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G-Convergence of Linear Differential Equations

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Abstract. We discuss G-convergence of linear integro-differential-algebraic equations in Hilbert spaces. We show under which assumptions it is generic for the limit equation to exhibit memory effects. Moreover, we investigate which classes of equations are closed under the process of G-convergence. The results have applications to the theory of homogenization. As an example we treat Maxwell's equation with the Drude-Born-Fedorov constitutive relation.
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1. Introduction

We discuss some issues occuring in the homogenization of linear integro-differential equations in Hilbert spaces. Similar to [20,21,24,25] we understand homogenization theory as the study of limits of sequences of equations in the sense of G-convergence introduced in [22, 23] with generalizations in [15]. Whereas in [20, 21] (non-linear) ordinary differential equations in finite-dimensional space are considered, we choose the perspective given in [1, 9, 10, 14, 16, 24, 25]. The abstract setting is the following.

Definition 1.1 (G-convergence, [32, p. 74], [22, 23, 29])**.** Let H be a Hilbert space. Let $(A_n : D(A_n) \subseteqq H \to H)_n$ be a sequence of continuously invertible linear operators onto H and let $B : D(B) \subseteqq H \to H$ be linear and one-to-one. We say that $(A_n)_n$ *G-converges to B* if $(A_n^{-1})_n$ converges in the weak operator topology to B^{-1}, i.e., for all $f \in H$ the sequence $(A_n^{-1}(f))_n$ converges weakly to some u, which satisfies $u \in D(B)$ and $B(u) = f$. B is called the *G-limit* of $(A_n)_n$ – which is uniquely determined (cf. [29, Proposition 4.1]) – and we write $A_n \xrightarrow{G} B$.

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Our starting point will be equations of the form

$$\partial_0 \mathcal{M} u + \mathcal{N} u = f,$$

where $\mathcal{M}, \mathcal{N}$ are suitable operators in space-time and ∂_0 is the time-derivative established in a Hilbert space setting to be specified below (see also [11, 18]). In the usual framework of homogenization theory, one assumes $\mathcal{M}$ and $\mathcal{N}$ to be multiplication operators in space-time, i.e., there are mappings a and b such that $\mathcal{M} = a(\cdot)$ and $\mathcal{N} = b(\cdot)$. Assuming well-posedness of the above equation, i.e., existence, uniqueness and continuous dependence on the right-hand side f in a suitable (Hilbert space) framework, one is interested in the sequence of equations

$$\partial_0 \mathcal{M}_n u_n + \mathcal{N}_n u_n = f \tag{1}$$

with $\mathcal{M}_n = a(n\cdot)$ and correspondingly for $\mathcal{N}_n$ yielding a sequence of solutions $(u_n)_n$. The question arises, whether the sequence $(u_n)_n$ converges and if so whether the respective limit u satisfies an equation of similar form. A formal computation in (1) reveals that

$$u_n = \left(\partial_0 \mathcal{M}_n + \mathcal{N}_n\right)^{-1} f.$$

Thus, if we show the convergence of $\left(\partial_0 \mathcal{M}_n + \mathcal{N}_n\right)^{-1}$ in the weak operator topology to some one-to-one mapping $C =: B^{-1}$, we deduce the weak convergence of $(u_n)_n$ the limit of which denoted by u satisfies

$$Bu = f.$$

In other words, $(\partial_0 \mathcal{M}_n + \mathcal{N}_n)$ G-converges to B.

In this article we think of $(\mathcal{M}_n)_n$ and $(\mathcal{N}_n)_n$ to be bounded sequences of bounded linear operators in space-time. We want to discuss assumptions on these sequences guaranteeing a compactness result with respect to G-convergence. Moreover, we outline possible assumptions yielding the closedness under G-convergence and give examples for equations, where the associated sequences of differential operators itself are G-convergent. We exemplify our findings with examples from the literature [9, 10, 14, 24, 25], highlight possible connections and give an example for a Drude-Born-Fedorov model in electro-magentism (see [7] and Example 3.12 below), where homogenization theorems are – to the best of the author's knowledge – not yet available in the literature. We will also underscore the reason of the limit equation to exhibit memory effects. An heuristic explanation is the lack of continuity of computing the inverse with respect to the weak operator topology.

In Section 2 we introduce the functional analytic setting used for discussing integro-differential-algebraic equations and state our main theorems. We successively apply the results from Section 2 to time-independent coefficients (Section 3), time-translation invariant coefficients (Section 4) and time-dependent coefficients (Section 5). In each of the Sections 3, 4 and 5 we give examples and discuss whether particular classes of equations are closed under limits with respect to G-convergence. In Section 6 we prove the main theorems of Section 2.

The respective proofs rely on elementary Hilbert space theory. In fact, the method of proof for the main underlying Theorems 2.5 and 2.7 is based on the observation that the time-derivative can be established as a continuously invertible operator with arbitrarily small norm bound for the inverse depending on the function spaces under consideration and appropriate Neumann series expansions. Due to the examples given, the present work may also underline the versatility of this simple idea. This particular idea has been used by the author on several occassions (see e.g. [27]). Note that, however, the situation in which this method is applied is more general than in [27], where only autonomous equations are treated, which are not of differential-algebraic type. In consequence, the limit expressions in the general setting are getting more involved as the (inverse of) the time-derivative does not commute anymore with the other operators under consideration. Hence, techniqual tools employed in [27] like the usage of the Fourier-Laplace transformation cannot be used in the derivation of the general result. Furthermore, we note here that the results in this article complement the work in [28, 30] as the results in the latter articles are of a partial differential type of nature. That is to say that in the equations under consideration in [28,30] despite the occurence of the time-derivative there is another unbounded operator, which has to be strictly unbounded in the sense that its resolvent needs to be compact. Thus the limit expressions are, though their derivation is more involved, conceptually easier to read. The present article focusses on the ordinary differential equation case with an infinite-dimensional state space. The results of the article show that in this situation the limit equation to exhibit memory effects is more likely than in the pde-case just sketched, see e.g. [28]. The consideration of differential-algebraic systems (Theorem 2.7) is – to the best of the author's knoweldge – new. Even in the time-translation invariant context the assumptions on the operators under consideration could be weakend in comparison to [29, Theorem 4.4]. As the main results of this article also cover the non-autonomous case, it is also of interest under which assumptions, there is no need to choose subsequences. This has been done in Section 5.

2. Setting and main theorems

The key fact giving way for computations is the possibility of establishing the time-derivative as a continuously invertible normal operator in an exponentially weighted Hilbert space. For $\nu > 0$ we define the operator

$$\partial_0 \colon H_{\nu,1}(\mathbb{R}) \subseteqq L^2_\nu(\mathbb{R}) \to L^2_\nu(\mathbb{R}), f \mapsto f',$$

where $L^2_\nu(\mathbb{R}) := L^2(\mathbb{R}, \exp(-2\nu\cdot)\lambda)$ is the space of square-integrable functions with respect to the weighted Lebesgue measure $\exp(-2\nu\cdot)\lambda$ and $H_{\nu,1}(\mathbb{R})$ is the space of $L^2_\nu(\mathbb{R})$-functions with distributional derivative in $L^2_\nu(\mathbb{R})$. We denote the scalar-product on $L^2_\nu(\mathbb{R})$ by $\langle\cdot,\cdot\rangle_\nu$ and the induced norm by $|\cdot|_\nu$. Of course the operator ∂_0 depends on the scalar ν. However, since it will be obvious from the context, which value of ν is chosen, we will omit the explicit reference to it in the notation of ∂_0. It can be shown that ∂_0 is continuously invertible ([18, Example 2.3] or [11, Corollary 2.5]). The norm bound of the inverse is $\frac{1}{\nu}$. Of course the latter construction can be extended to the Hilbert-space-valued case of $L^2_\nu(\mathbb{R}; H)$-functions[1]. We will use the same notation for the time-derivative. In order to formulate our main theorems related to the theory of homogenization of ordinary differential equations, we need to introduce the following notion.

Definition 2.1. Let H_0, H_1 be Hilbert spaces, $\nu_0 > 0$. We call a linear mapping

$$M\colon D(M) \subseteqq \bigcap_{\nu>0} L^2_\nu(\mathbb{R}; H_0) \to \bigcap_{\nu \geqq \nu_0} L^2_\nu(\mathbb{R}; H_1) \tag{2}$$

evolutionary (at $\nu_1 > 0$)[2] if $D(M)$ is dense in $L^2_\nu(\mathbb{R}; H)$ for all $\nu \geqq \nu_1$, if M extends to a bounded linear operator from $L^2_\nu(\mathbb{R}; H_0)$ to $L^2_\nu(\mathbb{R}; H_1)$ for all $\nu \geqq \nu_1$ and is such that[3]

$$\limsup_{\nu\to\infty} \|M\|_{L(L^2_\nu(\mathbb{R};H_0),L^2_\nu(\mathbb{R};H_1))} < \infty.$$

The continuous extension of M to some L^2_ν will also be denoted by M. In particular, we will not distinguish notationally between the different realizations of M as a bounded linear operator for different ν as these realizations coincide on a dense subset. We define the set

$$L_{\mathrm{ev},\nu_1}(H_0, H_1) := \{M; M \text{ is as in (2) and is evolutionary at } \nu_1\}.$$

We abbreviate $L_{\mathrm{ev},\nu_1}(H_0) := L_{\mathrm{ev},\nu_1}(H_0, H_0)$. A subset $\mathfrak{M} \subseteqq L_{\mathrm{ev},\nu_1}(H_0, H_1)$ is called *bounded* if $\limsup_{\nu\to\infty} \sup_{M\in\mathfrak{M}} \|M\|_{L(L^2_\nu)} < \infty$. A family $(M_\iota)_{\iota\in I}$ in $L_{\mathrm{ev},\nu_1}(H_0, H_1)$ is called *bounded* if $\{M_\iota; \iota \in I\}$ is bounded.

Note that $L_{\mathrm{ev},\nu_1}(H_0, H_1) \subseteqq L_{\mathrm{ev},\nu_2}(H_0, H_1)$ for all $\nu_1 \leqq \nu_2$. We give some examples of evolutionary mappings.

[1]We will also use the notation $\langle\cdot,\cdot\rangle_\nu$ and $|\cdot|_\nu$ for the scalar product and norm in $L^2_\nu(\mathbb{R}; H)$, respectively.

[2]The notion "evolutionary" is inspired by the considerations in [18, Definition 3.1.14, p. 91], where polynomial expressions in partial differential operators are considered.

[3]For a linear operator A from $L^2_\nu(\mathbb{R}; H_0)$ to $L^2_\nu(\mathbb{R}; H_1)$ we denote its operator norm by $\|A\|_{L(L^2_\nu(\mathbb{R};H_0),L^2_\nu(\mathbb{R};H_1))}$. If the spaces H_0 and H_1 are clear from the context, we shortly write $\|A\|_{L(L^2_\nu)}$.

Example 2.2. Let H be a Hilbert space and $M_0 \in L(H)$. Then there is a canonical extension M of M_0 to $L^2_\nu(\mathbb{R};H)$-functions such that $(M\phi)(t) := (M_0\phi(t))$ for all $\phi \in L^2_\nu(\mathbb{R};H)$ and a.e. $t \in \mathbb{R}$. In that way $M \in \bigcap_{\nu>0} L_{\mathrm{ev},\nu}(H)$. Henceforth, we shall not distinguish notationally between M and M_0.

Example 2.3. Let H be a Hilbert space and let $L^\infty_s(\mathbb{R};L(H))$ be the space of bounded strongly measurable functions from $\mathbb{R}$ to $L(H)$. For $A \in L^\infty_s(\mathbb{R};L(H))$ we denote the associated multiplication operator on $L^2_\nu(\mathbb{R};H)$ by $A(m_0)$. Thus, also in this case, $A(m_0) \in \bigcap_{\nu>0} L_{\mathrm{ev},\nu}(H)$.

Example 2.4. For $\nu_0 > 0$ let

$$g \in L^1_{\nu_0}(\mathbb{R}_{>0}) := \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}); g = 0 \text{ on } \mathbb{R}_{<0}, \int_{\mathbb{R}} |g(t)| e^{-\nu t} dt < \infty \right\}.$$

By Young's inequality or by Example 4.3 below, we deduce that $g* \in L_{\mathrm{ev},\nu_0}(\mathbb{C})$, where $g * f$ denotes the convolution of some function f with g.

To formulate our main theorems, we denote the weak operator topology by τ_{w}. Convergence within this topology is denoted by $\overset{\tau_{\mathrm{w}}}{\to}$. Limits within this topology are written as τ_{w}-lim. We will extensively use the fact that for a separable Hilbert space H bounded subsets of $L(H)$, which are τ_{w}-closed, are τ_{w}-sequentially compact. Our main theorems concerning the G-convergence of differential equations read as follows.

Theorem 2.5. *Let H be a separable Hilbert space, $\nu_0 > 0$. Let $(\mathcal{M}_n)_n$, $(\mathcal{N}_n)_n$ be bounded sequences in $L_{\mathrm{ev},\nu_0}(H)$. Assume there exists $c > 0$ such that for all $n \in \mathbb{N}$ and $\nu \geqq \nu_0$*

$$\mathfrak{Re}\langle \mathcal{M}_n\phi, \phi\rangle_\nu \geqq c\langle \phi, \phi\rangle_\nu \quad (\phi \in L^2_\nu(\mathbb{R};H)).$$

Then there exists $\nu \geqq \nu_0$ and a subsequence $(n_k)_k$ of $(n)_n$ such that

$$\partial_0 \mathcal{M}_{n_k} + \mathcal{N}_{n_k} \overset{G}{\longrightarrow} \partial_0 \mathcal{M}_{hom,0} + \partial_0 \sum_{j=1}^\infty \left(-\sum_{\ell=1}^\infty \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell} \right)^j \mathcal{M}_{hom,0},$$

as $k \to \infty$ in $L^2_\nu(\mathbb{R};H)$, where

$$\mathcal{M}_{hom,0} = \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \mathcal{M}_{n_k}^{-1} \right)^{-1}$$

and

$$\mathcal{M}_{hom,\ell} = \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \mathcal{M}_{n_k}^{-1} \left(-\partial_0^{-1} \mathcal{N}_{n_k} \mathcal{M}_{n_k}^{-1} \right)^\ell.$$

Remark 2.6. (a) We note here that $\mathcal{M}_{hom,0}$ is indeed well-defined. Indeed, from $\mathfrak{Re}\,\mathcal{M}_n \geqq c$, we read off that $\mathcal{M}_n^{-1}$ exists and satisfies $\mathfrak{Re}\,\mathcal{M}_n^{-1} \geqq \frac{c}{\|\mathcal{M}_n\|^2}$ (see also Lemma 6.1 below). Now, for a suitable subsequence, the limit $\mathcal{O} := \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \mathcal{M}_{n_k}^{-1}$ satisfies $\mathfrak{Re}\,\mathcal{O} \geqq \frac{c}{\sup_n \|\mathcal{M}_n\|^2}$. Hence, $\mathcal{O}^{-1} = \mathcal{M}_{hom,0}$ is well-defined.

(b) It should be noted that the positive-definiteness condition in Theorem 2.7 is a well-posedness condition, i.e., a condition for $\partial_0\mathcal{M}_{n_k} + \mathcal{N}_{n_k}$ to be continuously invertible for all ν sufficiently large. Indeed, for $f \in L_\nu^2(\mathbb{R}; H)$ and $u \in L_\nu^2(\mathbb{R}; H)$ with

$$(\partial_0\mathcal{M}_n + \mathcal{N}_n)\, u = f$$

we multiply by ∂_0^{-1} and get $\left(\mathcal{M}_n + \partial_0^{-1}\mathcal{N}_n\right) u = \partial_0^{-1} f$. The positive definiteness condition yields, see also Lemma 6.1, the invertibility of $\mathcal{M}_n$. Hence, we arrive at

$$\left(1 + \mathcal{M}_n^{-1}\partial_0^{-1}\mathcal{N}_n\right) u = \mathcal{M}_n^{-1}\partial_0^{-1} f.$$

Choosing $\nu > 0$ sufficiently large, we deduce that the operator $\left(1 + \mathcal{M}_n^{-1}\partial_0^{-1}\mathcal{N}_n\right)$ is continuously invertible with a Neumann series expression.

(c) If $\mathcal{N} = 0$ in Theorem 2.5, then we deduce that equations of the form $\partial_0\mathcal{M}u = f$ are closed under the process of G-convergence. If $\mathcal{N} \neq 0$, then the above theorem suggests that this is not true for equations of the form $(\partial_0\mathcal{M} + \mathcal{N})\, u = f$. However, if we consider $\partial_0\mathcal{M} + \mathcal{N}$ as $\partial_0\left(\mathcal{M} + \partial_0^{-1}\mathcal{N}\right)$, the equations under consideration in Theorem 2.5 are closed under G-limits. Indeed, the limit may be represented by

$$\partial_0\left(\mathcal{M}_{hom,0} + \sum_{k=1}^{\infty}\left(-\sum_{\ell=1}^{\infty}\mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\right)^k \mathcal{M}_{hom,0}\right).$$

In the forthcoming sections we will further elaborate the aspect of closedness under G-limits.

In system or control theory one is interested in differential-algebraic systems, see e.g. [8]. We, thus, formulate the analogous statement for (integro-differential-)algebraic systems.

Theorem 2.7. *Let H_0, H_1 be separable Hilbert spaces, $\nu_0 > 0$. Further, let $(\mathcal{M}_n)_n, (\mathcal{N}_n^{ij})_n$ be bounded sequences in $L_{\mathrm{ev},\nu_0}(H_0)$ and $L_{\mathrm{ev},\nu_0}(H_j, H_i)$, respectively ($i, j \in \{0,1\}$). Assume there exists $c > 0$ such that for all $n \in \mathbb{N}$ and $\nu \geqq \nu_0$ we have for all $(\phi, \psi) \in L_\nu^2(\mathbb{R}; H_0 \oplus H_1)$*

$$\mathfrak{Re}\langle\mathcal{M}_n\phi, \phi\rangle_\nu \geqq c\,|\phi|_\nu^2, \quad \mathfrak{Re}\langle\mathcal{N}_n^{11}\psi, \psi\rangle_\nu \geqq c\,|\psi|_\nu^2.$$

Then there exists $\nu \geqq \nu_0$ *and a subsequence* $(n_k)_k$ *of* $(n)_n$ *such that*

$$\begin{aligned}
&\partial_0 \begin{pmatrix} \mathcal{M}_{n_k} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{N}_{n_k}^{00} & \mathcal{N}_{n_k}^{01} \\ \mathcal{N}_{n_k}^{10} & \mathcal{N}_{n_k}^{11} \end{pmatrix} \\
&\xrightarrow{G} \begin{pmatrix} \partial_0 & 0 \\ 0 & 1 \end{pmatrix} \Bigg(\begin{pmatrix} \mathcal{M}_{hom,0,00} & 0 \\ 0 & \mathcal{N}_{hom,-1,11} \end{pmatrix} \\
&\qquad + \sum_{\ell=1}^{\infty} \left(- \begin{pmatrix} \mathcal{M}_{hom,0,00} & 0 \\ 0 & \mathcal{N}_{hom,-1,11} \end{pmatrix} \mathcal{M}^{(1)} \right)^{\ell} \begin{pmatrix} \mathcal{M}_{hom,0,00} & 0 \\ 0 & \mathcal{N}_{hom,-1,11} \end{pmatrix} \Bigg),
\end{aligned}$$

where we put

$$\mathcal{N}_n \coloneqq \mathcal{N}_n^{00} - \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} \quad (n \in \mathbb{N})$$

as well as

$$\mathcal{M}^{(1)} \coloneqq \begin{pmatrix} \sum_{\ell=1}^{\infty} \mathcal{M}_{hom,\ell,00} & \sum_{\ell=0}^{\infty} \mathcal{M}_{hom,\ell,01} \\ \sum_{\ell=0}^{\infty} \mathcal{M}_{hom,\ell,10} & \sum_{\ell=0}^{\infty} \mathcal{M}_{hom,\ell,11} \end{pmatrix}$$

and

$$\begin{aligned}
\mathcal{M}_{hom,0,00} &= \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \mathcal{M}_{n_k}^{-1}\right)^{-1}, \\
\mathcal{M}_{hom,\ell,00} &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \mathcal{M}_{n_k}^{-1} \left(-\partial_0^{-1}\mathcal{N}_{n_k}\mathcal{M}_{n_k}^{-1}\right)^{\ell}, (\ell \geqq 1) \\
\mathcal{M}_{hom,\ell,01} &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} -\mathcal{M}_{n_k}^{-1} \left(-\partial_0^{-1}\mathcal{N}_{n_k}\mathcal{M}_{n_k}^{-1}\right)^{\ell} \partial_0^{-1}\mathcal{N}_{n_k}^{01} \left(\mathcal{N}_{n_k}^{11}\right)^{-1}, \\
\mathcal{M}_{hom,\ell,10} &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} - \left(\mathcal{N}_{n_k}^{11}\right)^{-1} \mathcal{N}_{n_k}^{10}\mathcal{M}_{n_k}^{-1} \left(-\partial_0^{-1}\mathcal{N}_{n_k}\mathcal{M}_{n_k}^{-1}\right)^{\ell}, \\
\mathcal{M}_{hom,\ell,11} &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \left(\mathcal{N}_{n_k}^{11}\right)^{-1} \mathcal{N}_{n_k}^{10}\mathcal{M}_{n_k}^{-1} \left(-\partial_0^{-1}\mathcal{N}_{n_k}\mathcal{M}_{n_k}^{-1}\right)^{\ell} \partial_0^{-1}\mathcal{N}_{n_k}^{01} \left(\mathcal{N}_{n_k}^{11}\right)^{-1}, \\
\mathcal{N}_{hom,-1,11} &= \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \left(\mathcal{N}_{n_k}^{11}\right)^{-1}\right)^{-1}.
\end{aligned}$$

Remark 2.8. (a) As in Theorem 2.5 the positive definiteness conditions in Theorem 2.7 serve as well-posed conditions for the respective (integro-differential-algebraic) equations. We will compute the respective inverse in the proof of Theorem 2.7.

(b) Note that, by the definition of G-convergence, both the Theorems 2.5 and 2.7 implicitly assert that the limit equations are well-posed, i.e., that the limit operator is continuously invertible. In fact it will be the strategy of the respective proofs to compute the limit of the respective solution operators, which will be continuous linear operators and afterwards inverting the limit.

(c) Assume that in Theorem 2.7 the expressions $\mathcal{M}_{hom,\ell,00}$, $\mathcal{M}_{hom,\ell,01}$, $\mathcal{M}_{hom,\ell,10}$, $\mathcal{M}_{hom,\ell,11}$, $\mathcal{N}_{hom,-1,11}$ can be computed without choosing subsequences. Then the sequence

$$\left(\partial_0 \begin{pmatrix} \mathcal{M}_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{N}_n^{00} & \mathcal{N}_n^{01} \\ \mathcal{N}_n^{10} & \mathcal{N}_n^{11} \end{pmatrix}\right)_n$$

is G-convergent. Indeed, the latter follows with a subsequence argument.

(d) Assuming $H_1 = \{0\}$ and, as a consequence, $\mathcal{N}^{ij} = 0$ for all $i, j \in \{0, 1\}$ except $i = j = 0$, we see that Theorem 2.7 is more general than Theorem 2.5. The generalization in Theorem 2.7 is also needed in the theory of homogenization of partial differential equations, see e.g. [29, Theorem 4.4] for a more restrictive case. We give an example in the forthcoming sections.

For convenience, we include easy examples that show that the assumptions in the above theorems are reasonable.

Example 2.9 (Uniform positive definiteness condition does not hold, [29])**.** Let $H = \mathbb{C}$, $\nu > 0$ and, for $n \in \mathbb{N}$, let $\mathcal{M}_n = \partial_0^{-1}\frac{1}{n}$, $f \in L_\nu^2(\mathbb{R}) \setminus \{0\}$. For $n \in \mathbb{N}$, let $u_n \in L_\nu^2(\mathbb{R})$ be defined by

$$\partial_0 \mathcal{M}_n u_n = \frac{1}{n} u_n = f.$$

Then $(u_n)_n$ is not relatively weakly compact and contains no weakly convergent subsequence.

Example 2.10 (Boundedness assumption does not hold)**.** Let $H = \mathbb{C}$, $\nu > 0$ and, for $n \in \mathbb{N}$, let $\mathcal{M}_n = \partial_0^{-1} n$, $f \in L_\nu^2(\mathbb{R})$. For $n \in \mathbb{N}$, let $u_n \in L_\nu^2(\mathbb{R})$ be defined by

$$\partial_0 \mathcal{M}_n u_n = n u_n = f.$$

Then $(u_n)_n$ converges to 0. Thus, a limit "equation" would be in fact the relation $\{0\} \times L_\nu^2(\mathbb{R}) \subsetneqq L_\nu^2(\mathbb{R}) \oplus L_\nu^2(\mathbb{R})$.

We will now apply our main theorems to particular situations.

3. Time-independent coefficients

In this section, we treat time-independent coefficients. That is to say, we assume that the operators in the sequences under consideration only act on the "spatial" Hilbert spaces H_0 and H_1 in Theorem 2.7 or H in Theorem 2.5. More precisely and similar to Example 2.2, for a bounded linear operator $M \in L(H_0, H_1)$ there is a (canonical) extension to L_ν^2-functions in the way that $(M\phi)(t) := M(\phi(t))$ for $\phi \in L_\nu^2(\mathbb{R}; H_0)$ and a.e. $t \in \mathbb{R}$. Thus M is evolutionary (Example 2.2). The main property exploited here is that the (canonical) extensions commute with ∂_0^{-1}. We only state the specialization of this situation for Theorem 2.5. The result reads as follows.

Corollary 3.1. *Let H be a separable Hilbert space, $\nu_0 > 0$. Let $(M_n)_n$, $(N_n)_n$ be bounded sequences in $L(H)$. Assume there exists $c > 0$ such that for all $n \in \mathbb{N}$*

$$\mathfrak{Re}\langle M_n\phi, \phi\rangle_H \geqq c\langle \phi, \phi\rangle_H \quad (\phi \in H).$$

Then there exists $\nu \geqq \nu_0$ and a subsequence $(n_k)_k$ of $(n)_n$ such that

$$\partial_0 M_{n_k} + N_{n_k} \xrightarrow{G} \partial_0 M_{hom,0} + \partial_0 \sum_{j=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} M_{hom,0} M_{hom,\ell} \left(-\partial_0^{-1}\right)^{\ell} \right)^j M_{hom,0},$$

as $k \to \infty$ in $L^2_\nu(\mathbb{R}; H)$, where

$$M_{hom,0} = \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}^{-1}\right)^{-1} \quad \textit{and} \quad M_{hom,\ell} = \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}^{-1} \left(N_{n_k} M_{n_k}^{-1}\right)^{\ell}.$$

Proof. Recall that $M\partial_0^{-1} = \partial_0^{-1}M$ for all bounded linear operators $M \in L(H)$. Moreover, the estimate $\mathfrak{Re}\langle M\phi, \phi\rangle_H \geqq c\langle \phi, \phi\rangle_H$ for $\phi \in H$ also carries over to the analogous one for $\phi \in L^2_\nu(\mathbb{R}; H)$ and the extended M. Hence, the result follows from Theorem 2.5. □

Remark 3.2. As it has already been observed in [14, 25], the class of equations treated in Corollary 3.1 is *not* closed under G-convergence in general. The next example shows that this effect only occurs if the Hilbert space H is infinite-dimensional and the convergence of $(M_n)_n$ and $(N_n)_n$ is "weak enough" in a sense to be specified below.

Example 3.3. Assume for the moment that H is finite-dimensional. Then $(M_n)_n$ and $(N_n)_n$ are a mere bounded sequences of matrices with constant coefficients. In particular, the weak operator topology coincides with the topology induced by the operator norm. Hence, the processes of computing the inverse and computing the limit interchange and multiplication is a continuous process as well. Thus, assuming $(M_n)_n$ and $(N_n)_n$ to be convergent with the respective limits M and N, we compute

$$\begin{aligned}
&\partial_0 M_{hom,0} + \partial_0 \sum_{j=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} M_{hom,0} M_{hom,\ell} \left(-\partial_0^{-1}\right)^{\ell} \right)^j M_{hom,0} \\
&= \partial_0 \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}^{-1}\right)^{-1} \\
&\quad + \partial_0 \sum_{j=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}^{-1}\right)^{-1} \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}^{-1} \left(N_{n_k} M_{n_k}^{-1}\right)^{\ell}\right) \left(-\partial_0^{-1}\right)^{\ell} \right)^j \\
&\quad \times \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}^{-1}\right)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \partial_0 \left(\lim_{k\to\infty} M_{n_k}^{-1} \right)^{-1} \\
&\quad + \partial_0 \sum_{j=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} \left(\lim_{k\to\infty} M_{n_k}^{-1} \right)^{-1} \left(\lim_{k\to\infty} M_{n_k}^{-1} \left(N_{n_k} M_{n_k}^{-1} \right)^{\ell} \right) \left(-\partial_0^{-1} \right)^{\ell} \right)^{j} \\
&\quad \times \left(\lim_{k\to\infty} M_{n_k}^{-1} \right)^{-1} \\
&= \partial_0 M + \partial_0 \sum_{j=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} M \left(M^{-1} \left(N M^{-1} \right)^{\ell} \right) \left(-\partial_0^{-1} \right)^{\ell} \right)^{j} M \\
&= \partial_0 M + \partial_0 \sum_{j=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} \left(N M^{-1} \right)^{\ell} \left(-\partial_0^{-1} \right)^{\ell} \right)^{j} M \\
&= \partial_0 \sum_{j=0}^{\infty} \left(- \sum_{\ell=1}^{\infty} \left(N M^{-1} \right)^{\ell} \left(-\partial_0^{-1} \right)^{\ell} \right)^{j} M \\
&= \partial_0 \left(1 + \sum_{\ell=1}^{\infty} \left(N M^{-1} \right)^{\ell} \left(-\partial_0^{-1} \right)^{\ell} \right)^{-1} M = \partial_0 \left(\sum_{\ell=0}^{\infty} \left(-N M^{-1} \partial_0^{-1} \right)^{\ell} \right)^{-1} M \\
&= \partial_0 \left(\left(1 + N M^{-1} \partial_0^{-1} \right)^{-1} \right)^{-1} M \\
&= \partial_0 \left(M + N \partial_0^{-1} \right) \\
&= \partial_0 M + N.
\end{aligned}
$$

Thus, in finite-dimensional spaces, the above theorem restates the continuous dependence of the solution on the coefficients. Note that we only used that multiplication and computing the inverse are continuous operations. Hence, the above calculation literally expresses the fact of continuous dependence on the coefficients if H is infinite-dimensional and the sequences $(M_n)_n$ and $(N_n)_n$ converge in the strong operator topology. Thus, one can only expect that the limit expression differs from the one, which one might expect, if the actual convergence of the operators involved is strictly weaker than in the strong operator topology.

We will turn to a more sophisticated example. For this we recall the concept of periodicity in $\mathbb{R}^n$, see e.g. [4].

Definition 3.4. Let $a \colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ be bounded and measurable. a is called $]0,1[^n$-*periodic*, if for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$ we have $a(x+k) = a(x)$.

Moreover, recall the following well-known convergence result on periodic mappings, cf. e.g. [4, Theorem 2.6].

Theorem 3.5. *Let* $a\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ *be bounded and measurable and* $]0,1[^n$*-periodic. Then* $(a(k\cdot))_k$ *converges in* $L^\infty(\mathbb{R}^n)^{m\times m}$ $*$*-weakly to the integral mean* $\int_{[0,1]^n} a(y)dy$*.*

Remark 3.6. For any bounded measurable function $a\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ one can associate the corresponding multiplication operator in $L^2(\mathbb{R}^n)^m$. Hence, Theorem 3.5 states the fact that in case of periodic a the sequence of associated multiplication operators of $a(k\cdot)$ converges in the weak operator topology to the operator of multiplying with the respective integral mean. Indeed, this follows easily from $L^2(\mathbb{R}^n)\cdot L^2(\mathbb{R}^n) = L^1(\mathbb{R}^n)$. See also [5, Chapter IX, Theorem 8.10] .

Example 3.7. Let $H = L^2(\mathbb{R}^n)^m$ and let $a,b\colon \mathbb{R}^n \to \mathbb{C}^{m\times m}$ be bounded, measurable and $]0,1[^n$-periodic. We assume $\mathfrak{Re}\, a(x) \geqq c$ for all $x\in\mathbb{R}^n$. Observe that any polynomial in a and b is $]0,1[^n$-periodic and so is $a^{-1} := (x\mapsto a(x)^{-1})$. Thus, by Corollary 3.1, we deduce that

$$\begin{aligned}
&\partial_0 a(k\cdot) + b(k\cdot)\\
&\xrightarrow{G} \partial_0 \left(\int_{[0,1]^n} a(y)^{-1}dy\right)^{-1}\\
&\quad + \partial_0 \sum_{j=1}^{\infty}\left(-\sum_{\ell=1}^{\infty}\left(\int_{[0,1]^n} a(y)^{-1}dy\right)^{-1}\int_{[0,1]^n} a(y)^{-1}\left(b(y)a(y)^{-1}\right)^{\ell} dy\left(-\partial_0^{-1}\right)^{\ell}\right)^{j}\\
&\quad \times \left(\int_{[0,1]^n} a(y)^{-1}dy\right)^{-1},
\end{aligned}$$

as $k\to\infty$ in $L^2_\nu(\mathbb{R};H)$.

Remark 3.8. In [24, Theorem 1.2] or [25] the author considers the equation $(\partial_0 - b_k(\cdot))\,u_k = f$ with $(b_k)_k$ being a $[\alpha,\beta]$-valued (for some $\alpha,\beta\in\mathbb{R}$) sequence of bounded, measurable mappings depending on one spatial variable. Also in that exposition a memory effect is derived. However, the method uses the concept of Young measures. The reason for that is the representation of the solution being a function of the oscillating coefficent. More precisely, the convergence of the sequence $\left(e^{tb(k\cdot)}\right)_k$ is addressed. In order to let k tend to infinity in this expression one needs a result on the (weak-$*$) convergence of (continuous) functions of bounded functions. This is where the Young-measures come into play, see e.g. [2, Section 2] and the references therein or [25, p. 930]. The result used is the following. There exists a family of probabilty measures $(\nu_x)_x$ supported on $[\alpha,\beta]$ such that for (a subsequence of) $(k)_k$ and all real continuous functions G we have

$$G\circ b_k(\cdot) \to \left(\mathbb{R}\ni x\mapsto \int_{[\alpha,\beta]} G(\lambda)d\nu_x(\lambda)\right)$$

as $k \to \infty$ in $L^\infty(\mathbb{R})$ $*$-weakly. The family $(\nu_x)_x$ is also called the *Young-measure associated to* $(b_k)_k$. With the help of the family $(\nu_x)_x$ a convolution kernel is computed such that the respective limit equation can be written as

$$\partial_0 u(t,x) + b^0(x)u - \int_0^t K(x,t-s)u(x,s)ds = f(x,t),$$

where b^0 is a weak-$*$-limit of a subsequence of $(b_k)_k$ and

$$K(x,t) = \int_{\mathbb{R}_{>0}} e^{-\lambda t} d\nu_x(\lambda)$$

for a.e. $t \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$. The relation to our above considerations is as follows. The resulting limit equation within our approach can also be considered as an ordinary differential equation perturbed by a convolution term. Denoting limits with respect to the $\sigma(L_\infty, L_1)$-topology by $*$-lim, we realize that Corollary 3.1 in this particular situation states that the limit equation admits the form

$$\partial_0 + \partial_0 \sum_{k=1}^\infty \left(-\sum_{\ell=1}^\infty *\text{-}\lim_{k\to\infty} (b_k)^\ell \left(-\partial_0^{-1}\right)^\ell \right)^k$$
$$= \partial_0 + b^0 + \sum_{\ell=2}^\infty *\text{-}\lim_{k\to\infty} (b_k)^\ell \left(-\partial_0^{-1}\right)^{\ell-1} + \partial_0 \sum_{k=2}^\infty \left(-\sum_{\ell=1}^\infty *\text{-}\lim_{k\to\infty} (b_k)^\ell \left(-\partial_0^{-1}\right)^\ell \right)^k$$

as $k \to \infty$ in $L^2_\nu(\mathbb{R}; L^2(\mathbb{R}))$. Indeed, using [18, 6.2.6. Memory Problems, (b) p. 448] or [26, Theorem 1.5.6 and Remark 1.5.7], we deduce that the term

$$\sum_{\ell=2}^\infty *\text{-}\lim_{k\to\infty} (b_k)^\ell \left(-\partial_0^{-1}\right)^{\ell-1} + \partial_0 \sum_{k=2}^\infty \left(-\sum_{\ell=1}^\infty *\text{-}\lim_{k\to\infty} (b_k)^\ell \left(-\partial_0^{-1}\right)^\ell \right)^k$$

can be represented as a (temporal) convolution. Moreover, note that the choice of subsequences is the same. Indeed, in the above rationale with the Young measure approach, by a density argument, it suffices to choose a subsequence of $(b_k)_k$ such that any polynomial of $(b_k)_k$ converges $*$-weakly. This choice of subsequences also suffices to deduce G-convergence of the respective equations within the operator-theoretic perspective treated in this exposition.

Remark 3.9 (On initial value problems and semigroups associated to them)**.** In view of the latter remark we illustrate how initial value problems may be treated here. In its abstract form, for some $M \in L(H)$, H a separable Hilbert space, the equations discussed in Remark 3.8 may be written as follows

$$(\partial_0 + M)u = f.$$

for some right-hand side f. The initial value problem can be formulated in the distribution space associated to ∂_0 (i.e. Sobolev chain or Sobolev tower, see e.g. [11, 18]). More precisely, one can show that the Dirac-distribution δ of evaluation at 0 is an element of $H_{-1}(\partial_0)$, the completion of $L^2_\nu(\mathbb{R}; H)$ with respect to the norm $u \mapsto |\partial_0^{-1}u|_\nu$ or, equivalently, δ is a continuous linear functional on $H_1(\partial_0) \coloneqq D(\partial_0)$ endowed with the graph norm. Imposing the initial datum $u_0 \in H$, we rearrange the equation by replacing f by δu_0:

$$\partial_0 u + Mu = \delta u_0.$$

It can then be shown that u is actually continuous on $\mathbb{R}_{\geqq 0}$ and that $u(0+) = u_0$, see [11]. Turning back to the problem discussed in Remark 3.8 in its abstract form, we would then consider a bounded sequence $(M_k)_k$ in $L(H)$. The associated semigroups are then $(t \mapsto e^{-tM_k})_k$. Or, in the context of the present work, simply

$$t \mapsto \big(H \ni u_0 \mapsto \big((\partial_0 + M_k)^{-1}\delta u_0\big)(t)\big).$$

Now, in Remark 3.8, we have seen that, in general, the G-limit of $(\partial_0 + M_k)_k$ is of integro-differential type and thus cannot be written as a semigroup $(t \mapsto e^{-tM})$ for some $M \in L(H)$ even though one assumes that $(M_k)_k$ G-converges to some M. In fact, it can be shown that $(\partial_0 + M_k)_k$ is G-convergent *if and only if* for all $\ell \in \mathbb{N}$ the sequence $(M_k^\ell)_k$ converges in the weak operator topology, see [19, Section 2].

In the next example, we consider a partial differential equation, which can be reformulated as *ordinary* differential equation in an infinite-dimensional Hilbert space. More precisely, we treat Maxwell's equations with the Drude-Born-Fedorov material model, see e.g. [7]. In order to discuss this equation properly, we need to introduce several operators from vector analysis.

Definition 3.10. Let $\Omega \subseteqq \mathbb{R}^3$ be open. Then we define[4]

$$\begin{aligned}\operatorname{curl}_c\colon C_{\infty,c}(\Omega)^3 \subseteqq L^2(\Omega)^3 &\to L^2(\Omega)^3\\ \phi &\mapsto \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}\phi,\end{aligned}$$

where we denote by ∂_i the partial derivative with respect to the i'th variable, $i \in \{1, 2, 3\}$. Moreover, introduce

$$\begin{aligned}\operatorname{div}_c\colon C_{\infty,c}(\Omega)^3 \subseteqq L^2(\Omega)^3 &\to L^2(\Omega)\\ (\phi_1, \phi_2, \phi_3) &\mapsto \sum_{i=1}^3 \partial_i \phi_i.\end{aligned}$$

[4]We denote by $C_{\infty,c}(\Omega)$ the set of arbitrarily often differentiable functions with compact support in Ω.

We define $\mathrm{curl}_0 := \overline{\mathrm{curl}_c}$, $\mathrm{div}_0 := \overline{\mathrm{div}_c}$. The 0 serves as a reminder for (the generalization of) the electric and the Neumann boundary condition, respectively. If Ω is simply connected, we also introduce

$$\begin{aligned} \mathrm{curl}_\diamond \colon D(\mathrm{curl}_\diamond) \subseteqq L^2(\Omega)^3 &\to L^2(\Omega)^3 \\ \phi &\mapsto \mathrm{curl}\,\phi, \end{aligned}$$

where $D(\mathrm{curl}_\diamond) := \{\phi \in D(\mathrm{curl}); \mathrm{curl}\,\phi \in D(\mathrm{div}_0)\}$.

Remark 3.11. It can be shown that if Ω is simply connected with finite measure, then $\mathrm{curl}_\diamond$ is a selfadjoint operator, see [6, 7]. In that reference it is also stated that $\mathrm{curl}_\diamond$ has, except 0, only discrete spectrum. In particular, this means that the intersection of the resolvent set of $\mathrm{curl}_\diamond$, $\varrho(\mathrm{curl}_\diamond)$, with the set $\mathbb{R}$ is nonempty. For other geometric properties of Ω resulting in the selfadjointness of $\mathrm{curl}_\diamond$, we refer to [17].

We now treat a homogenization problem of the Drude-Born-Fedorov model as treated in [7].

Example 3.12. Assume that $\Omega \subseteqq \mathbb{R}^3$ is open, simply connected and has finite Lebesgue measure. Invoking Remark 3.11 and following [7, Theorem 2.1], the equation

$$\left(\partial_0\,(1+\eta\,\mathrm{curl}_\diamond)\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{curl}_\diamond \\ \mathrm{curl}_\diamond & 0 \end{pmatrix}\right)\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} J \\ 0 \end{pmatrix} \tag{3}$$

for $\eta \in \mathbb{R}$ such that $-\frac{1}{\eta} \in \varrho(\mathrm{curl}_\diamond)$, $J \in L^2_\nu(\mathbb{R}; L^2(\Omega)^3)$ and given $\varepsilon, \mu \in L(L^2(\Omega)^3)$ being strictly positive selfadjoint operators, admits a unique solution $(E, H) \in H_{\nu,1}(\mathbb{R}; L^2(\Omega)^3)$.[5] Indeed, multiplying (3) by $(1 + \eta\,\mathrm{curl}_\diamond)^{-1}$, we get that

$$\left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \mathrm{curl}_\diamond\,(1+\eta\,\mathrm{curl}_\diamond)^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} E \\ H \end{pmatrix} = (1+\eta\,\mathrm{curl}_\diamond)^{-1}\begin{pmatrix} J \\ 0 \end{pmatrix}.$$

Realizing that $\mathrm{curl}_\diamond\,(1 + \eta\,\mathrm{curl}_\diamond)^{-1}$ is a *bounded* linear operator by the spectral theorem for the selfadjoint operator $\mathrm{curl}_\diamond$, we get that $(E, H) \in H_{\nu,1}(\mathbb{R}; L^2(\Omega)^3)$ solves the above equation. Note that the equation derived from (3) is a mere ordinary differential equation in an infinite-dimensional Hilbert space. Assume we are given bounded sequences of selfadjoint operators $(\varepsilon_n)_n$ and $(\mu_n)_n$ satisfying $\varepsilon_n \geqq c$ and $\mu_n \geqq c$ for some $c > 0$ and all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ we consider the problem

$$\left(\partial_0 \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \mu_n \end{pmatrix} + \mathrm{curl}_\diamond\,(1+\eta\,\mathrm{curl}_\diamond)^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} E_n \\ H_n \end{pmatrix} = (1+\eta\,\mathrm{curl}_\diamond)^{-1}\begin{pmatrix} J \\ 0 \end{pmatrix}$$

[5]Note that for $(E, H) \in H_{\nu,1}(\mathbb{R}; L^2(\Omega)^3)$ being a solution of (3) can only be true in the distributional sense, which can be made more precise with the help of the extrapolation spaces of $\mathrm{curl}_\diamond$. We shall, however, not follow this reasoning here in more details and refer again to [7] or [18, Chapter 2].

and address the question of G-convergence of (a subsequence of)

$$(\mathrm{DBF}_n)_n := \left(\partial_0 \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \mu_n \end{pmatrix} + \operatorname{curl}_\diamond (1+\eta \operatorname{curl}_\diamond)^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)_n .$$

Clearly, Corollary 3.1 applies and we get that (a subsequence of) $(\mathrm{DBF}_n)_n$ G-converges to

$$\partial_0 M_{hom,0} + \partial_0 \sum_{k=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} M_{hom,0} M_{hom,\ell} \left(-\partial_0^{-1}\right)^{\ell} \right)^k M_{hom,0},$$

as $k \to \infty$ in $L^2_\nu(\mathbb{R}; H)$, where

$$M_{hom,0} = \left(\tau_{\mathrm{w}}\text{-} \lim_{k\to\infty} \begin{pmatrix} \varepsilon_{n_k}^{-1} & 0 \\ 0 & \mu_{n_k}^{-1} \end{pmatrix} \right)^{-1}$$

and

$$M_{hom,\ell} = \tau_{\mathrm{w}}\text{-} \lim_{k\to\infty} \begin{pmatrix} \varepsilon_{n_k}^{-1} & 0 \\ 0 & \mu_{n_k}^{-1} \end{pmatrix} \left(\operatorname{curl}_\diamond (1+\eta \operatorname{curl}_\diamond)^{-1} \begin{pmatrix} 0 & -\mu_{n_k}^{-1} \\ \varepsilon_{n_k}^{-1} & 0 \end{pmatrix} \right)^{\ell} .$$

We have seen that the class of problems discussed in Corollary 3.1 in this section is *not* closed under the G-convergence, unless $N = 0$.

4. Time-translation invariant coeffcients

In Corollary 3.1, we have seen that the limit equation can be described as a power series expression in ∂_0^{-1}. A possible way to generalize this is the introduction of holomorphic functions in ∂_0^{-1}, see [18, Section 6.1, p. 427]. To make this precise, we need the spectral representation for ∂_0^{-1}, the Fourier-Laplace transform $\mathcal{L}_\nu$, which is given as the unitary operator being the closure of

$$\begin{aligned} C_{\infty,c}(\mathbb{R}) \subseteqq L^2_\nu(\mathbb{R}) &\to L^2_\nu(\mathbb{R}) \\ \phi &\mapsto \left(x \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy-\nu y} \phi(y) dy \right). \end{aligned}$$

Denoting by $m\colon D(m) \subseteqq L^2(\mathbb{R}) \to L^2(\mathbb{R}), f \mapsto (x \mapsto xf(x))$ the multiplication-by-argument-operator with maximal domain $D(m)$, we arrive at the representation

$$\partial_0^{-1} = \mathcal{L}_\nu^* \left(\frac{1}{im+\nu} \right) \mathcal{L}_\nu .$$

Thus, for bounded and analytic functions $M\colon B(r,r) \to \mathbb{C}$ with $r > \frac{1}{2\nu}$ we define

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\nu^* M \left(\frac{1}{im+\nu} \right) \mathcal{L}_\nu ,$$

where $\left(M\left(\frac{1}{im+\nu}\right)\phi\right)(t) := M\left(\frac{1}{it+\nu}\right)\phi(t)$ for $\phi \in L^2(\mathbb{R})$ and a.e. $t \in \mathbb{R}$. We canonically extend the above definitions to the case of vector-valued functions $L^2_\nu(\mathbb{R}; H)$ with values in a Hilbert space H. In this way, the definition of $M\left(\partial_0^{-1}\right)$ can be generalized to bounded and operator-valued functions $M : B(r,r) \to L(H_0, H_1)$ for Hilbert spaces H_0 and H_1. We denote

$$\mathcal{H}^\infty(B(r,r); L(H_0,H_1)) := \{M : B(r,r) \to L(H_0,H_1); M \text{ bounded, analytic}\}.$$

A subset $\mathfrak{M} \subseteqq \mathcal{H}^\infty(B(r,r); L(H_0,H_1))$ is called *bounded*, if

$$\sup\{\|M(z)\|; z \in B(r,r), M \in \mathfrak{M}\} < \infty.$$

A family $(M_\iota)_{\iota\in I}$ in $\mathcal{H}^\infty(B(r,r); L(H_0,H_1))$ is *bounded*, if $\{M_\iota; \iota \in I\}$ is bounded.

We will treat some examples for $\mathcal{H}^\infty$-functions of ∂_0^{-1} below, see also [27]. In this reference, a homogenization theorem of problems of the kind treated in Theorem 2.5 with $(\mathcal{M}_n)_n = \left(M_n\left(\partial_0^{-1}\right)\right)_n$ for a bounded sequence $(M_n)_n$ in $\mathcal{H}^\infty$ has been presented, see [27, Theorem 5.2]. Moreover, in [29, Theorem 4.4] a special case of an analogous result of Theorem 2.7 has been presented and used. In order to state a G-convergence theorem in a more general situation, note that $\left\{M\left(\partial_0^{-1}\right); M \in \mathcal{H}^\infty(B(r,r); L(H_0,H_1))\right\} \subseteqq \bigcap_{\frac{1}{2r}<\nu} L_{\mathrm{ev},\nu}(H_0,H_1)$. The theorem reads as follows.

Theorem 4.1. *Let H_0, H_1 be separable Hilbert spaces, $\nu_0 > 0$, $r > \frac{1}{2\nu_0}$. Let $(M_n)_n, (N_n^{ij})_n$ be bounded sequences in*

$$\mathcal{H}^\infty(B(r,r); L(H_0)) \text{ and } \mathcal{H}^\infty(B(r,r); L(H_j,H_i)),$$

respectively ($i,j \in \{0,1\}$). Assume there exists $c > 0$ such that for all $n \in \mathbb{N}$ we have for all $(\phi,\psi) \in H_0 \oplus H_1$ and $z \in B(r,r)$

$$\mathfrak{Re}\langle M_n(z)\phi,\phi\rangle_{H_0} \geqq c\,|\phi|^2_{H_0}, \quad \mathfrak{Re}\langle N_n^{11}(z)\psi,\psi\rangle_{H_1} \geqq c\,|\psi|^2_{H_1}.$$

Then there exists $\nu > \nu_0$ and a subsequence $(n_k)_k$ of $(n)_n$ such that

$$\begin{aligned}
&\partial_0 \begin{pmatrix} M_{n_k}\left(\partial_0^{-1}\right) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N^{00}_{n_k}\left(\partial_0^{-1}\right) & N^{01}_{n_k}\left(\partial_0^{-1}\right) \\ N^{10}_{n_k}\left(\partial_0^{-1}\right) & N^{11}_{n_k}\left(\partial_0^{-1}\right) \end{pmatrix} \\
&\xrightarrow{G} \begin{pmatrix} \partial_0 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} M_{hom,0,00}\left(\partial_0^{-1}\right) & 0 \\ 0 & N_{hom,-1,11}\left(\partial_0^{-1}\right) \end{pmatrix} \right. \\
&\quad + \sum_{\ell=1}^\infty \left(- \begin{pmatrix} M_{hom,0,00}\left(\partial_0^{-1}\right) & 0 \\ 0 & N_{hom,-1,11}\left(\partial_0^{-1}\right) \end{pmatrix} M^{(1)}\left(\partial_0^{-1}\right) \right)^\ell \\
&\quad \times \left. \begin{pmatrix} M_{hom,0,00}\left(\partial_0^{-1}\right) & 0 \\ 0 & N_{hom,-1,11}\left(\partial_0^{-1}\right) \end{pmatrix} \right),
\end{aligned}$$

where we put

$$N_n := N_n^{00} - N_n^{01} \left(N_n^{11}\right)^{-1} N_n^{10} \quad (n \in \mathbb{N})$$

as well as

$$M^{(1)}\left(\partial_0^{-1}\right) := \begin{pmatrix} \sum_{\ell=1}^{\infty} M_{hom,\ell,00}\left(\partial_0^{-1}\right)\left(\partial_0^{-1}\right)^{\ell} & \sum_{\ell=0}^{\infty} M_{hom,\ell,01}\left(\partial_0^{-1}\right)\left(\partial_0^{-1}\right)^{\ell+1} \\ \sum_{\ell=0}^{\infty} M_{hom,\ell,10}\left(\partial_0^{-1}\right)\left(\partial_0^{-1}\right)^{\ell} & \sum_{\ell=0}^{\infty} M_{hom,\ell,11}\left(\partial_0^{-1}\right)\left(\partial_0^{-1}\right)^{\ell+1} \end{pmatrix}$$

and

$$\begin{aligned} M_{hom,0,00}(z) &= \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}(z)^{-1}\right)^{-1}, \\ M_{hom,\ell,00}(z) &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} M_{n_k}(z)^{-1}\left(-N_{n_k}(z)M_{n_k}(z)^{-1}\right)^{\ell}, \ell \geqq 1, \\ M_{hom,\ell,01}(z) &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} -M_{n_k}(z)^{-1}\left(-N_{n_k}(z)M_{n_k}(z)^{-1}\right)^{\ell} N_{n_k}^{01}(z)\left(N_{n_k}^{11}(z)\right)^{-1}, \\ M_{hom,\ell,10}(z) &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} -\left(N_{n_k}^{11}(z)\right)^{-1} N_{n_k}^{10}(z)M_{n_k}(z)^{-1}\left(-N_{n_k}(z)M_{n_k}(z)^{-1}\right)^{\ell}, \\ M_{hom,\ell,11}(z) &= \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \left(N_{n_k}^{11}(z)\right)^{-1} N_{n_k}^{10}(z)M_{n_k}(z)^{-1}\left(-N_{n_k}(z)M_{n_k}(z)^{-1}\right)^{\ell} \\ &\qquad \times N_{n_k}^{01}(z)\left(N_{n_k}^{11}(z)\right)^{-1}, \\ N_{hom,-1,11}(z) &= \left(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty}\left(N_{n_k}^{11}(z)\right)^{-1}\right)^{-1}, \end{aligned}$$

for all $z \in B\left(\frac{1}{2\nu_1}, \frac{1}{2\nu_1}\right)$ *for some* $\nu > \nu_1 \geqq \nu_0$.

Proof. Observe that bounded and analytic functions of ∂_0^{-1} commute with ∂_0^{-1}. Note that the only thing left to prove is that the operator-valued functions involved are indeed analytic functions of ∂_0^{-1}. For this we need to introduce a topology on $\mathcal{H}^\infty(B(r,r); L(H_0, H_1))$. Let τ be the topology induced by the mappings

$$\begin{aligned} \mathcal{H}^\infty(B(r,r); L(H_0, H_1)) &\to \mathcal{H}(B(r,r)) \\ M &\mapsto \langle \phi, M(\cdot)\psi \rangle, \end{aligned}$$

for all $(\phi, \psi) \in H_1 \oplus H_0$, where $\mathcal{H}(B(r,r))$ is the set of analytic functions endowed with the compact open topology. In [27, Theorem 3.4] or [30, Theorem 4.3] it is shown that closed and bounded subsets of $\mathcal{H}^\infty(B(r,r); L(H_0, H_1))$ are sequentially compact with respect to the τ-topology. Moreover, by the separability of $L_\nu^2(\mathbb{R}; H_0 \oplus H_1)$, closed and bounded subsets of $L\left(L_\nu^2(\mathbb{R}; H_0 \oplus H_1)\right)$ are sequentially compact with respect to the weak operator topology. Furthermore, by [27, Lemma 3.5], we have that if a bounded sequence $(T_n)_n$ in $\mathcal{H}^\infty(B(r,r); L(H_0, H_1))$ converges in the τ-topology then the operator sequence $\left(T_n\left(\partial_0^{-1}\right)\right)_n$ converges in the weak operator topology of $L\left(L_\nu^2(\mathbb{R}; H_0 \oplus H_1)\right)$. Putting all this together, we deduce that the assertion follows from Theorem 2.7. □

Remark 4.2. (a) Theorem 4.1 asserts that the time-translation invariant equations under consideration are closed under G-convergence. Though the formulas may become a bit cluttered, in principle, an iterated homogenization procedure is possible.

(b) In [29, Theorem 4.4] operator-valued functions that are analytic at 0 were treated. This assumption can be lifted. Indeed, we only require analyticity of the operator-valued functions under consideration on the open ball $B(r,r)$ for some radius $r > 0$ and do not assume that any of these functions have holomorphic extensions to 0.

We give several examples.

Example 4.3. Let $\nu_0 > 0$. In this example we treat integral equations of convolution type. Let $(g_n)_n$ be a bounded sequence in $L^1_{\nu_0}(\mathbb{R}_{>0})$ such that there is $h \in L^1_{\nu_0}(\mathbb{R}_{>0})$ with $\|g(t)\| \leqq h(t)$ for all $n \in \mathbb{N}$ and a.e. $t \in \mathbb{R}$. For $f \in C_{\infty,c}(\mathbb{R})$ consider the equation

$$u_n + g_n * u_n = f. \tag{4}$$

The latter equation fits into the scheme of Theorem 4.1 for $H = \mathbb{C}$. Indeed, using that the Fourier transform $\mathcal{F}$ translates convolutions into multiplication, we get for any $g \in L^1_\nu(\mathbb{R}_{>0})$ and $u \in L^2_\nu(\mathbb{R})$ for some $\nu > \nu_0$ that

$$\begin{aligned} g * u &= \sqrt{2\pi}\mathcal{L}_\nu^*\mathcal{L}_\nu g(m)\mathcal{L}_\nu u \\ &= \sqrt{2\pi}\mathcal{L}_\nu^*\left(\mathcal{F}g\right)(m - i\nu)\mathcal{L}_\nu u \\ &= \sqrt{2\pi}\mathcal{L}_\nu^*\left(\mathcal{F}g\right)\left(-i\frac{1}{(im+\nu)^{-1}}\right)\mathcal{L}_\nu u. \end{aligned}$$

The support and integrability condition of g implies analyticity of

$$M_g := \sqrt{2\pi}\left(\mathcal{F}g\right)\left(-i\frac{1}{(\cdot)}\right)$$

on $B(r,r)$ for $0 < r < \frac{1}{2\nu_0}$. The computation also shows that

$$\begin{aligned} |g * u|_\nu^2 &= \left|\sqrt{2\pi}\left(\mathcal{F}g\right)\left(-i\frac{1}{(im+\nu)^{-1}}\right)\mathcal{L}_\nu u\right|_{L^2}^2. \\ &\leqq 2\pi\left|\left(\mathcal{F}g\right)\left(-i\frac{1}{(i(\cdot)+\nu)^{-1}}\right)\right|_\infty^2 |\mathcal{L}_\nu u|_{L^2}^2 \\ &\leqq 2\pi\left|\left(\mathcal{F}g\right)((\cdot) - i\nu)\right|_\infty^2 |u|_\nu^2, \end{aligned}$$

where $2\pi\left|\left(\mathcal{F}g\right)((\cdot) - i\nu)\right|_\infty^2 := \sup_{t\in\mathbb{R}} 2\pi\left|\left(\mathcal{F}g\right)(t - i\nu)\right|^2 = \left|\int_{\mathbb{R}} e^{-i(t-i\nu)y}g(y)dy\right|^2$ $\leqq \left(\int_{\mathbb{R}} e^{-\nu y}|g(y)|\,dy\right)^2$, which tends to zero, if $\nu \to \infty$. Thus, by our assumption on the sequence $(g_n)_n$ having the uniform majorizing function h, there exists $\nu_1 > 0$ such that we have

$$\varepsilon := \sup_{n\in\mathbb{N}} \|g_n * \|_{L(L^2_{\nu_1}(\mathbb{R}))} < 1.$$

Hence, we can reformulate (4) as follows

$$\left(1 + M_{g_n}\left(\partial_0^{-1}\right)\right) u_n = f,$$

Thus with $H_0 = \{0\}$, $H_1 = H$ and $N^{11} = (1 + M_{g_n})_n$ Theorem 4.1 is applicable. (Note that $\mathfrak{Re}\, N_n^{11} \geqq 1 - \varepsilon > 0$ for all $n \in \mathbb{N}$). The assertion states that, for a suitable subsequence for which we will use the same notation, we have

$$\left(1 + M_{g_n}\left(\partial_0^{-1}\right)\right) \xrightarrow{G} N_{hom,-1,11}\left(\partial_0^{-1}\right)$$

with

$$\begin{aligned} N_{hom,-1,11}(z) &= \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} \left(1 + M_{g_n}(z)\right)^{-1} \\ &= \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} 1 + \sum_{\ell=1}^{\infty} M_{g_n}(z)^{\ell} \\ &= \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} 1 + \sum_{\ell=1}^{\infty} M_{(g_n)^{*\ell}}(z) \\ &= \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} 1 + M_{\sum_{\ell=1}^{\infty}(g_n)^{*\ell}}(z) \end{aligned}$$

for all $z \in B\left(\frac{1}{2\nu_1}, \frac{1}{2\nu_1}\right)$ for some $\nu > \nu_1 \geqq \nu_0$, where we denoted the ℓ-fold convolution with a function g by $g^{*\ell}$, $\ell \in \mathbb{N}$.

In [31] we discussed the following variant of Example 3.7.

Example 4.4. In the situation of Example 3.7, we let $(h_k)_k$ be a convergent sequence of positive real numbers with limit h. Then Theorem 4.1 gives

$$\begin{aligned} &\partial_0 a(k\cdot) + \tau_{-h_k} b(k\cdot) \\ &\xrightarrow{G} \partial_0 \left(\int_{[0,1]^n} a(y)^{-1} dy\right)^{-1} \\ &\quad + \partial_0 \sum_{k=1}^{\infty} \left(-\sum_{\ell=1}^{\infty} \left(\int_{[0,1]^n} a(y)^{-1} dy\right)^{-1} \right. \\ &\quad \left. \times \int_{[0,1]^n} a(y)^{-1} \left(\tau_{-h} b(y) a(y)^{-1}\right)^{\ell} dy \left(-\partial_0^{-1}\right)^{\ell} \right)^{k} \left(\int_{[0,1]^n} a(y)^{-1} dy\right)^{-1}. \end{aligned}$$

Indeed, it suffices to observe that $\tau_{-h} = \mathcal{L}_\nu^* e^{-h(im+\nu)} \mathcal{L}_\nu$.

Fractional differential equations are also admissible as the following example shows.

Example 4.5. Again in the situation of Example 3.7, let $(\alpha_k)_k$ and $(\beta_k)_k$ be convergent sequences in $]0,1]$ and $[-1,0]$ with limits α and β, resp. Then Theorem 4.1 gives

$$\begin{aligned}
&\partial_0^{\alpha_k} a(k\cdot) + \partial_0^{\beta_k} b(k\cdot) = \partial_0 \partial_0^{\alpha_k - 1} a(k\cdot) + \partial_0^{\beta_k} b(k\cdot) \\
&\xrightarrow{G} \partial_0^{\alpha} \left(\int_{[0,1]^n} a(y)^{-1} dy \right)^{-1} \\
&\quad + \partial_0^{\alpha} \sum_{k=1}^{\infty} \left(- \sum_{\ell=1}^{\infty} \partial_0^{\alpha-1} \left(\int_{[0,1]^n} a(y)^{-1} dy \right)^{-1} \right. \\
&\quad \left. \times \int_{[0,1]^n} a(y)^{-1} \partial_0^{1-\alpha} \left(\partial_0^{1+\beta-\alpha} b(y) a(y)^{-1} \right)^{\ell} dy \left(-\partial_0^{-1} \right)^{\ell} \right)^{k} \left(\int_{[0,1]^n} a(y)^{-1} dy \right)^{-1}.
\end{aligned}$$

Remark 4.6. Note that all the above theorems on homogenization of differential equations straightforwardly apply to higher order equations. For example the equation

$$\sum_{k=0}^{n} \partial_0^k a_k u = f$$

can be reformulated as a first order system in the standard way. Another way is to integrate $n-1$ times, to get that

$$\sum_{k=0}^{n} \partial_0^{1+k-n} a_k u = \partial_0^{-(n-1)} f,$$

which is by setting $M(\partial_0^{-1}) = a_n$ and $N(\partial_0^{-1}) = \sum_{k=0}^{n-1} \partial_0^{1+k-n} a_k$ of the form treated in Theorem 4.1.

5. Time-dependent coefficients

In this section we treat operators depending on temporal and spatial variables, which are, in contrast to the previous section, not time-translation invariant. Thus, the structural hypothesis of being analytic functions of ∂_0^{-1} has to be lifted. Consequently, the expressions for the limit equations do not simplify in the manner as they did in Corollary 3.1 and Theorem 4.1. Particular ((non)-linear) equations have been considered in [9, 10, 14, 16, 24]. The main objective of this section is to give a sufficient criterion under which the choice of subsequences in Theorem 2.7 is not required. We introduce the following notion.

Definition 5.1. Let H be a Hilbert space. A family $((T_{n,\iota})_{n\in\mathbb{N}})_{\iota\in I}$ of sequences of linear operators in $L(H)$ is said to have the *product-convergence property*, if for all $k\in\mathbb{N}$ and $(\iota_1,\ldots,\iota_k)\in I^k$ the sequence $\left(\prod_{i=1}^k T_{n,\iota_i}\right)_n$ converges in the weak operator topology of $L(H)$.

Example 5.2. Let $N,M\in\mathbb{N}$ and denote

$$\mathbb{P}:=\{a\colon \mathbb{R}^N\to\mathbb{C}^{M\times M}; a \text{ is } [0,1]^N\text{-periodic}\}.$$

Theorem 3.5 asserts that the family $\left((a(k\cdot))_{k\in\mathbb{N}}\right)_{a\in\mathbb{P}}$ has the product-convergence property in $L(L^2(\mathbb{R}^N)^M)$.

We refer to the notion of homogenization algebras for other examples, see e.g. [12, 13]. The main theorem of this section reads as follows. Recall from Example 2.3 the space $L_s^\infty(\mathbb{R};L(H))$ of strongly measurable bounded functions with values in $L(H)$ endowed with the sup-norm. Moreover, recall that for $A\in L_s^\infty(\mathbb{R};L(H))$ the associated multiplication operator $A(m_0)$ is evolutionary at ν for every $\nu>0$.

Theorem 5.3. *Let H be a Hilbert space, $\nu>0$. Let $\left((A_{\iota,n})_n\right)_\iota$ be a family of bounded sequences in $L_s^\infty(\mathbb{R};L(H))$. Assume that the family $\left((A_{\iota,n}(t))_n\right)_{\iota,t\in\mathbb{R}}$ has the product-convergence property. Then $\left((A_{\iota,n}(m_0))_n, (\partial_0^{-1})_n\right)_\iota$ has the product-convergence property.*

Remark 5.4. (a) With the latter result, it is possible to deduce that the choice of subsequences in Theorem 2.7 is not needed. Indeed, assume that

$$\begin{pmatrix}\mathcal{M}_n & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\mathcal{N}_n^{00} & \mathcal{N}_n^{01}\\ \mathcal{N}_n^{10} & \mathcal{N}_n^{11}\end{pmatrix}=\begin{pmatrix}M_n(m_0) & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}N_n^{00}(m_0) & N_n^{01}(m_0)\\ N_n^{10}(m_0) & N_n^{11}(m_0)\end{pmatrix}$$

for some strongly measurable and bounded $M_n^1, N_n^{00}, N_n^{01}, N_n^{10}, N_n^{11}$ and assume that the family

$$\Bigg(\begin{pmatrix}M_n(t) & 0\\ 0 & 0\end{pmatrix}_n, \begin{pmatrix}M_n(t)^{-1} & 0\\ 0 & 0\end{pmatrix}_n, \begin{pmatrix}N_n^{00}(t) & 0\\ 0 & 0\end{pmatrix}_n, \begin{pmatrix}0 & N_n^{01}(t)\\ 0 & 0\end{pmatrix}_n,$$
$$\begin{pmatrix}0 & 0\\ N_n^{10}(t) & 0\end{pmatrix}_n, \begin{pmatrix}0 & 0\\ 0 & N_n^{11}(t)\end{pmatrix}_n, \begin{pmatrix}0 & 0\\ 0 & N_n^{11}(t)^{-1}\end{pmatrix}_n\Bigg)_{t\in\mathbb{R}}$$

satisfies the product-convergence property. Then Theorem 5.3 ensures that the limit expressions in Theorem 2.7 converge without choosing subsequences.

(b) The crucial fact in Theorem 5.3 is that powers of ∂_0^{-1} are involved. Indeed, let H be a Hilbert space, $\nu > 0$. Let $\big((A_{\iota,n})_n\big)_\iota$ be a family of bounded sequences in $L_s^\infty(\mathbb{R}; L(H))$. Assume that, for every $t \in \mathbb{R}$, the family $\big((A_{\iota,n}(t))_n\big)_\iota$ has the product-convergence property. Then $\big((A_{\iota,n}(m_0))_n\big)_\iota$ has the product-convergence property. Showing the assertion for two sequences $(A_{1,n})_n$ and $(A_{2,n})_n$ and using the boundedness of the sequence $(A_{1,n}(m_0)A_{2,n}(m_0))_n$, we deduce that it suffices to show weak convergence on a dense subset. For this to show let $K, L \subseteqq \mathbb{R}$ be bounded and measurable and $\phi, \psi \in H$. We get for $n \in \mathbb{N}$ and $\nu > 0$ that

$$\begin{aligned}\langle \chi_K \phi, A_{1,n}(m_0)A_{2,n}(m_0)\chi_L\psi\rangle_\nu &= \int_{K\cap L} \langle \phi, A_{1,n}(t)A_{2,n}(t)\psi\rangle e^{-2\nu t}dt \\ &\to \int_{K\cap L} \lim_{n\to\infty} \langle \phi, A_{1,n}(t)A_{2,n}(t)\psi\rangle e^{-2\nu t}dt,\end{aligned}$$

by dominated convergence.

Lemma 5.5. *Let H be a Hilbert space, $\nu > 0$. Let $\big((A_{\iota,n})_n\big)_{\iota\in I}$ be a family of bounded sequences in $L_s^\infty(\mathbb{R}; L(H))$. Assume that the family $\big((A_{\iota,n}(t))_n\big)_{\iota,t\in\mathbb{R}}$ has the product-convergence property. Then*[6] *$\prod_{j=1}^k \left(A_{\iota_j,n}(m_0), \partial_0^{-1}\right)^{\ell_j}$ converges in the weak operator toplogogy for all $k\in\mathbb{N}$, $\ell_1, \dots, \ell_k \in \{0,1\}\times\mathbb{N}$ and $\iota_1, \dots, \iota_k \in I$.*

Proof. Let $k \in \mathbb{N}$, $\ell_1, \dots, \ell_k \in \{0,1\} \times \mathbb{N}$ and $\iota_1, \dots, \iota_k \in I$. Moreover, take $\phi, \psi \in H$ and $K, L \subseteqq \mathbb{R}$ be bounded and measurable. For $n \in \mathbb{N}$ and $\nu > 0$ we compute

$$\begin{aligned}&\left\langle \chi_K\phi, \prod_{j=1}^k \left(A_{\iota_j,n}(m_0), \partial_0^{-1}\right)^{\ell_j} \chi_L\psi \right\rangle_\nu \\
&= \int_K \left\langle \phi, A_{\iota_1,n}(s_0^0)^{\ell_{1,1}} \int_{-\infty}^{s_0^0}\int_{-\infty}^{s^1_{\ell_{1,2}-1}} \cdots \int_{-\infty}^{s_1^1} A_{\iota_2,n}(s_0^1)^{\ell_{2,1}} \int_{-\infty}^{s_0^1}\int_{-\infty}^{s^2_{\ell_{2,2}-1}}\cdots\int_{-\infty}^{s_1^2}\cdots \right.\\
&\quad \left. A_{\iota_k,n}(s_0^{k-1})^{\ell_{k,1}} \int_{-\infty}^{s_0^{k-1}}\int_{-\infty}^{s^k_{\ell_{k,2}-1}}\cdots\int_{-\infty}^{s_1^k} \chi_L(s_0^k)\psi \right\rangle \\
&\quad ds_0^k \cdots ds^k_{\ell_{k,2}-2} ds^k_{\ell_{k,2}-1} \cdots ds_0^2 \cdots ds^2_{\ell_{2,2}-2} ds^2_{\ell_{2,2}-1} ds_0^1 \cdots ds^1_{\ell_{1,2}-2} ds^1_{\ell_{1,2}-1} e^{-2\nu s_0^0} ds_0^0 \\
&= \int_K \int_{-\infty}^{s_0^0}\int_{-\infty}^{s^1_{\ell_{1,2}-1}}\cdots\int_{-\infty}^{s_1^1}\int_{-\infty}^{s_0^1}\int_{-\infty}^{s^2_{\ell_{2,2}-1}}\cdots\int_{-\infty}^{s_1^2}\cdots\int_{-\infty}^{s_0^{k-1}}\int_{-\infty}^{s^k_{\ell_{k,2}-1}}\cdots\int_{-\infty}^{s_1^k} \\
&\quad \left\langle \phi, A_{\iota_1,n}(t)^{\ell_{1,1}} A_{\iota_2,n}(s_0)^{\ell_{2,1}} \cdots \chi_L(s_0^k)\psi\right\rangle \\
&\quad ds_0^k \cdots ds^k_{\ell_{k,2}-2} ds^k_{\ell_{k,2}-1} \cdots ds_0^2 \cdots ds^2_{\ell_{2,2}-2} ds^2_{\ell_{2,2}-1} ds_0^1 \cdots ds^1_{\ell_{1,2}-2} ds^1_{\ell_{1,2}-1} e^{-2\nu s_0^0} ds_0^0.\end{aligned}$$

[6]In what follows we adopt multiindex notation: For two operators A, B and $k = (k_1, k_2) \in \mathbb{N}_0^2$ we denote $(A, B)^k := A^{k_1}B^{k_2}$. If k_j is a multiindex in $\mathbb{N}_0^2$, we denote its first and second component respectively by $k_{j,1}$ and $k_{j,2}$.

Using dominated convergence, we deduce the convergence of the latter expression. □

Proof of Theorem 5.3. The proof follows easily with Lemma 5.5. □

Theorem 5.3 serves as a possibility to deduce G-convergence of differential operators, where the coefficients take values in, for example, periodic mappings as in Example 5.2. Another instance is given in the following example.

Example 5.6. Let $A, B \in L^\infty(\mathbb{R})$ be 1-periodic, $f \in C_{\infty,c}(\mathbb{R})$. Assume that $A \geqq c$ for some $c > 0$. For $n \in \mathbb{N}$ and $\nu > 0$ consider

$$(\partial_0 A(n \cdot m_0) + B(n \cdot m_0))\, u_n = f.$$

Recall that from Theorem 2.5, in order to compute the limit equation, we have to compute expressions of the form

$$\mathcal{M}_{hom,\ell} = \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} \mathcal{M}_n^{-1}\left(-\partial_0^{-1}\mathcal{N}_n\mathcal{M}_n^{-1}\right)^\ell,\ \ell \geqq 1, \quad \text{and} \quad \mathcal{M}_{hom,0} = \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} \mathcal{M}_n^{-1},$$

where $\mathcal{M}_n = A(n \cdot m_0)$ and $\mathcal{N}_n = B(n \cdot m_0)$, $\ell \in \mathbb{N}$.

In order to deduce G-convergence in the latter example we need the following theorem.

Theorem 5.7. *Let $A_1, \ldots, A_k \in L^\infty(\mathbb{R})$ be 1-periodic. Then for every $\nu > 0$ we have*

$$\begin{aligned}\mathcal{A}_n &:= A_1(n \cdot m_0)\left(\prod_{j=1}^{k-1} \partial_0^{-1} A_{j+1}(n \cdot m_0)\right)\\ &\xrightarrow{\tau_{\mathrm{w}}, n\to\infty} \left(\partial_0^{-1}\right)^{k-1} \prod_{j=1}^{k} \int_0^1 A_j(y)dy \in L\left(L_\nu^2(\mathbb{R})\right).\end{aligned}$$

Proof. For $n \in \mathbb{N}$ and $K, L \subseteqq \mathbb{R}$ bounded, measurable we compute

$$\begin{aligned}&\langle \chi_K, \mathcal{A}_n \chi_L\rangle_\nu\\ &= \int_K A_1(nt_1) \int_{-\infty}^{t_1} A_2(nt_2) \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_{k-1}} A_k(nt_k)\chi_L(t_k)dt_k \cdots dt_2 e^{-2\nu t_1} dt_1\\ &= \int_K \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_{k-1}} \left(\prod_{j=1}^k A_j(nt_j)\right) \chi_L(t_k) e^{-2\nu t_1} dt_k \cdots dt_1\\ &= \underbrace{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}}}_{k\text{-times}} \left(\prod_{j=1}^k A_j(nt_j)\right) \chi_K(t_1) \left(\prod_{j=2}^k \chi_{\mathbb{R}_{>0}}(t_{j-1} - t_j)\right) \chi_L(t_k) e^{-2\nu t_1} dt_k \cdots dt_1.\end{aligned}$$

Now, observe that $(t_1,\ldots,t_k)\mapsto \chi_K(t_1)\left(\prod_{j=2}^k \chi_{\mathbb{R}_{>0}}(t_{j-1}-t_j)\right)\chi_L(t_k)e^{-2\nu t_1}\in L^1(\mathbb{R}^k)$. Moreover, the mapping $(t_1,\cdots,t_k)\mapsto \prod_{j=1}^k A_j(t_j)$ is $[0,1]^k$-periodic. Thus, by Theorem 3.5, we conclude that

$$\langle \chi_K, \mathcal{A}_n\chi_L\rangle_\nu \to \left\langle \chi_K, \left(\partial_0^{-1}\right)^{k-1}\prod_{j=1}^k\int_0^1 A_j(y)dy\chi_L\right\rangle$$

as $n\to\infty$ for all $K,L\subseteqq\mathbb{R}$ bounded and measurable. A density argument concludes the proof. □

Example 5.8 (Example 5.6 continued)**.** Thus, with the Theorems 5.7 and 2.5, we conclude that $(\partial_0 A(n\cdot m_0)+B(n\cdot m_0))$ G-converges to

$$\begin{aligned}
&\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\\
&+\partial_0\sum_{k=1}^\infty\left(-\sum_{\ell=1}^\infty\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\int_0^1\frac{1}{A(y)}dy\left(-\partial_0^{-1}\int_0^1\frac{B(y)}{A(y)}dy\right)^\ell\right)^k\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\\
&=\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\left(1+\sum_{k=1}^\infty\left(-\sum_{\ell=1}^\infty\left(-\partial_0^{-1}\int_0^1\frac{B(y)}{A(y)}dy\right)^\ell\right)^k\right)\\
&=\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\sum_{k=0}^\infty\left(-\sum_{\ell=1}^\infty\left(-\partial_0^{-1}\int_0^1\frac{B(y)}{A(y)}dy\right)^\ell\right)^k\\
&=\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\left(1+\sum_{\ell=1}^\infty\left(-\partial_0^{-1}\int_0^1\frac{B(y)}{A(y)}dy\right)^\ell\right)^{-1}\\
&=\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\left(\sum_{\ell=0}^\infty\left(-\partial_0^{-1}\int_0^1\frac{B(y)}{A(y)}dy\right)^\ell\right)^{-1}\\
&=\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\left(1+\partial_0^{-1}\int_0^1\frac{B(y)}{A(y)}dy\right)\\
&=\partial_0\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}+\left(\int_0^1\frac{1}{A(y)}dy\right)^{-1}\int_0^1\frac{B(y)}{A(y)}dy.
\end{aligned}$$

Remark 5.9. In [16], the authors consider an equation of the form

$$\left(\partial_0+a_n(m_0)\right)u_n=f$$

in the space $L^2(\mathbb{R};L^2(\mathbb{R}))$ with $(a_n)_n$ being a bounded sequence in $L^\infty(\mathbb{R}\times\mathbb{R})$. Assuming weak-$*$-convergence of $(a_n)_n$, the author shows weak convergence of $(u_n)_n$. The limit equation is a convolution equation involving the Young-measure associated to the sequence $(a_n)_n$. Within our reasoning, we cannot show that the whole sequence converges, unless any power of $(a_n)_n$ converges in the weak-$*$ topology of L^∞. However, as we illustrated above (see e.g. Example 3.12) our approach has a wide range of applications, where the method involving Young-measures might fail to work.

6. Proof of the main theorems

We will finally prove our main theorems. The proof relies on elementary Hilbert space concepts. We emphasize that the generality of the perspective hardly allows the introduction of Young-measures, which have proven to be useful in particular cases (see the sections above for a detailed discussion). Before we give a detailed account of the proofs of our main theorems, we state the following auxilary result, which we state without proof.

Lemma 6.1. *Let H be a Hilbert space, $T \in L(H)$. Assume that $\mathfrak{Re}\, T \geqq c$ for some $c > 0$. Then $\|T^{-1}\| \leqq \frac{1}{c}$ and $\mathfrak{Re}\, T^{-1} \geqq \frac{c}{\|T\|^2}$.*

Proof of Theorem 2.5. By the boundedness assumptions on $\mathcal{M}_n$ and $\mathcal{N}_n$ there exists $\nu_1 \geqq \nu_0$ such that

$$C_{\mathcal{N}} := \sup_{\eta \geqq \nu_1} \sup_{n\in\mathbb{N}} \|\mathcal{N}_n\|_{L(L^2_\eta)} < \infty, \quad C_{\mathcal{M}} := \sup_{\eta \geqq \nu_1} \sup_{n\in\mathbb{N}} \|\mathcal{M}_n\|_{L(L^2_\eta)} < \infty.$$

Further, choose $\nu > \nu_1$ such that

$$\nu > \frac{C_{\mathcal{M}}^2}{c^3} + \frac{C_{\mathcal{N}}}{c^2}. \tag{5}$$

Now, for $f \in C_{\infty,c}(\mathbb{R}; H)$ let u_n solve

$$(\partial_0 \mathcal{M}_n + \mathcal{N}_n)\, u_n = f.$$

This yields

$$\begin{aligned} u_n &= \mathcal{M}_n^{-1} \left(1 + \partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^{-1} \partial_0^{-1} f \\ &= \mathcal{M}_n^{-1} \sum_{\ell=0}^{\infty} \left(-\partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^{\ell} \partial_0^{-1} f \\ &= \left(\mathcal{M}_n^{-1} + \sum_{\ell=1}^{\infty} \mathcal{M}_n^{-1} \left(-\partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^{\ell} \right) \partial_0^{-1} f. \end{aligned}$$

Hence, choosing an appropriate subsequence, we arrive at an expression of the form

$$u = \left(\mathcal{M}_{hom,0}^{-1} + \sum_{\ell=1}^{\infty} \mathcal{M}_{hom,\ell}\right) \partial_0^{-1} f.$$

We remark here that due to the (standard) estimate $\|T\| \leqq \liminf_{k\to\infty} \|T_k\|$ for a sequence $(T_k)_k$ of bounded linear operators in some Hilbert space converging to T in the weak operator topology, the series $\sum_{\ell=1}^{\infty} \mathcal{M}_{hom,\ell}$ converges with respect to the operator norm by our choice of ν (see (5)). Indeed, for $n \in \mathbb{N}$ we have the estimate

$$\|\mathcal{M}_n^{-1}\left(-\partial_0^{-1}\mathcal{N}_n\mathcal{M}_n^{-1}\right)\| \leqq \frac{1}{\nu}\|\mathcal{M}_n^{-1}\|^2\|\mathcal{N}_n\| \leqq \frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}}, \tag{6}$$

where we used Lemma 6.1 to deduce that $\|\mathcal{M}_n\| \leqq \frac{1}{c}$. Hence, recalling that estimate (6) carries over to the weak limit, we infer that the norm bound of $\mathcal{M}_{hom,\ell}$ can be estimated by $(\frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}})^\ell$ and, from (5), we deduce that $\frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}} < 1$.

Now, using the positive definiteness of $\mathcal{M}_n$ for all $n \in \mathbb{N}$ and Lemma 6.1, we deduce that

$$\mathfrak{Re}\, \mathcal{M}_n^{-1} \geqq \frac{c}{C_{\mathcal{M}}^2}.$$

By $\|\mathcal{M}_n^{-1}\| \leqq \frac{1}{c}$, we conclude that

$$\mathfrak{Re}\, \mathcal{M}_{hom,0}^{-1} \geqq \frac{c}{C_{\mathcal{M}}^2} \quad \text{and} \quad \mathfrak{Re}\, \mathcal{M}_{hom,0} \geqq \frac{c^3}{C_{\mathcal{M}}^2}.$$

Observe that $\|\sum_{\ell=1}^{\infty} \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\| < 1$. Indeed, using again (5), we have

$$\begin{aligned}
\left\|\sum_{\ell=1}^{\infty} \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\right\| &\leqq \sum_{\ell=1}^{\infty} \|\mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\| \\
&\leqq \sum_{\ell=1}^{\infty} \frac{C_{\mathcal{M}}^2}{c} \|\mathcal{M}_{hom,\ell}\| \\
&\leqq \sum_{\ell=1}^{\infty} \frac{C_{\mathcal{M}}^2}{c} \left(\frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}}\right)^\ell \\
&= \frac{C_{\mathcal{M}}^2}{c}\left(\frac{1}{1-\frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}}} - 1\right) \\
&= \frac{C_{\mathcal{M}}^2}{c} \frac{\frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}}}{1-\frac{1}{\nu}\frac{1}{c^2}C_{\mathcal{N}}} \\
&= \frac{C_{\mathcal{M}}^2}{c} \frac{\frac{1}{c^2}C_{\mathcal{N}}}{\nu-\frac{1}{c^2}C_{\mathcal{N}}} \\
&< 1.
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned} f &= \partial_0 \left(1 + \sum_{\ell=1}^{\infty} \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\right)^{-1} \mathcal{M}_{hom,0}u \\ &= \partial_0 \sum_{k=0}^{\infty} \left(-\sum_{\ell=1}^{\infty} \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\right)^{k} \mathcal{M}_{hom,0}u \\ &= \partial_0 \left(1 + \sum_{k=1}^{\infty} \left(-\sum_{\ell=1}^{\infty} \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\right)^{k}\right) \mathcal{M}_{hom,0}u \\ &= \partial_0\mathcal{M}_{hom,0}u + \partial_0 \sum_{k=1}^{\infty} \left(-\sum_{\ell=1}^{\infty} \mathcal{M}_{hom,0}\mathcal{M}_{hom,\ell}\right)^{k} \mathcal{M}_{hom,0}u. \qquad \square \end{aligned}$$

Proof of Theorem 2.7. We observe

$$\begin{aligned} &\begin{pmatrix} \partial_0\mathcal{M}_n + \mathcal{N}_n^{00} & \mathcal{N}_n^{01} \\ \mathcal{N}_n^{10} & \mathcal{N}_n^{11} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_0\mathcal{M}_n + \mathcal{N}_n^{00} - \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} & 0 \\ 0 & \mathcal{N}_n^{11} \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 \\ \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} & 1 \end{pmatrix}. \end{aligned}$$

Thus, with $B := \left(\partial_0\mathcal{M}_n + \mathcal{N}_n^{00} - \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10}\right)^{-1}$

$$\begin{aligned} &\begin{pmatrix} \partial_0\mathcal{M}_n + \mathcal{N}_n^{00} & \mathcal{N}_n^{01} \\ \mathcal{N}_n^{10} & \mathcal{N}_n^{11} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -\left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} & 1 \end{pmatrix} \begin{pmatrix} \left(\partial_0\mathcal{M}_n + \mathcal{N}_n^{00} - \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10}\right)^{-1} & 0 \\ 0 & \left(\mathcal{N}_n^{11}\right)^{-1} \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & -\mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} B & 0 \\ -\left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} B & \left(\mathcal{N}_n^{11}\right)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} B & -B\mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \\ -\left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} B & \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} B \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} + \left(\mathcal{N}_n^{11}\right)^{-1} \end{pmatrix}. \end{aligned}$$

With the Neumann series expression derived in the previous theorem, i.e.,

$$B = \mathcal{M}_n^{-1}\partial_0^{-1} + \sum_{\ell=1}^{\infty} \mathcal{M}_n^{-1} \left(-\partial_0^{-1}\mathcal{N}_n\mathcal{M}_n^{-1}\right)^{\ell} \partial_0^{-1}$$

with $\mathcal{N}_n = \mathcal{N}_n^{00} - \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10}$, we get that

$$\begin{pmatrix} \partial_0 \mathcal{M}_n + \mathcal{N}_n^{00} & \mathcal{N}_n^{01} \\ \mathcal{N}_n^{10} & \mathcal{N}_n^{11} \end{pmatrix}^{-1} = \sum_{\ell=0}^{\infty} \begin{pmatrix} U_1^\ell & U_2^\ell \\ U_3^\ell & U_4^\ell \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \left(\mathcal{N}_n^{11}\right)^{-1} \end{pmatrix}.$$

Here

$$\begin{aligned} U_1^\ell &= \mathcal{M}_n^{-1} \left(-\partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^\ell \partial_0^{-1}, \\ U_2^\ell &= -\mathcal{M}_n^{-1} \left(-\partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^\ell \partial_0^{-1} \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1}, \\ U_3^\ell &= -\left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} \mathcal{M}_n^{-1} \left(-\partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^\ell \partial_0^{-1}, \\ U_4^\ell &= \left(\mathcal{N}_n^{11}\right)^{-1} \mathcal{N}_n^{10} \mathcal{M}_n^{-1} \left(-\partial_0^{-1} \mathcal{N}_n \mathcal{M}_n^{-1}\right)^\ell \partial_0^{-1} \mathcal{N}_n^{01} \left(\mathcal{N}_n^{11}\right)^{-1}. \end{aligned}$$

With Theorem 2.5, we deduce convergence of the top left corner in the latter matrix. Similarly, we deduce convergence of the other expressions. Thus, for a suitable choice of subsequences, we arrive at

$$\sum_{\ell=1}^{\infty} \begin{pmatrix} \mathcal{M}_{hom,\ell,00} \partial_0^{-1} & \mathcal{M}_{hom,\ell,01} \\ \mathcal{M}_{hom,\ell,10} \partial_0^{-1} & \mathcal{M}_{hom,\ell,11} \end{pmatrix} + \begin{pmatrix} \mathcal{M}_{hom,0,00} \partial_0^{-1} & \mathcal{M}_{hom,0,01} \\ \mathcal{M}_{hom,0,10} \partial_0^{-1} & \mathcal{M}_{hom,0,11} + \mathcal{N}_{hom,-1,11} \end{pmatrix}$$

Note that the invertiblility of $\mathcal{M}_{hom,0,00}$ and $\mathcal{N}_{hom,-1,11}$ follows with the same reasoning as in Remark 2.6(a). We observe that

$$\begin{aligned} &\sum_{\ell=1}^{\infty} \begin{pmatrix} \mathcal{M}_{hom,\ell,00} \partial_0^{-1} & \mathcal{M}_{hom,\ell,01} \\ \mathcal{M}_{hom,\ell,10} \partial_0^{-1} & \mathcal{M}_{hom,\ell,11} \end{pmatrix} + \begin{pmatrix} \mathcal{M}_{hom,0,00} \partial_0^{-1} & \mathcal{M}_{hom,0,01} \\ \mathcal{M}_{hom,0,10} \partial_0^{-1} & \mathcal{M}_{hom,0,11} + \mathcal{N}_{hom,-1,11} \end{pmatrix} \\ &= \left(\mathcal{M}^{(1)} + \begin{pmatrix} \mathcal{M}_{hom,0,00}^{-1} & 0 \\ 0 & \mathcal{N}_{hom,-1,11}^{-1} \end{pmatrix}\right) \begin{pmatrix} \partial_0^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Moreover, note that the operator $\mathcal{M}^{(1)}$ has norm arbitrarily small if ν was chosen large enough (see also the argument in the proof of Theorem 2.5). Hence, the operator

$$\begin{aligned} &\left(\mathcal{M}^{(1)} + \begin{pmatrix} \mathcal{M}_{hom,0,00}^{-1} & 0 \\ 0 & \mathcal{N}_{hom,-1,11}^{-1} \end{pmatrix}\right) \\ &= \begin{pmatrix} \mathcal{M}_{hom,0,00}^{-1} & 0 \\ 0 & \mathcal{N}_{hom,-1,11}^{-1} \end{pmatrix} \left(\begin{pmatrix} \mathcal{M}_{hom,0,00} & 0 \\ 0 & \mathcal{N}_{hom,-1,11} \end{pmatrix}^{-1} \mathcal{M}^{(1)} + 1 \right) \end{aligned}$$

is invertible. This gives

$$\left(\left(\mathcal{M}^{(1)}+\begin{pmatrix}\mathcal{M}_{hom,0,00}^{-1} & 0\\ 0 & \mathcal{N}_{hom,-1,11}^{-1}\end{pmatrix}\right)\begin{pmatrix}\partial_0^{-1} & 0\\ 0 & 1\end{pmatrix}\right)^{-1}$$

$$=\begin{pmatrix}\partial_0 & 0\\ 0 & 1\end{pmatrix}\sum_{\ell=0}^{\infty}\left(-\begin{pmatrix}\mathcal{M}_{hom,0,00}^{-1} & 0\\ 0 & \mathcal{N}_{hom,-1,11}^{-1}\end{pmatrix}^{-1}\mathcal{M}^{(1)}\right)^{\ell}\begin{pmatrix}\mathcal{M}_{hom,0,00} & 0\\ 0 & \mathcal{N}_{hom,-1,11}\end{pmatrix}$$

$$=\begin{pmatrix}\partial_0 & 0\\ 0 & 1\end{pmatrix}\left(\begin{pmatrix}\mathcal{M}_{hom,0,00} & 0\\ 0 & \mathcal{N}_{hom,-1,11}\end{pmatrix}\right.$$

$$\left.+\sum_{\ell=1}^{\infty}\left(-\begin{pmatrix}\mathcal{M}_{hom,0,00} & 0\\ 0 & \mathcal{N}_{hom,-1,11}\end{pmatrix}\mathcal{M}^{(1)}\right)^{\ell}\begin{pmatrix}\mathcal{M}_{hom,0,00} & 0\\ 0 & \mathcal{N}_{hom,-1,11}\end{pmatrix}\right). \qquad \square$$

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