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Stability of Traveling Wave Fronts for Nonlocal Delayed Reaction Diffusion Systems

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Abstract. This paper is concerned with the stability of traveling wave fronts for nonlocal delayed reaction diffusion system. The stability of traveling wave front is proved in some exponentially weighted L^∞-spaces, when the difference between initial data and traveling wave front decays exponentially as $x \to -\infty$, but the initial data can be arbitrary large in other locations. Moreover, the time decay rates are obtained by weighted energy estimates.

Keywords. Stability; Nonlocal delayed reaction diffusion system; traveling wave fronts; weighted energy

Mathematics Subject Classification (2010). Primary 35K57; 35B40, secondary 35B35; 34D23

1. Introduction

In this paper, we are interested in the following nonlocal delayed reaction-diffusion systems

$$\begin{cases} u_t - J_1 * u + u = r_1 u\big[1 - a_1 u + b_1 v(x, t-\tau)\big], & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ v_t - J_2 * v + v = r_2 v\big[1 + b_2 u(x, t-\tau) - a_2 v\big], & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \end{cases} \tag{1}$$

where J_i is a nonnegative even function, $J_i * 1 = 1$, $J_i * e^{\lambda \cdot} < \infty$, for $\lambda > 0$, $i = 1, 2$. We remark that the convolution in (1) is convolution in space. Lv-Wang [16, 18] obtained the existence and stability of traveling wave front for system (1) with $J_1 * u - u$ and $J_2 * v - v$ replaced by Δu and Δv, respectively, under the condition that $h(y) = \delta(y)$ where $\delta(y)$ is the Dirac δ-function. It is

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easy to see that the system (1) has four constant equilibria $(0,0)$, $(0,\frac{1}{a_2})$, $(\frac{1}{a_1},0)$, and $(k_1,k_2):=K$, where

$$k_1=\frac{b_1+a_2}{a_1a_2-b_1b_2},\quad k_2=\frac{a_1+b_2}{a_1a_2-b_1b_2}.$$

Traveling wave fronts of reaction-diffusion equations with local and nonlocal delays have been extensively studied by many authors in the last two decades. Schaaf [28] studied two scalar reaction-diffusion equations with a discrete delay for both Huxley nonlinearity and Fisher nonlinearity, and established the existence of traveling wave fronts and uniqueness of wave speeds by a phase plane analysis method. Wu-Zou [34] considered a more general reaction-diffusion system with finite delay and obtained the existence of traveling wave fronts by using the classical monotone iteration technique with sub-supersolution method. Ma [20] employed the Schauder's fixed point theorem to an operator used in [34] in a properly chosen subset in the Banach space $C(\mathbb{R},\mathbb{R}^n)$ equipped with the so-called exponential decay norm, and showed the existence of traveling wave fronts for a class of delayed systems with quasimonotonicity reaction terms. Recently, Li-Lin-Ruan [14] developed a new cross iteration scheme, which is different from that defined in [20,34], and established the existence of traveling wave solutions for Lotka-Volterra competition system with delays.

For reaction-diffusion equations with nonlocal delay, Britton [6, 7] made the first attempt to study the periodic traveling wave solutions. By using the perturbation theory of ordinary differential equations coupled with the Fredholm alternative, Al-Omari and Gourley [2] and Gourley [11] studied an age-structured reaction-diffusion model and a nonlocal Fisher equation, respectively. Ruan-Xiao [26] obtained the existence of traveling wave fronts for a vector disease model with nonlocal delays by the geometric singular perturbation theory [10], see Ai [1] and Gourley-Ruan [12] for this method. Wang-Li-Ruan [32] obtained the existence of traveling wave fronts for system (1) with $(J*u)(x,t)-(J*I)u(x,t)$ replaced by $\Delta u(x,t)$. Applying the theory of [32], Li-Wang [15] studied the existence of traveling wave fronts for cooperative Lotka-Volterra system with nonlocal delays.

The study of uniqueness and asymptotic stability of traveling wave fronts become relatively more difficult. Sattinger [27] studied a reaction diffusion system without delay. By detail spectral analysis, he proved that the traveling wave fronts were stable to perturbations in some exponentially weighted L^∞-spaces. Kapitula [13] also studied a reaction diffusion system without delay. Using detail semigroup estimates, Kapitula [13] showed that the wave fronts is stable in polynomially weighted L^∞-spaces. For the stability and uniqueness of traveling wave fronts in reaction-diffusion equations with discrete delay, we should mention the work of Smith-Zhao [31]. They first established the existence and comparison theorem of solution in a quasimontone reaction-diffusion bistable equation with a discrete delay and then obtained the stability of traveling wave fronts by using the elementary super-sub solution comparison and squeezing methods developed by Chen [8] (see also [9,32,33] for this technique). As far as we know, there is no result about the stability of traveling wave fronts for system (1).

Just recently, Mei et al. [21–23] considered the so-called Nicholson's blowflies equation with diffusion. They first established a comparison principle and then proved that traveling wave fronts of Nicholson's blowflies equation are asymptotic stable in some exponentially weighted L^2-spaces. Lv-Wang [17] considered some more general models and established the stability of traveling wave fronts using the method developed by Mei et al. [21].

As Murray [24] pointed out that the general reaction-diffusion equations are strictly only applicable to dilute systems and population models. But in many biological areas, such as the embryological development case, the densities of cells involved are not small and a local or short range diffusive flux proportional to the gradient is not sufficiently accurate. Meanwhile, the time delay seems to be inevitable in the real world. Hence system (1) is a more suitable model.

Schumacher [29,30] studied the traveling wave solutions of integro-differential equations. Bates et al. [5] considered the following equation

$$u_t = J * u - u + f(u), \tag{2}$$

where $J(x)$ is a nonnegative function and satisfies $J * 1 = 1$. They obtained the existence of traveling wave fronts for equation (2). Bates-Chen [3,4] established the stability of traveling wave fronts for equation (2). The asymptotic stability of traveling wave fronts for equation (2) was obtained by Chen [8] using squeezing method. Recently, Pan-Li-Lin [25] studied the following system

$$\frac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - (J * I)u(x,t) + f(u(x,t), u(x,t-\tau)), \tag{3}$$

where $\tau > 0$ and functions $J(x)$ are defined as system (1). They proved the existence of traveling wave fronts for system (3) using Schauder's fixed point theorem and upper-lower solution technique.

Encouraged by papers [17,21,25], in this paper we prove that the traveling wave fronts of nonlocal delayed cooperative Lotka-Volterra system are stable by using weighted energy estimates [21].

Throughout this paper, $C > 0$ denotes a generic constant, while C_i $(i = 1, 2, \ldots)$ represents a specific constant. Let I be an interval, typically $I = \mathbb{R}$. Denote by $L^2(I)$ the space of square integrable functions defined on I, and $H^k(I)$ $(k \geq 0)$ the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\frac{\mathrm{d}^i}{\mathrm{d}x^i} f$ $(i = 1, \ldots, k)$ also belong to $L^2(I)$. Let $L^2_w(I)$ be the weighted L^2-space with a weight function $w(x) > 0$ and its norm is defined by

$$\|f\|_{L^2_w(I)} = \left(\int_I w(x)|f(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Let $H^k_w(I)$ be the weighted Sobolev space with the norm given by

$$\|f\|_{H^k_w(I)} = \left(\sum_{i=0}^{k} \int_I w(x) \left| \frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Let $T > 0$ be a number and B be a Banach space. We denote by $C^0([0,T]; B)$ the space of the B-valued continuous function on $[0,T]$, and by $L^2([0,T]; B)$ the space of the B-valued L^2-functions on $[0,T]$. The corresponding spaces of the B-valued L^2-functions on $[0,\infty)$ are defined similarly.

This paper is organized as follows. In Section 2, we recall some results for systems (1). In Section 3, the stability of traveling wave fronts is proved by using the weighted energy estimates. This paper ends with a short discussion.

2. Stability of traveling wave front

In this section, the stability of traveling wave front is established by using the weighted energy method [18, 21]. The existence of traveling wave fronts of (1) can be obtained by using the method of [25].

A traveling wave front of system (1) connecting with the constant states $(u_\pm, v_\pm)$ is a solution (u, v) with the special form $u(x,t) = \phi(x+ct)$, $v(x,t) = \psi(x+ct)$ and satisfies

$$\begin{cases} c\phi'(\xi) - (J_1 * \phi)(\xi) + \phi(\xi) = r_1\phi(\xi)\big[1 - a_1\phi(\xi) + b_1\psi(\xi - c\tau)\big], \\ c\psi'(\xi) - (J_2 * \psi)(\xi) + \psi(\xi) = r_2\psi(\xi)\big[1 + b_2\phi(\xi - c\tau) - a_2\psi(\xi)\big], \end{cases}$$

where $\xi := x + ct$, $' = \frac{\mathrm{d}}{\mathrm{d}\xi}$ and

$$\begin{cases} \phi(\pm\infty) = u_\pm, & \psi(\pm\infty) = v_\pm, \\ \phi'(\xi) \geq 0, & \psi'(\xi) \geq 0. \end{cases}$$

Now, we consider the system (1) with the nonnegative initial data

$$\begin{cases} u(x,s) = u_0(x,s), & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ v(x,s) = v_0(x,s), & (x,s) \in \mathbb{R} \times [-\tau, 0], \end{cases} \tag{4}$$

It is well-known that the functions $u(x,t)$ and $v(x,t)$ represent two different kinds of species in system (1) with (4). So we are only interested in the nonnegative solution of system (1) with (4). It follows from $k_1, k_2 > 0$ that $a_1a_2 > b_1b_2$. In this paper, we assume that the coefficients satisfy

$$\begin{cases} 2r_1\left[\left(2a_1 - \dfrac{b_1}{2}\right)k_1 - 1 - 2b_1k_2\right] - r_2b_2k_2 > 0, \\ 2r_2\left[\left(2a_2 - \dfrac{b_2}{2}\right)k_2 - 1 - 2b_2k_1\right] - r_1b_1k_1 > 0. \end{cases}$$

We note that the above assumption is possible because $2(a_1a_2 - b_1b_2) - b_1(b_1 + a_2) > 0$ and $2(a_1a_2 - b_1b_2) - b_1(b_1 + a_2) > 0$ if b_1 and b_2 are suitable small. Then it is easy to see that there exists a positive constant η such that

$$\begin{cases} 2r_1\left[\left(2a_1 - \dfrac{b_1}{2}\right)k_1 - 1 - 2b_1k_2\right] - r_2b_2k_2 - \displaystyle\int_0^\infty J_1(y)e^{\eta y}\mathrm{d}y + \frac{1}{2} > 0, \\ 2r_2\left[\left(2a_2 - \dfrac{b_2}{2}\right)k_2 - 1 - 2b_2k_1\right] - r_1b_1k_1 - \displaystyle\int_0^\infty J_2(y)e^{\eta y}\mathrm{d}y + \frac{1}{2} > 0. \end{cases} \tag{5}$$

We define a weight function as

$$w(\xi) = w_{\eta,\xi_0}(\xi) = \begin{cases} \mathrm{e}^{-\eta(\xi-\xi_0)} & \text{for } \xi \le \xi_0, \\ 1 & \text{for } \xi > \xi_0, \end{cases} \tag{6}$$

where ξ_0 is chosen to be large enough such that

$$\begin{cases} 2r_1\left[\left(2a_1 - \dfrac{b_1}{2}\right)\phi(\xi_0) - 1 - 2b_1k_2\right] - r_2b_2k_2 - \displaystyle\int_{-\infty}^0 J_1(y)e^{\eta y}\mathrm{d}y + \frac{1}{2} > 0, \\ 2r_2\left[\left(2a_2 - \dfrac{b_2}{2}\right)\psi(\xi_0) - 1 - 2b_2k_1\right] - r_1b_1k_1 - \displaystyle\int_{-\infty}^0 J_2(y)e^{\eta y}\mathrm{d}y + \frac{1}{2} > 0. \end{cases}$$

Theorem 2.1. *Assume that* $a_1a_2 > b_1b_2$, *and* (5) *holds. Denote* $\theta = r_1b_1k_1 + r_2b_2k_2$ *and let* c^* *be the critical speed and*

$$\bar{c} = \frac{1}{\eta}\max\left\{2r_1(1 + 2b_1k_2) + \int_{-\infty}^\infty \mathrm{e}^{\eta y}J_1(y)\mathrm{d}y,\ 2r_2(1 + 2b_2k_1) + \int_{-\infty}^\infty \mathrm{e}^{\eta y}J_2(y)\mathrm{d}y\right\} + \frac{1}{\eta}\left(\theta - \frac{1}{2}\right).$$

For any given wave front (ϕ,ψ) *of* (1) *with speed* $c > \max\{c^*, \bar{c}\}$, *if the initial data satisfies*

$$0 \le u_0(x,s) \le k_1, \quad 0 \le v_0(x,s) \le k_2 \quad \text{for } (x,s) \in \mathbb{R} \times [-\tau, 0],$$

and the initial perturbations $u_0(x,s)-\phi(x+cs)$ *and* $v_0(x,s)-\psi(x+cs)$ *belong to* $C([-\tau,0],H_w^1(\mathbb{R}))$*, then the nonnegative solution of the initial value problem* (1) *with* (4) *satisfies*

$$0\le u(x,t)\le k_1,\quad 0\le v(x,t)\le k_2\quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}_+,$$

and

$$u(x,t)-\phi(x+ct),\ v(x,t)-\psi(x+ct)\in C([0,+\infty);H_w^1(\mathbb{R})),$$

where the function $w(x)$ *was defined by* (6)*. Moreover, the nonnegative solution* $(u(x,t),v(x,t))$ *of the initial value problem* (1) *with* (4) *convergence to the wave front* $(\phi(x+ct),\psi(x+ct)$ *exponentially in time:*

$$\sup_{x\in\mathbb{R}}|u(x,t)-\phi(x+ct)|\le C\mathrm{e}^{-\mu t},$$
$$\sup_{x\in\mathbb{R}}|v(x,t)-\psi(x+ct)|\le C\mathrm{e}^{-\mu t}$$

for some positive constant μ*.*

In order to prove Theorem 2.1, we first establish a comparison principle for initial problem (1) with (4). For this, we need the following lemma.

Lemma 2.2 ([18])**.** *Let* $T>0$ *and* $Q_T=\mathbb{R}\times(0,T]$*. Assume that the nonnegative function* $c(x,t)$ *is bounded for* $(x,t)\in Q_T$*. If the function* u *satisfies*

$$\begin{cases} u_t-J*u+u+c(x,t)u\ge 0\ (\le 0), & (x,t)\in Q_T,\\ u(x,0)\ge 0\ (\le 0), & x\in\mathbb{R}.\end{cases}$$

then $u(x,t)\ge 0\ (\le 0)$ *for* $(x,t)\in Q_T$*.*

Lemma 2.3. *Assume that* $a_1a_2-b_1b_2>0$ *and the initial data satisfies*

$$(0,0)\le(u_0(x,s),v_0(x,s))\le(k_1,k_2)\quad \text{for } (x,s)\in\mathbb{R}\times[-\tau,0].$$

Then the nonnegative solution $(u(x,t),v(x,t))$ *of the problem* (1) *with* (4) *satisfies*

$$(0,0)\le(u(x,t),v(x,t))\le(k_1,k_2)\quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}_+.$$

Lemma 2.4. (*Comparison Principle*) *Let* $(u^\pm(x,t),v^\pm(x,t))$ *be the nonnegative solutions of equations* (1) *with initial data* $(u_0^\pm(x,s),v_0^\pm(x,s))$*, respectively. If*

$$(0,0)\le(u_0^-(x,s),v_0^-(x,s))\le(u_0^+(x,s),v_0^+(x,s))\le(k_1,k_2)$$

for $(x,s)\in\mathbb{R}\times[-\tau,0]$*. Then*

$$(0,0)\le(u^-(x,t),v^-(x,t))\le(u^+(x,t),v^+(x,t))\le(k_1,k_2)$$

for $(x,t)\in\mathbb{R}\times\mathbb{R}_+$*.*

Lemmas 2.3 and 2.4 can be proved by an argument similar to [18, Lemmas 3.2, 3.3] and [19, Lemmas 3.1, 3.2].

Following the idea of [17, 21], we shall use the comparison principle and weighted energy method to prove Theorem 2.1. Let $(u_0(x,s), v_0(x,s))$ satisfy

$$(0,0) \le (u_0(x,s), v_0(x,s)) \le (k_1, k_2) \quad \text{for } (x,s) \in \mathbb{R} \times [-\tau, 0].$$

Define

$$\begin{cases} u_0^-(x,s) = \min\{u_0(x,s), \phi(x+cs)\}, & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ u_0^+(x,s) = \max\{u_0(x,s), \phi(x+cs)\}, & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ v_0^-(x,s) = \min\{v_0(x,s), \psi(x+cs)\}, & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ v_0^+(x,s) = \max\{v_0(x,s), \psi(x+cs)\}, & (x,s) \in \mathbb{R} \times [-\tau, 0]. \end{cases}$$

Obviously,

$$\begin{cases} 0 \le u_0^-(x,s) \le u_0(x,s) \le u_0^+(x,s) \le k_1, & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ 0 \le u_0^-(x,s) \le \phi(x+cs) \le u_0^+(x,s) \le k_1, & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ 0 \le v_0^-(x,s) \le v_0(x,s) \le v_0^+(x,s) \le k_2, & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ 0 \le v_0^-(x,s) \le \psi(x+cs) \le v_0^+(x,s) \le k_2, & (x,s) \in \mathbb{R} \times [-\tau, 0]. \end{cases}$$

Let $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ be the nonnegative solutions of equations (1) with initial data $(u_0^-(x,s), v_0^-(x,s))$ and $(u_0^+(x,s), v_0^+(x,s))$, respectively.

It follows from comparison principle that

$$\begin{cases} 0 \le u^-(x,t) \le u(x,t) \le u^+(x,t) \le k_1, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ 0 \le u^-(x,t) \le \phi(x+ct) \le u^+(x,t) \le k_1, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ 0 \le v^-(x,t) \le v(x,t) \le v^+(x,t) \le k_2, & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ 0 \le v^-(x,t) \le \psi(x+ct) \le v^+(x,t) \le k_2, & (x,t) \in \mathbb{R} \times \mathbb{R}_+. \end{cases} \tag{7}$$

Now we prove Theorem 2.1 in three steps.

Step 1. We first prove the convergence of $u^+(x,t)$ to $\phi(x+ct)$ and $v^+(x,t)$ to $\psi(x+ct)$.

Let $\xi = x + ct$ and

$$m(\xi,t) = u^+(x,t) - \phi(x+ct), \quad m_0(\xi,s) = u_0^+(x,s) - \phi(x+cs),$$
$$n(\xi,t) = v^+(x,t) - \psi(x+ct), \quad n_0(\xi,s) = v_0^+(x,s) - \psi(x+cs).$$

It follows from (7) that

$$(m(\xi,t), n(\xi,t)) \ge (0,0), \quad (m_0(\xi,s), n_0(\xi,s)) \ge (0,0).$$

Moreover, $(m(\xi,t),n(\xi,t))$ satisfies

$$\begin{cases} m_t+cm_\xi-J_1*m+m+r_1\big[2a_1\phi-1-b_1n(\xi-c\tau,t-\tau)-b_1\psi(\xi-c\tau)\big]m\\ =-a_1r_1m^2+r_1b_1\phi n(\xi-c\tau,t-\tau),\\ n_t+cn_\xi-J_2*n+n+r_2\big[2a_2\psi-1-b_2m(\xi-c\tau,t-\tau)+\phi(\xi-c\tau)\big]n\\ =-a_2r_2n^2+r_2b_2\psi m(\xi-c\tau,t-\tau),\\ m(\xi,s)=m_0(\xi,s),\quad n(\xi,s)=n_0(\xi,s),\quad (\xi,s)\in\mathbb{R}\times[-\tau,0]. \end{cases} \tag{8}$$

Let $w(\xi)>0$ be the weight function defined by (6). Multiplying differential equations of (8) by $e^{2\mu t}w(\xi)m(\xi,t)$ and $e^{2\mu t}w(\xi)n(\xi,t)$, respectively, where $\mu>0$ will be specified later in Lemma 2.6, we have

$$\begin{aligned} &\Big(\frac{1}{2}e^{2\mu t}wm^2\Big)_t+\Big(\frac{c}{2}e^{2\mu t}wm^2\Big)_\xi-e^{2\mu t}wm\int_{-\infty}^{\infty}J_1(y)m(\xi-y,t)dy\\ &+\Big(-\frac{c}{2}\frac{w'}{w}-\mu+1+r_1\big[2a_1\phi-1-b_1n(\xi-c\tau,t-\tau)-b_1\psi(\xi-c\tau)\big]\Big)wm^2e^{2\mu t}\\ &=-a_1r_1m^3we^{2\mu t}+r_1b_1\phi n(\xi-c\tau,t-\tau)mwe^{2\mu t}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\Big(\frac{1}{2}e^{2\mu t}wn^2\Big)_t+\Big(\frac{c}{2}e^{2\mu t}wn^2\Big)_\xi-e^{2\mu t}wn\int_{-\infty}^{\infty}J_2(y)n(\xi-y,t)dy\\ &+\Big(-\frac{c}{2}\frac{w'}{w}-\mu+1+r_2\big[2a_2\psi-1-b_2m(\xi-c\tau,t-\tau)+\phi(\xi-c\tau)\big]\Big)wn^2e^{2\mu t}\\ &=-a_2r_2n^3we^{2\mu t}+r_2b_2\psi nwe^{2\mu t}m(\xi-c\tau,t-\tau). \end{aligned} \tag{10}$$

Integrating (9) and (10) with respect to (ξ,t) over $\mathbb{R}\times[0,t]$ and dropping the negative terms

$$-a_1r_1\int_0^t\int_{-\infty}^{\infty}m^3(\xi,s)w(\xi)e^{2\mu s}\quad\text{and}\quad -a_2r_2\int_0^t\int_{-\infty}^{\infty}n^3(\xi,s)w(\xi)e^{2\mu s},$$

we obtain

$$\begin{aligned} &e^{2\mu t}\|m(t)\|^2_{L^2_w}-2\int_0^t\int_{-\infty}^{\infty}e^{2\mu s}w(\xi)m(\xi,s)\int_{-\infty}^{\infty}J_1(y)m(\xi-y,s)dyd\xi ds\\ &+\int_0^t\int_{-\infty}^{\infty}e^{2\mu s}\Big[c\frac{w'}{w}-2\mu+2+2r_1\big[2a_1\phi-1-b_1n(\xi-c\tau,s-\tau)\\ &-b_1\psi(\xi-c\tau)\big]\Big]w(\xi)m^2(\xi,s)d\xi ds\\ &\le\|m_0(0)\|^2_{L^2_w}+2r_1b_1\int_0^t\int_{-\infty}^{\infty}e^{2\mu s}\phi(\xi)n(\xi-c\tau,s-\tau)m(\xi,s)w(\xi)d\xi ds, \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
& e^{2\mu t}\|n(t)\|_{L_w^2}^2 - 2\int_0^t\int_{-\infty}^{\infty} e^{2\mu s} w(\xi) n(\xi,s)\int_{-\infty}^{\infty} J_2(y) n(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
& + \int_0^t\int_{-\infty}^{\infty} e^{2\mu s}\Big[-c\frac{w'}{w} - 2\mu + 2 + 2r_2\big(2a_2\psi - 1 - b_2 m(\xi-c\tau,t-\tau) \\
& + \phi(\xi-c\tau)\big)\Big] w(\xi) n^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \le \|n_0(0)\|_{L_w^2}^2 + 2r_2 b_2\int_0^t\int_{-\infty}^{\infty} e^{2\mu s}\psi(\xi) n(\xi,s) w(\xi) m(\xi-c\tau,s-\tau)\mathrm{d}\xi\mathrm{d}s.
\end{aligned} \tag{12}
$$

By using the Cauchy-Schwarz inequality $2xy \le x^2 + y^2$, we can estimate that

$$
\begin{aligned}
& 2\int_0^t\int_{-\infty}^{\infty} e^{2\mu s} w(\xi) m(\xi,s)\int_{-\infty}^{\infty} J_1(y) m(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
& \le \int_0^t\int_{-\infty}^{\infty} e^{2\mu s} w(\xi) m^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \quad + \int_0^t\int_{-\infty}^{\infty} e^{2\mu s} w(\xi)\int_{-\infty}^{\infty} J_1(y) m^2(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
& = \int_0^t\int_{-\infty}^{\infty} e^{2\mu s} w(\xi) m^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \quad + \int_0^t\int_{-\infty}^{\infty} e^{2\mu s} w(\xi) m^2(\xi,s)\int_{-\infty}^{\infty} J_1(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y\mathrm{d}\xi\mathrm{d}s,
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
& 2r_1 b_1\int_0^t\int_{-\infty}^{\infty}\phi(\xi) n(\xi-c\tau,s-\tau) m(\xi,s) w(\xi) e^{2\mu t}\mathrm{d}\xi\mathrm{d}s \\
& \le r_1 b_1\int_0^t\int_{-\infty}^{\infty} e^{2\mu s}\phi(\xi) w(\xi)\big[m^2(\xi,s) + n^2(\xi-c\tau,s-\tau)\big]\mathrm{d}\xi\mathrm{d}s \\
& = r_1 b_1\int_0^t\int_{-\infty}^{\infty} e^{2\mu s}\phi(\xi) w(\xi) m^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \quad + r_1 b_1 e^{2\mu\tau}\int_{-\tau}^{t-\tau}\int_{-\infty}^{\infty} e^{2\mu s}\phi(\xi+c\tau) w(\xi+c\tau) n^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \le r_1 b_1\int_0^t\int_{-\infty}^{\infty} e^{2\mu s}\phi(\xi) w(\xi) m^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \quad + r_1 b_1 e^{2\mu\tau}\int_0^t\int_{-\infty}^{\infty} e^{2\mu s}\phi(\xi+c\tau) w(\xi+c\tau) n^2(\xi,s)\mathrm{d}\xi\mathrm{d}s \\
& \quad + r_1 b_1 e^{2\mu\tau}\int_{-\tau}^0\int_{-\infty}^{\infty} e^{2\mu s}\phi(\xi+c\tau) w(\xi+c\tau) n_0^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\,.
\end{aligned} \tag{14}
$$

Similarly, we have

$$\begin{aligned}
&2\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}w(\xi)n(\xi,s)\int_{-\infty}^{\infty}J_2(y)n(\xi-y,s)\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
&\le\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}w(\xi)n^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
&\quad+\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}w(\xi)n^2(\xi,s)\int_{-\infty}^{\infty}J_2(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y\mathrm{d}\xi\mathrm{d}s,
\end{aligned}\tag{15}$$

and

$$\begin{aligned}
&2r_2b_2\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}\psi(\xi)n(\xi,s)w(\xi)m(\xi-c\tau,s-\tau)\mathrm{d}\xi\mathrm{d}s\\
&\le r_2b_2\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}\psi(\xi)w(\xi)n^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\
&\quad+r_2b_2\mathrm{e}^{2\mu\tau}\int_0^t\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}w(\xi)m^2(\xi,s)\frac{w(\xi+c\tau)}{w(\xi)}\psi(\xi+c\tau)\mathrm{d}\xi\mathrm{d}s\\
&\quad+r_2b_2\mathrm{e}^{2\mu\tau}\int_{-\tau}^0\int_{-\infty}^{\infty}\mathrm{e}^{2\mu s}w(\xi)m_0^2(\xi,s)\frac{w(\xi+c\tau)}{w(\xi)}\psi(\xi+c\tau)\mathrm{d}\xi\mathrm{d}s\,.
\end{aligned}\tag{16}$$

Substituting (13), (14) and (15), (16) into (11) and (12), respectively, and adding the resulting inequalities, we have

$$\begin{aligned}
&\mathrm{e}^{2\mu t}\left(\|m(t)\|_{L_w^2}^2+\|n(t)\|_{L_w^2}^2\right)\\
&+\int_0^t\int_{\mathbb{R}}\mathrm{e}^{2\mu s}\big[B_{\mu,w}(\xi)m^2(\xi,s)+D_{\mu,w}(\xi)n^2(\xi,s)\big]w(\xi)\mathrm{d}\xi\mathrm{d}s\\
&\le C_1\left(\|m_0(0)\|_{L_w^2}^2+\|n_0(0)\|_{L_w^2}^2+\int_{-\tau}^0\left(\|m_0(s)\|_{L_w^2}^2+\|n_0(s)\|_{L_w^2}^2\right)\mathrm{d}s\right),
\end{aligned}\tag{17}$$

where

$$\begin{aligned}
B_{\mu,w}(\xi)&=A_w(\xi)-2\mu-r_2b_2(\mathrm{e}^{2\mu\tau}-1)\frac{w(\xi+c\tau)}{w(\xi)}\psi(\xi+c\tau),\\
D_{\mu,w}(\xi)&=E_w(\xi)-2\mu-r_1b_1(\mathrm{e}^{2\mu\tau}-1)\phi(\xi+c\tau)\frac{w(\xi+c\tau)}{w(\xi)},
\end{aligned}$$

and

$$\begin{aligned}
A_w(\xi)=&-c\frac{w'}{w}+1+2r_1\big[2a_1\phi-1-b_1n(\xi-c\tau,s-\tau)-b_1\psi(\xi-c\tau)\big]\\
&-\int_{-\infty}^{\infty}\frac{w(\xi+y)}{w(\xi)}J_1(y)\mathrm{d}y-r_1b_1\phi-r_2b_2\frac{w(\xi+c\tau)}{w(\xi)}\psi(\xi+c\tau),
\end{aligned}$$

$$\begin{aligned}E_w(\xi) = & -c\frac{w'}{w} + 1 + 2r_2\big[2a_2\psi - 1 - b_2 m(\xi - c\tau, t-\tau) + \phi(\xi - c\tau)\big] \\ & - \int_{-\infty}^{\infty} \frac{w(\xi+y)}{w(\xi)} J_2(y)\mathrm{d}y - r_2 b_2 \psi - r_1 b_1 \phi(\xi + c\tau)\frac{w(\xi+c\tau)}{w(\xi)}.\end{aligned}$$

In order to get the basic estimate, we must prove $B_{\mu,w}(\xi) \geq C > 0$ and $D_{\mu,w}(\xi) \geq C > 0$ for some constant C. For this we need the following key Lemma.

Lemma 2.5. *Let* (5) *holds, then there exists positive constants C_2 and C_3 such that*

$$A_w(\xi) \geq C_2 > 0,\ E_w(\xi) \geq C_3 > 0 \quad \xi \in \mathbb{R}.$$

Proof. We only prove that $A_w(\xi) \geq C_2 > 0$. The proof of $E_w(\xi) \geq C_2 > 0$ can be treated in a similar way.

Case 1: $\xi \leq \xi_0$.

It follows from (6) that $\frac{w(\xi+c\tau)}{w(\xi)} \leq 1$. Note that $n(\xi - c\tau, s-\tau) \leq k_2$, by the direct calculation we have

$$\begin{aligned}A_w(\xi) = & -c\frac{w'}{w} + 1 + 2r_1\big[2a_1\phi - 1 - b_1 n(\xi - c\tau, s-\tau) - b_1\psi(\xi - c\tau)\big] \\ & - \int_{-\infty}^{\infty} \frac{w(\xi+y)}{w(\xi)} J_1(y)\mathrm{d}y - r_1 b_1 \phi - r_2 b_2 \frac{w(\xi+c\tau)}{w(\xi)}\psi(\xi+c\tau) \\ \geq & \, c\eta + 1 - 2r_1(1+2b_1k_2) - \mathrm{e}^{\eta(\xi-\xi_0)} \\ & - \int_{-\infty}^{\xi_0-\xi} J_1(y)(\mathrm{e}^{-\eta y} - \mathrm{e}^{\eta(\xi-\xi_0)})\mathrm{d}y - r_1b_1k_1 - r_2b_2k_2 \\ \geq & \, c\eta - 2r_1(1+2b_1k_2) - \int_{-\infty}^{\infty} J_1(y)\mathrm{e}^{-\eta y}\mathrm{d}y - r_1b_1k_1 - r_2b_2k_2 + \frac{1}{2} \\ =: & \, C_4 > 0,\end{aligned}$$

where we have used $c > \bar{c}$.

Case 2: $\xi > \xi_0$.

In this case, $w(\xi) = w(\xi + c\tau) = 1$. Note that $n(\xi - c\tau, s-\tau) \leq k_2$ and $J(-y) = J(y)$, using the monotonicity of $\phi(\xi)$ and $\psi(\xi)$ on $\mathbb{R}$, we have

$$\begin{aligned}A_w(\xi) = & \, 1 + 2r_1\big[2a_1\phi(\xi) - 1 - b_1 n(\xi - c\tau, s-\tau) - b_1\psi(\xi - c\tau)\big] \\ & - \int_{-\infty}^{\infty} w(\xi+y) J_1(y)\mathrm{d}y - r_1b_1\phi(\xi) - r_2b_2 w(\xi+c\tau)\psi(\xi+c\tau) \\ \geq & \, 2r_1\big[(2a_1 - \frac{b_1}{2})\phi(\xi_0) - 1 - 2b_1k_2\big] - \int_{-\infty}^{0} (e^{-\eta y} - 1)J_1(y)\mathrm{d}y - r_2b_2k_2 \\ = & \, 2r_1\big[(2a_1 - \frac{b_1}{2})\phi(\xi_0) - 1 - 2b_1k_2\big] - r_2b_2k_2 - \int_{-\infty}^{0} e^{-\eta y}J_1(y)\mathrm{d}y + \frac{1}{2} \\ =: & \, C_5 > 0\end{aligned}$$

by the condition (5). Finally, let $C_2 := \min\{C_4, C_5\}$. Then we have $A_{\eta,w}(\xi) \geq C_2 > 0$. □

Lemma 2.6. *Let $\mu_1 > 0$ be the unique root of the following equation*

$$2\hat{C} - 2\mu_1 - h(\mathrm{e}^{2\mu_1\tau} - 1) = 0,$$

where $\hat{C} = \frac{1}{2}\min\{C_2, C_3\}$ and $h = \max\{r_1b_1k_1, r_2b_2k_2\}$. Then for $0 < \mu \leq \mu_1$, $B_{\mu,w}(\xi) \geq C_6 > 0$ and $D_{\mu,w}(\xi) \geq C_6 > 0$ on $\mathbb{R}$.

Proof. Note that $(0,0) \leq (\phi(\xi), \psi(\xi)) \leq (k_1, k_2)$ on $\mathbb{R}$ and

$$\frac{w(\xi + c\tau)}{w(\xi)} = \begin{cases} \mathrm{e}^{-\eta c\tau} < 1 & \text{for } \xi \leq \xi_0 - c\tau, \\ \mathrm{e}^{\eta(\xi-\xi_0)} < 1 & \text{for } \xi_0 - c\tau < \xi \leq \xi_0, \\ 1 & \text{for } \xi > \xi_0. \end{cases}$$

It is easy to see that, for $0 < \mu \leq \mu_1$,

$$\begin{aligned} B_{\mu,w}(\xi) &= A_w(\xi) - 2\mu - r_2b_2(\mathrm{e}^{2\mu\tau} - 1)\int_{-\infty}^{\infty} \frac{w(\xi + c\tau)}{w(\xi)} h(y)\psi(\xi + y + c\tau)\mathrm{d}y \\ &\geq C_2 - 2\mu - r_2b_2k_2(\mathrm{e}^{2\mu\tau} - 1) \\ &\geq 2\hat{C} - 2\mu - h(\mathrm{e}^{2\mu\tau} - 1) \\ &=: C_6 > 0. \end{aligned}$$

Similarly, we can prove that $D_{\mu,w}(\xi) \geq C_6 > 0$. □

Submitting this result into (17), we have

$$\begin{aligned} &\mathrm{e}^{2\mu t}\left(\|m(t)\|^2_{L^2_w} + \|n(t)\|^2_{L^2_w}\right) + C_6\int_0^t \mathrm{e}^{2\mu s}\left(\|m(s)\|^2_{L^2_w} + \|n(s)\|^2_{L^2_w}\right)\mathrm{d}s \\ &\leq C_1\left(\|m_0(0)\|^2_{L^2_w} + \|n_0(0)\|^2_{L^2_w} + \int_{-\tau}^0 \left(\|m_0(s)\|^2_{L^2_w} + \|n_0(s)\|^2_{L^2_w}\right)\mathrm{d}s\right). \end{aligned} \tag{18}$$

Dropping the positive terms

$$\int_0^t \mathrm{e}^{2\mu s}\left(\|m(s)\|^2_{L^2_w} + \|n(s)\|^2_{sL^2_w}\right)\mathrm{d}s,$$

we obtain the basic estimates

$$\begin{cases} \mathrm{e}^{2\mu t}\|m(t)\|^2_{L^2_w} \leq C_1\Big[\|m_0(0)\|^2_{L^2_w} + \|n_0(0)\|^2_{L^2_w} \\ \qquad\qquad + \int_{-\tau}^0 \left(\|m_0(s)\|^2_{L^2_w} + \|n_0(s)\|^2_{L^2_w}\right)\mathrm{d}s\Big], \\ \mathrm{e}^{2\mu t}\|n(t)\|^2_{L^2_w} \leq C_1\Big[\|m_0(0)\|^2_{L^2_w} + \|n_0(0)\|^2_{L^2_w} \\ \qquad\qquad + \int_{-\tau}^0 \left(\|m_0(s)\|^2_{L^2_w} + \|n_0(s)\|^2_{L^2_w}\right)\mathrm{d}s\Big]. \end{cases} \tag{19}$$

Next, we differentiate (8) with respect to ξ, and then multiply the resulting equation by $e^{2\mu t}w(\xi)m_\xi(\xi,t)$ and $e^{2\mu t}w(\xi)n_\xi(\xi,t)$, respectively, where $w(\xi)$ again denotes the function in (6), with the same values ξ_0, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}e^{2\mu t}wm_\xi^2\right)_t + \left(\frac{c}{2}e^{2\mu t}wm_\xi^2\right)_\xi - e^{2\mu t}wm_\xi\int_{-\infty}^{\infty}J_1(y)m_\xi(\xi-y,t)\mathrm{d}y\\
&+\left(-\frac{cw'}{2w}-\mu+1+r_1\big[2a_1\phi-1-b_1n(\xi-c\tau,t-\tau)-b_1\psi(\xi-c\tau)\big]\right)wm_\xi^2e^{2\mu t}\\
&= r_1\big(b_1n(\xi-c\tau,t-\tau)+b_1\psi(\xi-c\tau)-2a_1\phi\big)_\xi wmm_\xi e^{2\mu t}-2a_1r_1mm_\xi^2we^{2\mu t}\\
&\quad + r_1b_1\phi_\xi n(\xi-c\tau,t-\tau)wm_\xi e^{2\mu t}+r_1b_1\phi n_\xi(\xi-c\tau,t-\tau)wm_\xi e^{2\mu t},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{1}{2}e^{2\mu t}wn_\xi^2\right)_t + \left(\frac{c}{2}e^{2\mu t}wn_\xi^2\right)_\xi - e^{2\mu t}wn_\xi\int_{-\infty}^{\infty}J_2(y)n_\xi(\xi-y,t)\mathrm{d}y\\
&+\left(-\frac{cw'}{2w}-\mu+r_2\big[2a_2\psi-1-b_2m(\xi-c\tau,t-\tau)+\phi(\xi-c\tau)\big]\right)wn_\xi^2e^{2\mu t}\\
&= r_2\big[b_2m(\xi-c\tau,t-\tau)+\phi(\xi-c\tau))-2a_2\psi\big]_\xi wnn_\xi e^{2\mu t}-2a_2r_2nn_\xi^2we^{2\mu t}\\
&\quad + r_2b_2\psi_\xi wn_\xi e^{2\mu t}m(\xi-c\tau,t-\tau)+r_2b_2\psi wn_\xi e^{2\mu t}m_\xi(\xi-c\tau,t-\tau),
\end{aligned}$$

Integrating over $\mathbb{R}\times[0,t]$ and carrying out similar steps to those that led to (18), we obtain

$$\begin{aligned}
&e^{2\mu t}\left(\|m_\xi(t)\|_{L_w^2}^2+\|n_\xi(t)\|_{L_w^2}^2\right)+C_6\int_0^t e^{2\mu s}\left(\|m_\xi(s)\|_{L_w^2}^2+\|n_\xi(s)\|_{L_w^2}^2\right)\mathrm{d}s\\
&\le C_1\left(\|m_{0\xi}(0)\|_{L_w^2}^2+\|n_{0\xi}(0)\|_{L_w^2}^2+\int_{-\tau}^0\left(\|m_{0\xi}(s)\|_{L_w^2}^2+\|n_{0\xi}(s)\|_{L_w^2}^2\right)\mathrm{d}s\right) \qquad (20)\\
&\quad+\int_0^t\int_{\mathbb{R}}Q(\xi,s)w(\xi)e^{2\mu s}\mathrm{d}\xi\mathrm{d}s,
\end{aligned}$$

where

$$\begin{aligned}
Q(\xi,s) = {} & r_1\big(b_1n(\xi-c\tau,t-\tau)+b_1\psi(\xi-c\tau)-2a_1\phi\big)_\xi mm_\xi\\
&+r_1b_1\phi_\xi n(\xi-c\tau,t-\tau)m_\xi+r_2\big[b_2m(\xi-c\tau,t-\tau)\\
&+\phi(\xi-c\tau)-2a_2\psi\big]_\xi wnn_\xi e^{2\mu t}+r_2b_2\psi_\xi wn_\xi e^{2\mu t}m(\xi-c\tau,t-\tau).
\end{aligned}$$

Note that $(0,0)\le(m(\xi,t),n(\xi,t))\le(k_1,k_2)$ and ϕ_ξ, ψ_ξ are bounded on $\mathbb{R}$, using Young-inequality and similar to [19, Section 3], we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}Q(\xi,s)w(\xi)e^{2\mu s}\mathrm{d}\xi\mathrm{d}s\\
&\le C\left(\|m_0(0)\|_{L_w^2}^2+\|n_0(0)\|_{L_w^2}^2+\int_{-\tau}^0\left(\|m_0(s)\|_{L_w^2}^2+\|n_0(s)\|_{L_w^2}^2\right)\mathrm{d}s\right).
\end{aligned}$$

Substituting the inequality into (20) and dropping the positive terms

$$\int_0^t e^{2\mu s}\left(\|m_\xi(s)\|_{L_w^2}^2+\|n_\xi(s)\|_{L_w^2}^2\right)ds,$$

we obtain

$$\begin{cases} e^{2\mu t}\|m_\xi(t)\|_{L_w^2}^2 \le C_7\Big[\|m_0(0)\|_{H_w^1}^2+\|n_0(0)\|_{H_w^1}^2 \\ \qquad\qquad + \int_{-\tau}^0\left(\|m_0(s)\|_{H_w^1}^2+\|n_0(s)\|_{H_w^1}^2\right)ds\Big], \\ e^{2\mu t}\|n_\xi(t)\|_{L_w^2}^2 \le C_7\Big[\|m_0(0)\|_{H_w^1}^2+\|n_0(0)\|_{H_w^1}^2 \\ \qquad\qquad + \int_{-\tau}^0\left(\|m_0(s)\|_{H_w^1}^2+\|n_0(s)\|_{H_w^1}^2\right)ds\Big]. \end{cases} \tag{21}$$

Combining (19) with (21) and noting that $w(\xi)\ge 1$ on $\mathbb{R}$, we obtain the following decay rate result.

Lemma 2.7. *It holds that, for $t>0$,*

$$\begin{aligned} \|m(t)\|_{H^1} &\le \|m(t)\|_{H_w^1} \\ &\le C_8 e^{-\mu t}\left[\|m_0(0)\|_{H_w^1}^2+\|n_0(0)\|_{H_w^1}^2+\int_{-\tau}^0\left(\|m_0(s)\|_{H_w^1}^2+\|n_0(s)\|_{H_w^1}^2\right)ds\right], \\ \|n(t)\|_{H^1} &\le \|n(t)\|_{H_w^1} \\ &\le C_8 e^{-\mu t}\left[\|m_0(0)\|_{H_w^1}^2+\|n_0(0)\|_{H_w^1}^2+\int_{-\tau}^0\left(\|m_0(s)\|_{H_w^1}^2+\|n_0(s)\|_{H_w^1}^2\right)ds\right], \end{aligned}$$

where $C_8=\max\left\{\sqrt{C_1},\sqrt{C_7}\right\}$.

Using Sobolev embedding theorem $H^1(\mathbb{R})\hookrightarrow C^0(\mathbb{R})$ and Lemma 2.7, we have the following stability result

Lemma 2.8. *It holds that*

$$\begin{aligned} \sup_{x\in\mathbb{R}}|u^+(x,t)-\phi(x+ct)| &= \sup_{\xi\in\mathbb{R}}|m(\xi,t)|\le C_9 e^{-\mu t}, \quad t>0, \\ \sup_{x\in\mathbb{R}}|v^+(x,t)-\psi(x+ct)| &= \sup_{\xi\in\mathbb{R}}|n(\xi,t)|\le C_9 e^{-\mu t}, \quad t>0, \end{aligned}$$

where $C_9>0$.

Step 2. Next, we prove the convergence of $u^-(x,t)$ to $\phi(x+ct)$ and $v^-(x,t)$ to $\psi(x+ct)$.

Let $\xi = x + ct$ and

$$\begin{aligned} m(\xi,t) &= u^-(x,t) - \phi(x+ct), \quad m_0(\xi,s) = u_0^-(x,s) - \phi(x+cs), \\ n(\xi,t) &= v^-(x,t) - \psi(x+ct), \quad n_0(\xi,s) = v_0^-(x,s) - \psi(x+cs). \end{aligned}$$

Similar to Step 1, we have the following stability result

Lemma 2.9. *It holds that*

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |u^-(x,t) - \phi(x+ct)| = \sup_{\xi\in\mathbb{R}} |m(\xi,t)| \le C_{10} \mathrm{e}^{-\mu t}, \quad t > 0, \\ &\sup_{x\in\mathbb{R}} |v^-(x,t) - \psi(x+ct)| = \sup_{\xi\in\mathbb{R}} |n(\xi,t)| \le C_{10} \mathrm{e}^{-\mu t}, \quad t > 0, \end{aligned}$$

where $C_{10} > 0$.

Step 3. In the last step, we prove the convergence of $(u(x,t), v(x,t))$ to $(\phi(x+ct), \psi(x+ct))$.

Using Lemmas 2.8 and 2.9, similar to the proof of Lemma 3.10 in [21], we can prove the convergence of $(u(x,t), v(x,t))$ to $(\phi(x+ct), \psi(x+ct))$, that is

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct)| \le C_{11} \mathrm{e}^{-\mu t}, \quad t > 0, \\ &\sup_{x\in\mathbb{R}} |v(x,t) - \psi(x+ct)| \le C_{11} \mathrm{e}^{-\mu t}, \quad t > 0 \end{aligned}$$

for some $C_{11} > 0$.

The proof of Theorem 2.1 is completed.

3. Discussion

In this paper, we have established the existence and nonlinear stability of traveling wave fronts for delayed Lotka-Volterra cooperative system. It is remarked that the upper and lower solution used to establish the existence of traveling wave front is same as that in [16]. Moreover, it is easily seen that our method is also suitable to nonlocal delayed Lotka-Volterra competition system and nonlocal delayed Belousov-Zhabotinskii system.

Consider the following nonlocal dealyed Lotka-Volterra competition system

$$\begin{cases} u_t - J_1 * u + u = u(a_1 - b_1 u - c_1 v(x, t-\tau)), & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ v_t - J_2 * v + v = v(a_2 - b_2 u(x,t-\tau) - c_2 v), & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \end{cases} \tag{22}$$

with initial data (4) and nonlocal delayed Belousov-Zhabotinskii system

$$\begin{cases} u_t - J_1 * u + u = u[1 - u - rv(x, t-\tau)], & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ v_t - J_2 * v + v = -buv, & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \end{cases} \tag{23}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), & (x,s) \in \mathbb{R} \times [-\tau, 0], \\ v(x,s) = v_0(x,s), & (x,s) \in \mathbb{R} \times [-\tau, 0], \end{cases} \tag{24}$$

where $r > 0$ and $b > 0$ are constants, functions $J_i, h(y)$ $(i = 1, 2)$ are defined as Section 3. The existence of traveling wave front for system (22) can be proved similar to Section 3. Just Recently, Pan-Li-Lin [25] obtained the existence of traveling wave solution of system (23) by the fixed point theorem. The stability of traveling wave solutions for system (22) and system (23), connecting $(0, \frac{a_2}{c_2})$ with $(\frac{a_1}{b_1}, 0)$ and $(1, 0)$ with $(0, \frac{1}{r})$, can be proved similarly to Section 4, respectively. We leave the details to reader.

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