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# Convolution in Rearrangement-Invariant Spaces Defined in Terms of Oscillation and the Maximal Function

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**Abstract.** We characterize boundedness of a convolution operator with a fixed kernel between the classes $S^p(v)$, defined in terms of oscillation, and weighted Lorentz spaces $\Gamma^q(w)$, defined in terms of the maximal function, for $0 < p, q \leq \infty$. We prove corresponding weighted Young-type inequalities of the form

$$\|f * g\|_{\Gamma^q(w)} \leq C\|f\|_{S^p(v)}\|g\|_Y$$

and characterize the optimal rearrangement-invariant space $Y$ for which these inequalities hold.

**Keywords.** Convolution, Young inequality, weighted Lorentz spaces, oscillation
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## 1. Introduction

The classical Young inequality

$$\|f * g\|_q \leq \|f\|_p\|g\|_r,$$

where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ and $f * g$ is the convolution given by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x, \quad t \in \mathbb{R},$$

is one of the fundamental results related to the convolution and function spaces. It has been already modified and generalized for classes of function spaces that

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are wider than the Lebesgue spaces in the original Young inequality. O'Neil [14] extended the result for the two-parametric Lorentz spaces $L_{p,q}$. Precisely, he proved that, for $1 < p, q, r < \infty$ and $1 \le a, b, c \le \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, the inequality

$$\|f * g\|_{L_{q,a}} \le C\|f\|_{L_{p,b}}\|g\|_{L_{r,c}}, \quad f \in L_{p,b},\ g \in L_{r,c},$$

holds. This problem was further studied e.g. in [3,10,18] and the result was also improved up to the range $1 < p, q, r < \infty$ and $0 < a, b, c \le \infty$. Nursultanov and Tikhonov [13] recently studied the same question considering convolution of periodic functions.

In the preceding paper [11] the author studied the boundedness of the operator $T_g$ given by

$$T_g f(t) := (f * g)(t)$$

between weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ with given weights $v, w$ and exponents $p, q$. It turned out that the result could be expressed by Young-type inequalities of the form

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Lambda^p(v)}\|g\|_Y, \quad f \in \Lambda^p(v),\ g \in Y,$$

where the best r.i. space $Y$, such that this inequality holds, was characterized.

In this paper we deal with similar questions with $S^p(v)$ in place of $\Lambda^p(v)$. The class $S^p(v)$ is defined in terms of $f^{**} - f^*$, where $f^*$ is the nonincreasing rearrangement of $f$ and $f^{**}$ is the maximal function of $f$ (for precise definitions see Section 2 below). The quantity $f^{**} - f^*$ naturally represents the oscillation of $f$ (see the fundamental paper of Bennett, DeVore and Sharpley [1]) and has appeared in numerous applications, particularly within the theory of Sobolev embeddings (see e.g. [4] and the references therein).

We are going to solve the following problems: At first, given exponents $p, q \in (0, \infty]$ and weights $v, w$, we provide conditions on the kernel $g \in L^1$ under which $T_g$ is bounded between $S^p(v)$ and $\Gamma^q(w)$, written $T_g : S^p(v) \to \Gamma^q(w)$. Precisely, we will show that there exists an r.i. space $Y$ such that $T_g : S^p(v) \to \Gamma^q(w)$ if (and in reasonable cases also only if) $g \in Y$ and characterize the norm of $Y$. Next, we write these results in the form of Young-type convolution inequalities

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{S^p(v)}\|g\|_Y, \quad f \in S^p(v),\ g \in L^1 \cap Y. \tag{1}$$

The constant $C$ here in general depends on $p, q$ but is independent of $f, g, v, w$. We will also show that the space $Y$ we obtained is the essentially largest (optimal) r.i. space for which the inequality (1) is valid.

To get the desired results, we employ a similar technique as in [11]. We represent the investigated convolution-related inequalities by certain Hardy-type weighted inequalities and then treat the problem by working with the latter ones. This is done in Section 3. The final result shaped as the Young-type inequality (1) is presented in Section 4.

## 2. Preliminaries

Let us present some definitions and technical results we are going to use. The set of all measurable functions on $\mathbb{R}$ is denoted by $\mathscr{M}(\mathbb{R})$. The symbols $\mathscr{M}_+(0,\infty)$ and $\mathscr{M}_+(\mathbb{R})$ stand for the sets of all nonnegative measurable functions on $(0,\infty)$ and $\mathbb{R}$, respectively. If $p \in (1,\infty)$, we define $p' := \frac{p}{p-1}$. The notation $A \lesssim B$ means that $A \le CB$ where $C$ is a positive constant independent of relevant quantities. Unless specified else, $C$ actually depends only on the exponents $p$ and $q$, if they are involved. If $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$. The *optimal constant* $C$ in an inequality $A \le CB$ is the least $C$ such that the inequality holds. By writing inequalities in the form

$$A(f) \lesssim B(f), \quad f \in X,$$

we always mean that $A(f) \lesssim B(f)$ is satisfied for all $f \in X$.

A *weight* is any nonnegative function on $(0,\infty)$. such that $0 < W(t) < \infty$ for all $t > 0$, where $W(t) := \int_0^t w(s)\,\mathrm{d}s$.

If $f \in \mathscr{M}(\mathbb{R})$, we define the *nonincreasing rearrangement* of $f$ by

$$f^*(t) := \inf\{s > 0;\ |\{x \in \mathbb{R};\ |f(x)| > s\}| \le t\}, \quad t > 0,$$

and the *Hardy-Littlewood maximal function* of $f$ by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0.$$

If $u$ is a weight, then a generalized version of the maximal function is defined by

$$f_u^{**}(t) := \frac{1}{U(t)}\int_0^t f^*(s)u(s)\,\mathrm{d}s, \quad t > 0.$$

By $L^1$ we denote the Lebesgue-integrable functions on $\mathbb{R}$. The symbol $L^1_{\mathrm{loc}}$ stands for locally integrable functions on $\mathbb{R}$. If $q \in (0,\infty]$ and $w$ is a weight, then $L^q(w)$ denotes the Lebesgue $L^q$-space over the interval $(0,\infty)$ with the measure $w(t)\,\mathrm{d}t$.

Let $\varrho : \mathscr{M}(\mathbb{R}) \to [0,\infty]$ be a functional with the following properties:

(i) $E \subset \mathbb{R},\ |E| < \infty \ \Rightarrow\ \varrho(\chi_E) < \infty$,

(ii) $f \in \mathscr{M}(\mathbb{R}),\ c \ge 0 \ \Rightarrow\ \varrho(cf) = c\varrho(f)$ (positive homogeneity),

(iii) $f, g \in \mathscr{M}(\mathbb{R}),\ 0 \le f \le g$ a.e. $\ \Rightarrow\ \varrho(f) \le \varrho(g)$ (lattice property),

(iv) $f, g \in \mathscr{M}(\mathbb{R}),\ f^* = g^* \ \Rightarrow\ \varrho(f) = \varrho(g)$ (r.i. property).

The set $X = X(\varrho) := \{f \in \mathscr{M}(\mathbb{R}),\ \varrho(f) < \infty\}$ is called a *rearrangement-invariant* (*r.i.*) *lattice*. For such $X$ we define $\|f\|_X := \varrho(|f|)$ for all $f \in X$.

For the definition of a *rearrangement-invariant space* see [2, p. 59].

Let $p \in (0,\infty]$ and $u, v$ be weights. The *weighted Lorentz spaces* are defined by what follows:

$$\Lambda^p(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Lambda^p(v)} := \left( \int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0,\infty),$$

$$\Lambda^\infty(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f^*(t) v(t) < \infty \right\}, \quad p = \infty,$$

$$\Gamma_u^p(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Gamma_u^p(v)} := \left( \int_0^\infty (f_u^{**}(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0,\infty),$$

$$\Gamma_u^\infty(v) := \left\{ f \in \mathscr{M}(\mathbb{R});\ \|f\|_{\Gamma_u^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f_u^{**}(t) v(t) < \infty \right\}, \quad p = \infty.$$

If $u \equiv 1$, we write just $\Gamma^p(v)$, $\Gamma^\infty(v)$. Next, we denote

$$\mathbb{A} := \{ f \in \mathscr{M}(\mathbb{R}); f^*(\infty) = 0 \}.$$

Clearly, any function $f \in \mathbb{A}$ satisfies $f^{**}(\infty) = 0$.

The class $S^p(v)$ is given by

$$S^p(v) := \left\{ f \in \mathbb{A};\ \|f\|_{S^p(v)} := \left( \int_0^\infty (f^{**}(t) - f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0,\infty),$$

$$S^\infty(v) := \left\{ f \in \mathbb{A};\ \|f\|_{S^\infty(v)} := \operatorname*{ess\,sup}_{t>0} (f^{**}(t) - f^*(t)) v(t) < \infty \right\}, \quad p = \infty.$$

The $\Gamma$-spaces with $u \equiv 1$ are linear and the functional $\|\cdot\|_{\Gamma^p(v)}$ is at least a quasi-norm. In fact, for $p \in [1,\infty]$ it is a norm. The key property is the sublinearity of the maximal function (see e.g. [2, p. 54]), i.e.

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad t > 0.$$

On the other hand, the rearrangement itself is not sublinear and the $\Lambda$-"spaces" need not to be linear [7]. However, they are always at least r.i. lattices.

In contrast with that, $S^p(v)$ in general does not even have the lattice property. A detailed study of this and other functional properties of $S^p(v)$ was published in [4].

Obviously, $\Gamma^p(v) \subset S^p(v)$ for any $p \in (0,\infty]$ and any weight $v$. In case of $p \in (0,\infty)$, we will work with weights $v$ satisfying the conditions

$$\int_\varepsilon^\infty \frac{v(t)}{t^p}\,\mathrm{d}t < \infty \text{ for every } \varepsilon > 0 \quad \text{and} \quad \int_0^\infty \frac{v(t)}{t^p}\,\mathrm{d}t = \infty. \tag{2}$$

It can be checked easily that if the first part of (2) is not satisfied, then $\Gamma^p(v) = S^p(v) = \{0\}$, while failing the other part implies that $L^1 \subset \Gamma^p(v) \subset S^p(v)$. By the symbol $\mathscr{V}_p$ we denote the set of all weights $v$ satisfying (2) with $p \in (0, \infty)$. Similarly, $\mathscr{V}_\infty$ stands for the set of all weights satisfying

$$\operatorname*{ess\,sup}_{t>\varepsilon} \frac{v(t)}{t} < \infty \text{ for every } \varepsilon > 0 \quad \text{and} \quad \operatorname*{ess\,sup}_{t>0} \frac{v(t)}{t} = \infty.$$

A useful tool for investigation of convolution inequalities is the O'Neil inequality [14, Lemma 2.5]:

**Lemma 2.1.** *Let $f, g \in L^1_{\mathrm{loc}}$. Then, for every $t \in (0, \infty)$ it holds*

$$(f * g)^{**}(t) \leq t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s) \, \mathrm{d}s.$$

We are going to use this inequality with an alternative expression of its right-hand side from [11, Proposition 4.1]:

**Lemma 2.2.** *Let $f, g \in L^1_{\mathrm{loc}}$. Then for every $t \in (0, \infty)$ it holds*

$$\begin{aligned} & t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s) \, \mathrm{d}s \\ & = \limsup_{s \to \infty} s f^{**}(s) g^{**}(s) + \int_t^\infty (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s)) \, \mathrm{d}s. \end{aligned}$$

In particular, if $f \in \mathbb{A}$ and $g \in L^1$, then $\lim_{s \to \infty} s f^{**}(s) g^{**}(s) = 0$. Thus, Lemmas 2.1 and 2.2 together yield

$$(f * g)^{**}(t) \leq \int_t^\infty (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s)) \, \mathrm{d}s, \quad t > 0. \tag{3}$$

As observed already in [14], O'Neil inequality has also a converse form (for the proof of the following statement see e.g. [11, Lemma 2.3]).

**Lemma 2.3.** *Let $f, g \in L^1_{\mathrm{loc}}$ be nonnegative even functions which are nonincreasing on $(0, \infty)$. Then for every $t \in (0, \infty)$ it holds*

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(y) g^*(y) \, \mathrm{d}y \leq 12 (f * g)^{**}(t).$$

From now on we denote the "positive symmetrically decreasing" functions by

$$PSD := \{f;\ f \in \mathscr{M}_+(\mathbb{R}),\ f \text{ is even},\ f \text{ is nonincreasing on } (0, \infty)\}.$$

Applying Lemmas 2.2, 2.3 and the observation (3), we reach the following conclusion: Let $f \in \mathbb{A}$, $g \in L^1$ and assume that both $f, g \in PSD$. Then

$$\int_t^\infty (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s)) \, \mathrm{d}s \leq 12 (f * g)^{**}(t), \quad t > 0. \tag{4}$$

The last preliminary result is the proposition below (cf. e.g. [16, Lemma 1.2], [5, Proposition 7.2]).

**Proposition 2.4.** *Let $h$ be a nonnegative and nonincreasing real-valued function on $(0,\infty)$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n \in \mathscr{M}(\mathbb{R})$ such that for a.e. $t>0$ it holds*

$$\frac{f_n^{**}\left(\frac{1}{t}\right) - f_n^{*}\left(\frac{1}{t}\right)}{t} \uparrow h(t), \quad n\to\infty.$$

*Proof.* There exists a nonnegative Radon measure $\nu$ on $(0,\infty)$ such that for a.e. $t>0$ it is

$$h(t) = \int_{[t,\infty)} \frac{\mathrm{d}\nu(x)}{x}. \tag{5}$$

For any $n\in\mathbb{N}$ we can find a function $f_n \in \mathscr{M}(\mathbb{R})$ such that

$$f_n^*(t) = \int_{\left(0,\frac{1}{t}\right)} \chi_{\left(\frac{1}{n},\infty\right)}(x)\mathrm{d}\nu(x)$$

for all $t>0$. Now choose any $t>0$ such that (5) holds. By Fubini theorem,

$$\begin{aligned}
\frac{f_n^{**}\left(\frac{1}{t}\right) - f_n^{*}\left(\frac{1}{t}\right)}{t} &= \int_0^{\frac{1}{t}} \int_{\left(0,\frac{1}{s}\right)} \chi_{\left(\frac{1}{n},\infty\right)}(x)\mathrm{d}\nu(x)\,\mathrm{d}s - \frac{1}{t}\int_{(0,t)} \chi_{\left(\frac{1}{n},\infty\right)}(x)\mathrm{d}\nu(x)\\
&= \int_{(0,\infty)}\int_0^{\min\left\{\frac{1}{x},\frac{1}{t}\right\}} \mathrm{d}s\,\chi_{\left(\frac{1}{n},\infty\right)}(x)\mathrm{d}\nu(x) - \frac{1}{t}\int_{(0,t)} \chi_{\left(\frac{1}{n},\infty\right)}(x)\mathrm{d}\nu(x)\\
&= \int_{[t,\infty)} \frac{\chi_{\left(\frac{1}{n},\infty\right)}(x)}{x}\mathrm{d}\nu(x) \uparrow h(t), \quad n\to\infty.
\end{aligned}$$ □

## 3. Inequalities with $f^{**}-f^*$ and boundedness of the convolution operator

As mentioned in the introduction, we are going to describe when $T_g : S^p(v) \to \Gamma^q(w)$ is bounded and, above all, what is the optimal r.i. space $Y$ such that the inequality $\|f*g\|_{\Gamma^q(w)} \lesssim \|f\|_{S^p(v)}\|g\|_Y$ holds for all $f\in S^p(v)$ and $g\in L^1\cap Y$. The problem is connected to inequalities involving the expression $f^{**}-f^*$ which are shown in the following lemma. It is a direct consequence of the O'Neil inequality (3).

**Lemma 3.1.** *Let $p,q\in(0,\infty]$. Let $v,w$ be weights, $v\in\mathscr{V}_p$. Let $g\in L^1$.*

(i) *If $p,q\in(0,\infty)$ and*

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le C_{(6)}\left(\int_0^\infty (f^{**}(x)-f^*(x))^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}, \quad f\in S^p(v),
\end{aligned} \tag{6}$$

*then* $T_g : S^p(v) \to \Gamma^q(w)$ *and, moreover, the optimal constant* $C_{(6)}$ *satisfies* $\|T_g\|_{S^p(v)\to\Gamma^q(w)} \le C_{(6)}$.

(ii) *If* $0 < p < \infty = q$ *and*

$$\begin{aligned}&\operatorname*{ess\,sup}_{x>0} \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t))\,\mathrm{d}t\ w(x)\\ &\le C_{(7)} \left(\int_0^\infty (f^{**}(x) - f^*(x))^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}}, \quad f \in S^p(v),\end{aligned} \tag{7}$$

*then* $T_g : S^p(v) \to \Gamma^\infty(w)$ *and, moreover, the optimal constant* $C_{(8)}$ *satisfies* $\|T_g\|_{S^p(v)\to\Gamma^\infty(w)} \le C_{(8)}$.

(iii) *If* $0 < q < \infty = p$ *and*

$$\begin{aligned}&\left(\int_0^\infty \left(\int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t))\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\le C_{(8)} \operatorname*{ess\,sup}_{x>0} (f^{**}(x) - f^*(x))v(x), \quad f \in S^\infty(v),\end{aligned} \tag{8}$$

*then* $T_g : S^\infty(v) \to \Gamma^q(w)$ *and, moreover, the optimal constant* $C_{(7)}$ *satisfies* $\|T_g\|_{S^\infty(v)\to\Gamma^q(w)} \le C_{(7)}$.

(iv) *If* $p = q = \infty$ *and*

$$\begin{aligned}&\operatorname*{ess\,sup}_{x>0} \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t))\,\mathrm{d}t\ w(x)\\ &\le C_{(9)} \operatorname*{ess\,sup}_{x>0} (f^{**}(x) - f^*(x))v(x), \quad f \in S^\infty(v),\end{aligned} \tag{9}$$

*then* $T_g : S^\infty(v) \to \Gamma^\infty(w)$ *and, moreover, the optimal constant* $C_{(9)}$ *satisfies* $\|T_g\|_{S^\infty(v)\to\Gamma^\infty(w)} \le C_{(9)}$.

The next result is inverse to the previous lemma, showing that the validity of the inequalities with $f^{**} - f^*$ from that lemma is also necessary for the boundedness of $T_g$, given that $g \in PSD$.

**Lemma 3.2.** *Let* $p, q \in (0, \infty]$. *Let* $v, w$ *be weights,* $v \in \mathscr{V}_p$. *Let* $g \in L^1 \cap PSD$.

(i) *If* $p, q \in (0, \infty)$ *and* $T_g : S^p(v) \to \Gamma^q(w)$, *then* (6) *holds and the optimal constant* $C_{(6)}$ *satisfies* $C_{(6)} \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}$.

(ii) *If* $0 < p < \infty = q$ *and* $T_g : S^p(v) \to \Gamma^\infty(w)$, *then* (7) *holds and the optimal constant* $C_{(7)}$ *satisfies* $C_{(7)} \lesssim \|T_g\|_{S^p(v)\to\Gamma^\infty(w)}$.

(iii) *If* $0 < q < \infty = p$ *and* $T_g : S^\infty(v) \to \Gamma^q(w)$, *then* (8) *holds and the optimal constant* $C_{(8)}$ *satisfies* $C_{(8)} \lesssim \|T_g\|_{S^\infty(v)\to\Gamma^q(w)}$.

(iv) *If* $p = q = \infty$ *and* $T_g : S^\infty(v) \to \Gamma^\infty(w)$, *then* (9) *holds and the optimal constant* $C_{(9)}$ *satisfies* $C_{(9)} \lesssim \|T_g\|_{S^\infty(v)\to\Gamma^\infty(w)}$.

*Proof.* Let us show (i), the other cases are analogous. By (4), for the optimal constant $C_{(6)}$ we get

$$\begin{aligned} C_{(6)} &= \sup_{\|f\|_{S^p(v)}\le 1} \left(\int_0^\infty \left(\int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &= \sup_{\substack{\|f\|_{S^p(v)}\le 1\\ f\in PSD}} \left(\int_0^\infty \left(\int_x^\infty (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\le 12 \sup_{\substack{\|f\|_{S^p(v)}\le 1\\ f\in PSD}} \left(\int_0^\infty ((f*g)^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \\ &\le \|T_g\|_{S^p(v)\to\Gamma^q(w)}. \end{aligned}$$

□

Now we characterize under which conditions on weights and exponents the inequalities of Lemma 3.1 are satisfied.

**Theorem 3.3.** *Let $p,q\in(0,\infty)$. Let $v,w$ be weights, $v\in\mathscr{V}_p$. Let $g\in L^1$.*

(i) *If $1<p\le q<\infty$, then (6) holds if and only if*

$$A_{(10)} := \sup_{x>0}\left(\int_x^\infty (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_x^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-\frac{1}{p}}<\infty \tag{10}$$

*and*

$$A_{(11)} := \sup_{x>0} W^{\frac{1}{q}}(x)\left(\int_x^\infty (g^{**}(t))^{p'}\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-p'}\frac{v(t)}{t^p}\,\mathrm{d}t\right)^{\frac{1}{p'}}<\infty. \tag{11}$$

*The optimal constant $C_{(6)}$ satisfies $C_{(6)}\simeq A_{(10)}+A_{(11)}$.*

(ii) *If $0<p\le 1,\ 0<p\le q<\infty$, then (6) holds if and only if $A_{(10)}<\infty$ and*

$$A_{(12)} := \sup_{x>0} g^{**}(x) W^{\frac{1}{q}}(x)\left(\int_x^\infty v(t)\,\mathrm{d}t\right)^{-\frac{1}{p}}<\infty. \tag{12}$$

*The optimal constant $C_{(6)}$ satisfies $C_{(6)}\simeq A_{(10)}+A_{(12)}$.*

(iii) *If $1<p<\infty,\ 0<q<p$, then (6) holds if and only if*

$$A_{(13)} := \left(\int_0^\infty\left(\int_x^\infty (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{r}{q}}\left(\int_x^\infty \frac{v(t)}{t^p}\,\mathrm{d}t\right)^{-\frac{r}{q}}\frac{v(x)}{x^p}\,\mathrm{d}x\right)^{\frac{1}{r}}<\infty \tag{13}$$

*and*

$$\begin{aligned} A_{(14)} := \Bigg(\int_0^\infty & W^{\frac{r}{p}}(x) w(x) \\ &\times \left(\int_x^\infty (g^{**}(t))^{p'}\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-p'}\frac{v(t)}{t^p}\,\mathrm{d}t\right)^{\frac{r}{p'}}\mathrm{d}x\Bigg)^{\frac{1}{r}} \\ &<\infty. \end{aligned} \tag{14}$$

*The optimal constant* $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(13)} + A_{(14)}$.

(iv) *If* $0 < q < p \le 1$, *then* (6) *holds if and only if* $A_{(13)} < \infty$ *and*

$$A_{(15)} := \left( \int_0^\infty \sup_{x \le t < \infty} (g^{**}(t))^r \left( \int_0^t \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-\frac{r}{p}} W^{\frac{r}{p}}(x) w(x) \, \mathrm{d}x \right)^{\frac{1}{r}} < \infty. \quad (15)$$

*The optimal constant* $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(13)} + A_{(15)}$.

*Proof.* Let us show (i). After the change of variable $x \mapsto \frac{1}{x}$, inequality (6) is written as

$$\begin{aligned} &\left( \int_0^\infty \left( \int_0^x \frac{f^{**}\left(\frac{1}{t}\right) - f^{*}\left(\frac{1}{t}\right)}{t} \cdot \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \right)^q \frac{w\left(\frac{1}{x}\right)}{x^2} \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\le C_{(6)} \left( \int_0^\infty \left( \frac{f^{**}\left(\frac{1}{x}\right) - f^{*}\left(\frac{1}{x}\right)}{x} \right)^p v\left(\frac{1}{x}\right) x^{p-2} \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad f \in \mathscr{M}(\mathbb{R}). \end{aligned} \quad (16)$$

Let us denote by $\mathscr{M}_+^{\downarrow}(0,\infty)$ the cone of nonnegative and nonincreasing functions on $(0,\infty)$. We claim that (16) is true if and only if

$$\begin{aligned} &\left( \int_0^\infty \left( \int_0^x \varphi(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \right)^q \frac{w\left(\frac{1}{x}\right)}{x^2} \, \mathrm{d}x \right)^{\frac{1}{q}} \\ &\le C_{(6)} \left( \int_0^\infty \varphi^p(x) \frac{v\left(\frac{1}{x}\right)}{x^{2-p}} \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad \varphi \in \mathscr{M}_+^{\downarrow}(0,\infty). \end{aligned} \quad (17)$$

Indeed, every function $t \mapsto \frac{f^{**}\left(\frac{1}{t}\right) - f^{*}\left(\frac{1}{t}\right)}{t}$ is nonnegative and nonincreasing on $(0,\infty)$, hence (17) implies (16). On the other hand, if $\varphi \in \mathscr{M}_+^{\downarrow}(0,\infty)$ is given, by Proposition 2.4 we find $f_n \in \mathscr{M}(\mathbb{R})$ such that $\frac{f_n^{**}\left(\frac{1}{t}\right) - f_n^{*}\left(\frac{1}{t}\right)}{t} \uparrow \varphi(t)$ for a.e. $t \in (0,\infty)$. Since (16) holds for every $f_n$ in place of $f$, by the monotone convergence theorem we get (17) for the given $\varphi$. Hence, (16) implies (17).

Inequality (17) defines the embedding

$$\Lambda^p(\widetilde{v}) \hookrightarrow \Gamma_u^q(\widetilde{w}) \quad (18)$$

with

$$\widetilde{v}(x) := v\left(\frac{1}{x}\right) x^{p-2}, \quad \widetilde{w}(x) := w\left(\frac{1}{x}\right) x^{q-2}, \quad u(x) := \frac{g^{**}\left(\frac{1}{x}\right) - g^{*}\left(\frac{1}{x}\right)}{x}.$$

By [8, Theorem 3.1(iii)] or a modified version of [6, Theorem 4.1(i)], (18) (as well as (17)) holds if and only if $A_{(10)} + A_{(11)} < \infty$ and the optimal $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(10)} + A_{(11)}$, which is the result.

In cases (ii)–(iv) we proceed in the same way, the only difference being the conditions characterizing (18) for different settings of $p$ and $q$. These characterizations of (18) may be found in [8, Theorem 3.1] or, alternatively, in [6, Theorem 4.1] for (ii) and (iii) and [5, Theorem 3.1] for (iv). Note that in [5,6] the results are given just for $u = 1$. $\square$

**Remark 3.4.** For $1 \leq p < \infty$, Theorem 3.3 can be alternatively obtained using the reduction theorem [9, Theorem 2.2] and Hardy inequalities for nonnegative functions (see e.g. [12, 15]).

In the case $q = \infty$, i.e. for (7), we get

**Theorem 3.5.** *Let $p \in (0, \infty)$. Let $v, w$ be weights, $v \in \mathscr{V}_p$. Let $g \in L^1$. Then*

(i) *If $0 < p \leq 1$, then (7) holds if and only if*

$$A_{(19)} := \operatorname*{ess\,sup}_{x>0} w(x) \sup_{t>x} g^{**}(t) \left( \int_t^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-\frac{1}{p}} < \infty. \tag{19}$$

*Moreover, the optimal constant $C_{(7)}$ satisfies $C_{(7)} \simeq A_{(19)}$.*

(ii) *If $1 < p < \infty$, then (7) holds if and only if*

$$\begin{aligned} A_{(20)} := &\operatorname*{ess\,sup}_{x>0} w(x) \left[ \left( \int_x^\infty (g^{**}(t))^{p'} \left( \int_t^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-p'} \frac{v(t)}{t^p} \, \mathrm{d}t \right)^{\frac{1}{p'}} \right. \\ &\left. + \, g^{**}(x) \left( \int_x^\infty \frac{v(s)}{s^p} \, \mathrm{d}s \right)^{-\frac{1}{p}} \right] \\ &< \infty. \end{aligned} \tag{20}$$

*Moreover, the optimal constant $C_{(7)}$ satisfies $C_{(7)} \simeq A_{(20)}$.*

*Proof.* Following the same reasoning as in the proof of Theorem 3.3, the inequality (7) is equivalent to

$$\begin{aligned} &\operatorname*{ess\,sup}_{x>0} \int_0^x \varphi(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \; w\left(\frac{1}{x}\right) \\ &\leq C_{(7)} \left( \int_0^\infty \varphi^p(x) v\left(\frac{1}{x}\right) x^{p-2} \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad \varphi \in \mathscr{M}_+^{\downarrow}(0,\infty). \end{aligned}$$

Denote $v_p(x) := v\left(\frac{1}{x}\right) x^{p-2}$. The optimal $C_{(7)}$ satisfies

$$\begin{aligned} C_{(7)} &= \sup_{\|f\|_{\Lambda^p(v_p)} \leq 1} \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right) \int_0^x f^*(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t \\ &= \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right) \sup_{\|f\|_{\Lambda^p(v_p)} \leq 1} \int_0^x f^*(t) \frac{g^{**}\left(\frac{1}{t}\right) - g^{*}\left(\frac{1}{t}\right)}{t} \, \mathrm{d}t. \end{aligned} \tag{21}$$

In the following calculations, we are going to use the condition (2) without further comment.

(i) If $0 < p \le 1$, [6, Theorem 3.1(i)] gives

$$\sup_{\|f\|_{\Lambda^p(v_p)}\le 1}\int_0^x f^*(t)\frac{g^{**}\left(\frac{1}{t}\right)-g^*\left(\frac{1}{t}\right)}{t}\,\mathrm{d}t \simeq \sup_{t\in(0,x)}\int_0^t \frac{g^{**}\left(\frac{1}{s}\right)-g^*\left(\frac{1}{s}\right)}{s}\,\mathrm{d}s\left(\int_0^t v_p(s)\,\mathrm{d}s\right)^{-\frac{1}{p}}.$$

Hence, we get

$$\begin{aligned} C_{(7)} &\simeq \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right)\sup_{t\in(0,x)}\int_0^t \frac{g^{**}\left(\frac{1}{s}\right)-g^*\left(\frac{1}{s}\right)}{s}\,\mathrm{d}s\left(\int_0^t v_p(s)\,\mathrm{d}s\right)^{-\frac{1}{p}} \\ &= \operatorname*{ess\,sup}_{x>0} w\left(\frac{1}{x}\right)\sup_{t\in\left(\frac{1}{x},\infty\right)} g^{**}(t)\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-\frac{1}{p}} \\ &= A_{(19)}. \end{aligned}$$

(ii) If $1 < p < \infty$, by [6, Theorem 3.1(ii)] we have

$$\begin{aligned} &\sup_{\|f\|_{\Lambda^p(v_p)}\le 1}\int_0^x f^*(t)\frac{g^{**}\left(\frac{1}{t}\right)-g^*\left(\frac{1}{t}\right)}{t}\,\mathrm{d}t \\ &\simeq \left(\int_0^x\left(\int_0^t \frac{g^{**}\left(\frac{1}{s}\right)-g^*\left(\frac{1}{s}\right)}{s}\,\mathrm{d}s\right)^{p'}\left(\int_0^t v_p(s)\,\mathrm{d}s\right)^{-p'} v_p(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \\ &\quad+\int_0^x \frac{g^{**}\left(\frac{1}{s}\right)-g^*\left(\frac{1}{s}\right)}{s}\,\mathrm{d}s\left(\int_x^\infty\left(\int_0^t v_p(s)\,\mathrm{d}s\right)^{-p'} v_p(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \\ &= \left(\int_{\frac{1}{x}}^\infty (g^{**}(t))^{p'}\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-p'}\frac{v(t)}{t^p}\,\mathrm{d}t\right)^{\frac{1}{p'}} \\ &\quad+ g^{**}\left(\frac{1}{x}\right)\left(\int_0^{\frac{1}{x}}\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-p'}\frac{v(t)}{t^p}\,\mathrm{d}t\right)^{\frac{1}{p'}} \\ &= \left(\int_{\frac{1}{x}}^\infty (g^{**}(t))^{p'}\left(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-p'}\frac{v(t)}{t^p}\,\mathrm{d}t\right)^{\frac{1}{p'}} + g^{**}\left(\frac{1}{x}\right)\left(\int_{\frac{1}{x}}^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-\frac{1}{p}}. \end{aligned}$$

Hence, (21) implies $C_{(7)} \simeq A_{(20)}$ for the optimal $C_{(7)}$. □

For the last case, $p = \infty$, which covers the inequalities (8) and (9), we have the following theorem.

**Theorem 3.6.** *Let $v, w$ be weights, $v \in \mathscr{V}_\infty$. Let $g \in L^1$. Then*

(i) *For $0 < q < \infty$, the inequality (8) holds and only if*

$$A_{(22)} := \left( \int_0^\infty \left( \int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} \,\mathrm{d}t \right)^q w(x) \,\mathrm{d}x \right)^{\frac{1}{q}} < \infty. \quad (22)$$

*Moreover, the optimal constant $C_{(8)}$ satisfies $C_{(8)} \simeq A_{(22)}$.*

(ii) *The inequality (9) holds if and only if*

$$A_{(23)} := \operatorname*{ess\,sup}_{x>0} \int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} \,\mathrm{d}t \; w(x) < \infty. \quad (23)$$

*Moreover, the optimal constant $C_{(9)}$ satisfies $C_{(9)} \simeq A_{(23)}$.*

*Proof.* Here we use the same technique as in Theorems 3.3 and 3.5. During the process we apply e.g. the result of [17, Proposition 2.7]. We omit the details. □

**Remark 3.7.** In each of the particular settings of the exponents $p, q$ in Theorem 3.3(i)–(iv), the functionals $A_{(10)}, \dots, A_{(15)}$ are r.i. norms of $g$, with the following exceptions: In (iii) and (iv), if $0 < q < 1$, then $A_{(13)}$ is in general just an r.i. quasi-norm, the same applies to $A_{(15)}$ in (iv) if $r < 1$. Similarly, the functionals $A_{(19)}$ and $A_{(20)}$ in Theorem 3.5 are r.i. norms of $g$. For a detailed proof of this, see e.g. [11, Proposition 5.6].

In Theorem 3.6, the functional $A_{(23)}$ acting on $g \in L^1$ is an r.i. norm of $g$. The functional $A_{(22)}$ is, in general, an r.i. quasi-norm, for $q \geq 1$ an r.i. norm. Let us prove the claim about $A_{(22)}$. At first, since $t \mapsto \left(\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}\right)^{-1}$ is nondecreasing, its derivative, which we denote by

$$\delta(t) := \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}},$$

exists and is nonnegative for a.e. $t \in (0, \infty)$. Let $x \in (0, \infty)$. Suppose that

$$\int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} \,\mathrm{d}t < \infty.$$

Then, by monotonicity of $\left(\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}\right)^{-1}$, we have

$$\begin{aligned} \frac{g^{**}(t)}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} &= \frac{1}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}} \int_t^\infty \frac{g^{**}(y) - g^*(y)}{y} \,\mathrm{d}y \\ &\leq \int_t^\infty \frac{g^{**}(y) - g^*(y)}{y \operatorname{ess\,sup}_{s\in(y,\infty)} v(s)s^{-1}} \,\mathrm{d}y \overset{t\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence, by partial integration and the previous, we get

$$\begin{aligned}\int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t &= \left[\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\right]_{t=x}^{\infty} + \int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t\\ &= \int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t - \frac{g^{**}(x)}{\operatorname{ess\,sup}_{s\in(x,\infty)} v(s)s^{-1}}\\ &< \infty.\end{aligned}$$

Now assume, on the other hand, that $\int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t < \infty$. Then,

$$\int_x^\infty \frac{g^{**}(t)-g^*(t)}{t\ \operatorname{ess\,sup}_{s\in(t,\infty)} v(s)s^{-1}}\,\mathrm{d}t = \frac{g^{**}(x)}{\operatorname{ess\,sup}_{s\in(x,\infty)} v(s)s^{-1}} + \int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t < \infty.$$

Thus, we see that $A_{(22)}$ is equal to

$$\left(\int_0^\infty \left(\frac{g^{**}(x)}{\operatorname{ess\,sup}_{s\in(x,\infty)} v(s)s^{-1}} + \int_x^\infty g^{**}(t)\delta(t)\,\mathrm{d}t\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

This expression is an r.i. quasi-norm of $g$, for $q \geq 1$ it is an r.i. norm. To check this, we refer again to [11].

In the same way as above, we may show that $A_{(23)}$ is an r.i. norm.

## 4. Young-type convolution inequalities with the class $S$ on the right-hand side

In the previous part we obtained the conditions for boundedness of $T_g$. Let us now summarize these results and apply them to get the desired convolution inequalities. Note that, in what follows, if we define $\|\cdot\|_Y$ first, then the space $Y$ is naturally defined as $Y := \{f \in \mathscr{M}(\mathbb{R});\ \|f\|_Y < \infty\}$.

**Theorem 4.1.** *Let $p, q \in (0,\infty]$. Let $v, w$ be weights, $v \in \mathscr{V}_p$. For $g \in L^1$ define $\|g\|_Y$ by what follows:*

$$\|g\|_Y := \begin{cases} A_{(10)} + A_{(11)} & \text{if} \quad 1 < p \leq q < \infty;\\ A_{(10)} + A_{(12)} & \text{if} \quad 0 < p \leq 1,\ 0 < p \leq q < \infty;\\ A_{(13)} + A_{(14)} & \text{if} \quad 1 < p < \infty,\ 0 < q < p;\\ A_{(13)} + A_{(15)} & \text{if} \quad 0 < q < p \leq 1;\\ A_{(19)} & \text{if} \quad 0 < p \leq 1,\ q = \infty;\\ A_{(20)} & \text{if} \quad 1 < p < \infty,\ q = \infty;\\ A_{(22)} & \text{if} \quad p = \infty,\ 0 < q < \infty;\\ A_{(23)} & \text{if} \quad p = q = \infty.\end{cases}$$

*Then*

(i) *If* $g \in Y$, *then* $T_g : S^p(v) \to \Gamma^q(w)$ *and*

$$\|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim \|g\|_Y.$$

(ii) *If* $g \in PSD$ *and* $T_g : S^p(v) \to \Gamma^q(w)$, *then* $g \in Y$ *and*

$$\|g\|_Y \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}.$$

(iii) *The inequality*

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{S^p(v)}\|g\|_Y, \quad f \in S^p(v),\ g \in L^1 \cap Y, \tag{24}$$

*is satisfied. Moreover, if* $\widetilde{Y}$ *is any r.i. lattice such that* (24) *holds with* $\widetilde{Y}$ *in place of* $Y$, *then* $L^1 \cap \widetilde{Y} \hookrightarrow L^1 \cap Y$.

*Proof.* Let us prove the assertions for the case $1<p\leq q<\infty$. In the other cases, the only difference is that we work with another appropriate functional $A_{(\dots)}$.

(i) Let $g \in Y$, thus $A_{(10)} + A_{(11)} < \infty$. Then, by Theorem 3.3(i), the inequality (6) holds. Thus, from Lemma 3.1(i) it follows that $T_g : S^p(v) \to \Gamma^q(w)$ and $\|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim C_{(6)} \simeq \|g\|_Y$.

(ii) Assume that $g \in PSD$ and $T_g : S^p(v) \to \Gamma^q(w)$. By Lemma 3.2(i), inequality (6) holds and the optimal $C_{(6)}$ satisfies $C_{(6)} \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}$. Theorem 3.1(i) now yields that $A_{(10)} + A_{(11)} < \infty$, i.e. $g \in Y$. Moreover, we also get $\|g\|_Y \simeq C_{(6)} \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}$.

(iii) The inequality (24) follows from (i) and the relation $\|T_g f\|_{\Gamma^q(w)} \leq \|T_g\|_{S^p(v)\to\Gamma^q(w)}\|f\|_{S^p(v)}$. Let us prove the optimality of $Y$. Assume that $\widetilde{Y}$ is an r.i. lattice such that

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{S^p(v)}\|g\|_{\widetilde{Y}}, \quad f \in S^p(v),\ g \in L^1 \cap \widetilde{Y}. \tag{25}$$

Let $h \in L^1\cap\widetilde{Y}$. We can find a function $g \in L^1\cap\widetilde{Y}\cap PSD$ such that $g^* = h^*$. The inequality (25) yields that $\|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}$. Thus, $T_g : S^p(v) \to \Gamma^q(w)$ and by (ii) it holds $\|g\|_Y \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)}$. Together we get

$$\|g\|_Y \lesssim \|T_g\|_{S^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}.$$

The functionals $\|\cdot\|_Y$ and $\|\cdot\|_{\widetilde{Y}}$ are r.i., thus we obtain

$$\|h\|_Y \lesssim \|h\|_{\widetilde{Y}}.$$

Since $h$ was chosen arbitrarily, we got the desired embedding $L^1\cap\widetilde{Y}\hookrightarrow L^1\cap Y$. □

**Remark 4.2.** For given weights $v, w$ and exponents $p, q$, the optimal space $Y$ may equal $\{0\}$. (Let us formally consider $\{0\}$ to be an r.i. space.) In that case, the operator $T_g$ with a nonnegative kernel $g$ is bounded between $S^p(v)$ and $\Gamma^q(w)$ if and only if $g = 0$ a.e. (cf. [11, Corollary 3.3]).

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