \ \|u_{xx}(t)\|^2 \leq K^2, \ 0 \leq t \leq \delta \right\}.$$

Since

$$\|u_{xx}(0)\|^2 \leq K^2,$$

and $\|u_{xx}(t)\|$ is continuous, we have $T_0 > 0$. We shall prove that $T_0 = +\infty$. For this, we argue by *contradiction.* If $T_0 < +\infty$, then we have

$$\|u_{xx}(t)\|^2 \leq K^2, \quad \forall -\tau \leq t \leq T_0$$

and

$$\|u_{xx}(T_0)\|^2 = K^2. \tag{6}$$

Multiplying (1) by u, then integrating on $(0,1)$ with respect to x, we obtain

$$\frac{d}{dt}\|u(t)\|^2 + 2\|u_{xx}(t)\|^2 + 2\|u(t)\|^2 = -2\int_0^1 u_{xx}(t)u(t)dx - 2\int_0^1 u(t)u(t-\tau)u_x(t)dx.$$

By Young inequality and $\int_0^1 u^2(t)u_x(t)dx = 0$, we have

$$\frac{d}{dt}\|u(t)\|^2 + \|u_{xx}(t)\|^2 + \|u(t)\|^2 \leq \Phi, \tag{7}$$

where

$$\Phi = -2\int_0^1 u(t)[u(t-\tau) - u(t)]u_x(t)dx.$$

We now majorize Φ in the right hand side of (7). Firstly, since

$$|u(x,t)| \leq \|u_x(x,t)\|, \quad 0 \leq x \leq 1,$$

we have

$$\begin{aligned}
\Phi &\le 2\int_0^1 |u(t-\tau)-u(t)||u(t)||u_x(t)|dx \\
&\le 2\int_0^1 |u(t-\tau)-u(t)|\|u_x(t)\||u_x(t)|dx \\
&\le 2\|u_x(t)\|\int_0^1 |u(t-\tau)-u(t)||u_x(t)|dx \\
&\le 2\|u_x(t)\|\left(\int_0^1 |u(t-\tau)-u(t)|^2dx\right)^{\frac{1}{2}}\left(\int_0^1 |u_x(t)|^2dx\right)^{\frac{1}{2}} \\
&= 2\|u_x(t)\|^2\left(\int_0^1 |u(t-\tau)-u(t)|^2dx\right)^{\frac{1}{2}} \\
&= 2\|u_x(t)\|^2\left(\int_0^1 \left|\int_{t-\tau}^t u_s(s)ds\right|^2 dx\right)^{\frac{1}{2}} \\
&\le 2\sqrt{\tau}\|u_x(t)\|^2\left(\int_0^1\int_{t-\tau}^t u_s^2(s)dsdx\right)^{\frac{1}{2}}.
\end{aligned}$$

Let

$$\Psi = \left(\int_0^1\int_{t-\tau}^t u_s^2(s)dsdx\right)^{\frac{1}{2}},$$

we have

$$\Phi \le 2\sqrt{\tau}\|u_x(t)\|^2\Psi. \tag{8}$$

We now want to estimate Ψ. To this end, multiplying (1) by u_t, then integrating on $(0,1)$ with respect to x, we obtain $2\int_0^1 u_t^2(t)dx + 2\int_0^1 u_{xx}(t)u_t(t)dx + \frac{d}{dt}\int_0^1 u_{xx}^2(t)dx + 2\int_0^1 u_t(t)u(t-\tau)u_x(t)dx + 2\int_0^1 u(t)u_t(t)dx = 0$. Integrating on $0 \le t \le T_0$, we obtain

$$\begin{aligned}
&2\int_{t-\tau}^t\int_0^1 u_s^2(s)dxds + 2\int_{t-\tau}^t\int_0^1 u_{xx}(s)u_s(s)dxds + \int_0^1 u_{xx}^2(t)dx \\
&-\int_0^1 u_{xx}^2(t-\tau)dx + 2\int_{t-\tau}^t\int_0^1 u_s(s)u(s-\tau)u_x(s)dxds + 2\int_{t-\tau}^t\int_0^1 u(s)u_s(s)dxds \\
&= 0,
\end{aligned}$$

which implies that

$$2\Psi^2 \le E + F + G + H, \tag{9}$$

where

$$E = -2\int_{t-\tau}^{t}\int_{0}^{1} u_{xx}(s)u_s(s)dxds, \qquad F = \int_0^1 u_{xx}^2(t-\tau)dx \le K^2,$$
$$G = -2\int_{t-\tau}^{t}\int_{0}^{1} u_s(s)u(s-\tau)u_x(s)dxds, \quad H = -2\int_{t-\tau}^{t}\int_{0}^{1} u(s)u_s(s)dxds.$$

We now majorize E, G and H:

$$\begin{aligned}
E &\le 2\left(\int_{t-\tau}^{t}\int_{0}^{1} u_s^2(s)dxds\right)^{\frac{1}{2}}\left(\int_{t-\tau}^{t}\int_{0}^{1} u_{xx}^2(s)dxds\right)^{\frac{1}{2}}\\
&\le \frac{1}{4}\Psi^2 + 4\int_{t-\tau}^{t}\int_{0}^{1} u_{xx}^2(s)dxds\\
&\le \frac{1}{4}\Psi^2 + 4\int_{t-\tau}^{t} K^2 ds\\
&\le \frac{1}{4}\Psi^2 + 4K^2\tau, \qquad (10)
\end{aligned}$$

$$\begin{aligned}
G &\le 2\int_{t-\tau}^{t}\int_{0}^{1} |u_s(s)||u(s-\tau)||u_x(s)|dxds\\
&\le 2\int_{t-\tau}^{t}\int_{0}^{1} \|u_x(s-\tau)\|(|u_s(s)||u_x(s)|)dxds\\
&\le 2K\int_{t-\tau}^{t}\int_{0}^{1} |u_s(s)||u_x(s)|dxds\\
&\le 2K\left(\int_{t-\tau}^{t}\int_{0}^{1} u_s^2(s)dxds\right)^{\frac{1}{2}}\left(\int_{t-\tau}^{t}\int_{0}^{1} u_x^2(s)dxds\right)^{\frac{1}{2}}\\
&\le \frac{1}{2}\Psi^2 + 2K^2\int_{t-\tau}^{t}\int_{0}^{1} u_x^2(s)dxds\\
&\le \frac{1}{2}\Psi^2 + 2K^2\int_{t-\tau}^{t} K^2 ds\\
&\le \frac{1}{2}\Psi^2 + 2K^4\tau, \qquad (11)
\end{aligned}$$

$$H \le \frac{1}{4}\Psi^2 + 4\int_{t-\tau}^{t}\int_{0}^{1} u^2(s)dxds \le \frac{1}{4}\Psi^2 + 4K^2\tau. \qquad (12)$$

Thus by (9)–(12), we have

$$\Psi \le \sqrt{K^2 + 8K^2\tau + 2K^4\tau}, \quad \forall 0 \le t \le T_0. \qquad (13)$$

Then (8) and (13) implies that

$$\Phi \leq 2\|u_x(t)\|^2\sqrt{\tau(K^2+8K^2\tau+2K^4\tau)} \leq 2\|u_{xx}(t)\|^2\sqrt{\tau(K^2+8K^2\tau+2K^4\tau)}. \quad (14)$$

Thus, by (7) and (14), we obtain

$$\frac{d}{dt}\|u(t)\|^2 + \|u_{xx}(t)\|^2 \left(1 - 2\sqrt{\tau(K^2 + 8K^2\tau + 2K^4\tau)}\right) \leq 0,$$

that is $\frac{d}{dt}\|u(t)\|^2 + \omega\|u_{xx}(t)\|^2 \leq 0$, where ω is defined by (5). Furthermore,

$$\frac{d}{dt}\|u(t)\|^2 + \omega\|u_x(t)\|^2 \leq 0, \quad (15)$$

$$\frac{d}{dt}\|u(t)\|^2 + \omega\|u(t)\|^2 \leq 0.$$

Solving the above inequality gives

$$\|u(t)\|^2 \leq \|u_0(x,0)\|^2 e^{-2\omega t}, \quad 0 \leq t \leq T_0.$$

On the other hand, multiplying (15) by $e^{\omega t}$, we have

$$\frac{d}{dt}(e^{\omega t}\|u(t)\|^2) + \omega e^{\omega t}\|u_x(t)\|^2 \leq \omega e^{\omega t}\|u(t)\|^2 \leq \omega e^{-\omega t}\|u_0(x,0)\|^2.$$

Integrating the above inequality from 0 to T_0 gives

$$e^{\omega T_0}\|u(T_0)\|^2 + \omega \int_0^{T_0} e^{\omega t}\|u_x(t)\|^2 dt \leq (2 - e^{-\omega T_0})\|u_0(x,0)\|^2.$$

which implies that

$$\int_0^{T_0} e^{\omega t}\|u_x(t)\|^2 dt \leq \frac{2}{\omega}\|u_0(x,0)\|^2. \quad (16)$$

Consequently, we get

$$\begin{aligned}\int_0^{T_0} e^{\omega t}\|u_x(t-\tau)\|^2 dt &= \int_{-\tau}^{T_0-\tau} e^{\omega(s+\tau)}\|u_x(s)\|^2 ds \quad (\text{by } s = t-\tau)\\ &= \left(\int_{-\tau}^{0} + \int_0^{T_0} + \int_{T_0}^{T_0-\tau}\right) e^{\omega(s+\tau)}\|u_x(s)\|^2 ds\\ &= \left(\int_{-\tau}^{0} + \int_0^{T_0} - \int_{T_0-\tau}^{T_0}\right) e^{\omega(s+\tau)}\|u_x(s)\|^2 ds\\ &\leq \left(\int_{-\tau}^{0} + \int_0^{T_0}\right) e^{\omega(s+\tau)}\|u_x(s)\|^2 ds.\end{aligned}$$

Here, by (16) $\int_0^{T_0} e^{\omega(s+\tau)}\|u_x(s)\|^2 ds = e^{\omega\tau}\int_0^{T_0} e^{\omega s}\|u_x(s)\|^2 ds \le \frac{2e^{\omega\tau}}{\omega}\|u_0(x,0)\|^2$, and $\int_{-\tau}^0 e^{\omega(s+\tau)}\|u_x(s)\|^2 ds \le \int_{-\tau}^0 e^{\omega(s+\tau)}\|u_{0x}(s)\|^2 ds \le e^{\omega\tau}\int_{-\tau}^0 \|u_{0x}(s)\|^2 ds$, hence

$$\int_0^{T_0} e^{\omega t}\|u_x(t-\tau)\|^2 dt \le e^{\omega\tau}\int_{-\tau}^0 \|u_{0x}(s)\|^2 ds + \frac{2e^{\omega\tau}}{\omega}\|u_0(x,0)\|^2.$$

Furthermore we have

$$\int_0^{T_0} \|u_x(t-\tau)\|^2 dt \le e^{\omega\tau}\int_{-\tau}^0 \|u_{0x}(s)\|^2 ds + \frac{2e^{\omega\tau}}{\omega}\|u_0(x,0)\|^2. \tag{17}$$

On the other hand, integrating by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\|u_{xx}(t)\|^2 &= -2\int_0^1 u_{xxxx}(t)u_{xx}(t)dx - 2\|u_{xxxx}(t)\|^2 \\
&\quad - 2\int_0^1 u_{xxxx}(t)u(t-\tau)u_x(t)dx - 2\|u_{xx}(t)\|^2 \\
&\le 2\int_0^1 |u_{xxxx}(t)||u_{xx}(t)|dx - 2\|u_{xxxx}(t)\|^2 \\
&\quad + 2\|u_x(t-\tau)\|\int_0^1 |u_{xxxx}(t)||u_x(t)|dx - 2\|u_{xx}(t)\|^2 \\
&\le \|u_{xxxx}(t)\|^2 + \|u_{xx}(t)\|^2 - 2\|u_{xxxx}(t)\|^2 \\
&\quad + \|u_{xxxx}(t)\|^2 + \|u_x(t-\tau)\|^2\|u_x(t)\|^2 - 2\|u_{xx}(t)\|^2 \\
&\le \|u_x(t-\tau)\|^2\|u_x(t)\|^2 \\
&\le \|u_x(t-\tau)\|^2\|u_{xx}(t)\|^2.
\end{aligned} \tag{18}$$

By (16)–(18), using Lemma 3.1, we obtain

$$\begin{aligned}
&\|u_{xx}(t)\|^2 \\
&\le \big(2\|u_0(x,0)\|^2 + \|u_{0xx}(x,0)\|^2\big)\exp\left\{e^{\omega t}\int_{-\tau}^0 \|u_{0x}(s)\|^2 ds + \frac{2e^{\omega\tau}}{\omega}\|u_0(x,0)\|^2\right\}e^{-\omega t} \\
&\le \frac{K^2}{4}e^{-\omega t}.
\end{aligned} \tag{19}$$

Hence

$$\|u_{xx}(T_0)\|^2 \le K^2 e^{-\omega T_0},$$

which is in contradiction with (6). Therefore, we have proved that $T_0 = +\infty$ and then (4) follows from (19). Thus the proof of Theorem 1.2 is completed. □

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