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Interpolation of Closed Subspaces and Invertibility of Operators

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Abstract. Let (Y_0, Y_1) be a Banach couple and let X_j be a closed complemented subspace of Y_j, $(j = 0, 1)$. We present several results for the general problem of finding necessary and sufficient conditions on the parameters (θ, q) such that the real interpolation space $(X_0, X_1)_{\theta,q}$ is a closed subspace of $(Y_0, Y_1)_{\theta,q}$. In particular, we establish conditions which are necessary and sufficient for the equality $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$, with the proof based on a previous result by Asekritova and Kruglyak on invertibility of operators. We also generalize the theorem by Ivanov and Kalton where this problem was solved under several rather restrictive conditions, such as that $X_1 = Y_1$ and X_0 is a subspace of codimension one in Y_0.

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1. Introduction

Interpolation of subspaces is an important and difficult problem that arose at the beginning of modern interpolation theory, in applications of interpolation to boundary value problems for partial differential equations. In the papers by Triebel [12] and Wallstén [14], it was shown that interpolation of subspaces may behave badly. Lions and Magenes wrote in their book (see [9, p. 107]) that "the main difficulties of the use of interpolation is that the interpolated space between closed subspaces is not necessarily a closed subspace in the interpolated space" and later on the same page "It would be of great interest to obtain criteria allowing to affirm a priory that, except for certain values of the parameters, the interpolation space is closed". This is exactly the problem that we consider below.

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Let (X_0, X_1) be a subcouple of a Banach couple (Y_0, Y_1), i.e. let X_j be a closed subspace of Y_j. Then the general problem for the real method can be formulated as follows:

Problem 1. *Find necessary and sufficient conditions on the parameters* (θ, q) *such that the real interpolation space* $(X_0, X_1)_{\theta,q}$ *is a closed subspace of the space* $(Y_0, Y_1)_{\theta,q}$.

Problem 1 was solved by Ivanov and Kalton in [6] (a slightly weaker result was obtained earlier by Löfström [10, 11]) under the following conditions:

a) the couple (Y_0, Y_1) is regular (that is, $Y_0 \cap Y_1$ is dense in Y_i for $i = 0, 1$) and $X_1 = Y_1$,
b) X_0 is a closed subspace of codimension one in Y_0,
c) $X_0 \cap X_1$ is dense in X_1,
d) $1 \le q < \infty$.

The most restrictive condition in the Ivanov-Kalton theorem is b), i.e. that the codimension of X_0 in Y_0 is equal to one. It is interesting to generalize this theorem to the case when X_0 is a closed subspace of a *finite* codimension in Y_0. Such a generalization is presented in Section 2. Since the proof is based on the Ivanov-Kalton theorem, we cannot omit the other restrictions.

In Section 3, we establish necessary and sufficient conditions for the equality

$$(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q} \tag{1.1}$$

under the single condition that the space X_j is complemented in Y_j $(j = 0, 1)$. Thus the codimension of the space X_j in Y_j $(j = 0, 1)$ as well as the parameter q can be equal to infinity. The proof is based on a quite recent result on invertibility of operators in spaces of real interpolation [1]. Note that the condition that the space X_j is complemented in Y_j $(j = 0, 1)$ is fulfilled in many cases, for example, when the codimension of space X_j in Y_j is finite or if X_j and Y_j are Hilbert spaces. We also establish in this section some results for the couple $(L^2(\Omega), W^{1,2}(\Omega))$.

In Section 4, we use duality to establish a connection between our characterization of equality (1.1) and the generalization of the Ivanov-Kalton theorem obtained in Section 2.

Everywhere below, we will freely use standard notions and facts of real interpolation (see the books [2–4], or [13]).

2. Generalization of the Ivanov-Kalton theorem

We start by formulating the Ivanov-Kalton theorem from [6].

Let us consider a regular couple (Y_0, Y_1). Then there exists a conjugate couple (Y_0^*, Y_1^*) and for a non-zero element $\psi \in Y_0^*$ we can define indices (with respect to the couple (Y_0^*, Y_1^*))

$$\alpha_0(\psi) = \sup\left\{\theta \in [0,1] : \frac{K(s,\psi; Y_0^*, Y_1^*)}{K(t,\psi; Y_0^*, Y_1^*)} \le \gamma \left(\frac{s}{t}\right)^\theta, \text{ for all } 0 < s < t \le 1\right\},$$
$$\beta_0(\psi) = \inf\left\{\theta \in [0,1] : \frac{K(s,\psi; Y_0^*, Y_1^*)}{K(t,\psi; Y_0^*, Y_1^*)} \ge \gamma \left(\frac{s}{t}\right)^\theta, \text{ for all } 0 < s < t \le 1\right\},$$

where $\gamma = \gamma(\theta, \psi)$ is a constant independent of s and t. Clearly,

$$0 \le \alpha_0(\psi) \le \beta_0(\psi) \le 1.$$

Theorem 2.1 (Ivanov-Kalton)**.** *Suppose that the following conditions are satisfied:*

a) *the couple (Y_0, Y_1) is regular and $X_1 = Y_1$,*
b) *X_0 is a closed subspace of codimension one in Y_0,*
c) *$X_0 \cap X_1$ is dense in X_1,*
d) *$1 \le q < \infty$.*

Let $\psi \in Y_0^*$ be such that $X_0 = \ker\psi$, then $(X_0, X_1)_{\theta,q}$ is a closed subspace of $(Y_0, Y_1)_{\theta,q}$ if and only if

$$\theta \notin [\alpha_0(\psi), \beta_0(\psi)].$$

Moreover, for $\theta \in (0, \alpha_0(\psi))$ the space $(X_0, X_1)_{\theta,q}$ is a closed subspace of codimension one in $(Y_0, Y_1)_{\theta,q}$ and for $\theta \in (\beta_0(\psi), 1)$ the space $(X_0, X_1)_{\theta,q}$ coincides with $(Y_0, Y_1)_{\theta,q}$.

Remark 2.2. In the paper by Ivanov and Kalton [6], dilation indices were used instead of the indices $\alpha_0(\psi)$, $\beta_0(\psi)$:

$$\sigma_0 = \lim_{k\to\infty}\left(\inf_{n\ge 0} \frac{1}{k}\ln_2\left(\frac{K(2^{-n},\psi; Y_0^*, Y_1^*)}{K(2^{-n-k},\psi; Y_0^*, Y_1^*)}\right)\right),$$
$$\sigma_1 = \lim_{k\to\infty}\left(\sup_{n\in\mathbb{Z}} \frac{1}{k}\ln_2\left(\frac{K(2^{-n},\psi; Y_0^*, Y_1^*)}{K(2^{-n-k},\psi; Y_0^*, Y_1^*)}\right)\right).$$

However, it is not hard to show that $\sigma_0 = \alpha_0(\psi)$ and $\sigma_1 = \beta_0(\psi)$. The proof of the second equality uses the fact that $\psi \in Y_0^*$.

In this section we obtain a generalization of the Ivanov-Kalton theorem in which instead of the condition "X_0 is a closed subspace of codimension one in Y_0"

we assume that X_0 is a subspace of codimension n in Y_0, i.e. $\dim(Y_0/X_0) = n$. Note that since

$$(Y_0/X_0)^* = X_0^{\perp} = \{\psi \in Y_0^* : \psi(X_0) = 0\}$$

(see [5, Theorem III.10.2 (p. 91)]), the annihilator $X_0^{\perp}$ also has dimension n.

To formulate the result we will need the following lemma.

Lemma 2.3. *Suppose that the couple (Y_0, Y_1) is regular and X_0 is a closed subspace of codimension n in Y_0. Let $\psi_1, \ldots, \psi_n$ be a basis in $X_0^{\perp}$. Then there exists a system of vectors $e_1, \ldots, e_n \in Y_0 \cap Y_1$ such that $\psi_i(e_j) = \delta_{ij}$ and*

$$Y_0 = X_0 \oplus \operatorname{span}\{e_1, \ldots, e_n\}.$$

Proof. Since vectors $\psi_1, \ldots, \psi_n$ form a basis in $X_0^{\perp}$, there exit vectors $u_1, \ldots, u_n$ in Y_0 such that $\psi_i(u_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. Put M_0 for the subspace generated by $\{u_1, \ldots, u_n\}$. Clearly, $u_1, \ldots, u_n$ is a basis in M_0. Furthermore, since the operator $P_0 : Y_0 \to M_0$ defined by the formula

$$P_0(y) = \sum_{i=1}^{n} \psi_i(y) u_i$$

is a continuous linear projection onto M_0 with the kernel X_0, we have that $Y_0 = X_0 \oplus M_0$. From the regularity of the couple (Y_0, Y_1) it follows that the linear space $Y_0 \cap Y_1$ is dense in Y_0 and hence $P_0(Y_0 \cap Y_1)$ is a linear space dense in M_0. Since the dimension of the space M_0 is finite, we have that $P_0(Y_0 \cap Y_1) = M_0$. Thus it is possible to find vectors $e_1, \ldots, e_n \in Y_0 \cap Y_1$ such that $P_0(e_j) = u_j$, $j = 1, \ldots, n$. Let

$$e_j = x_j + u_j, \quad x_j \in X_0, \quad j = 1, \ldots, n.$$

Since $\psi_i(e_j) = \delta_{ij}$, the operator $P : Y_0 \to \operatorname{span}\{e_1, \ldots, e_n\}$ defined by the formula

$$P(y) = \sum_{i=1}^{n} \psi_i(y) e_i$$

is a continuous linear projection on $\operatorname{span}\{e_1, \ldots, e_n\}$ with the kernel X_0. Consequently, $Y_0 = X_0 \oplus \operatorname{span}\{e_1, \ldots, e_n\}$. □

To formulate the result let us fix a basis $\psi_1, \ldots, \psi_n$ in $X_0^{\perp}$ and fix a system of vectors $e_1, \ldots, e_n \in Y_0 \cap Y_1$ such that $\psi_i(e_j) = \delta_{ij}$ and $Y_0 = X_0 \oplus \operatorname{span}\{e_1, \ldots, e_n\}$ (the existence of such a system follows from Lemma 2.3). Then

$$Y_0^* = X_0^* \oplus \operatorname{span}\{\psi_1, \ldots, \psi_n\},$$

where

$$X_0^* = \{y_* \in Y_0^* : y_*(e_i) = 0,\ i = 1, \ldots, n\}.$$

For the element ψ_i $(i = 1, \ldots, n)$ we can consider the indices $\alpha_0(\psi_i)$, $\beta_0(\psi_i)$ defined with respect to the couple

$$\vec{U}_i = (X_0^* \oplus \operatorname{span}\{\psi_1, \ldots, \psi_i\}, Y_1^*),$$

i.e.

$$\alpha_0(\psi_i) = \sup\left\{\theta \in [0,1] : \frac{K(s, \psi_i; \vec{U}_i)}{K(t, \psi_i; \vec{U}_i)} \le \gamma \left(\frac{s}{t}\right)^\theta, 0 < s < t \le 1\right\}, \tag{2.1}$$

$$\beta_0(\psi_i) = \inf\left\{\theta \in [0,1] : \frac{K(s, \psi_i; \vec{U}_i)}{K(t, \psi_i; \vec{U}_i)} \ge \gamma \left(\frac{s}{t}\right)^\theta, 0 < s < t \le 1\right\}. \tag{2.2}$$

We are now ready to formulate the theorem.

Theorem 2.4. *Suppose that the following conditions are satisfied:*

a) *the couple* (Y_0, Y_1) *is regular and* $X_1 = Y_1$*,*
b) X_0 *is a closed subspace of codimension* n *in* Y_0*,*
c) $X_0 \cap X_1$ *is dense in* X_1*,*
d) $0 < \theta < 1$ *and* $1 \le q < \infty$*.*

Then the space $(X_0, X_1)_{\theta,q}$ *is a closed subspace in* $(Y_0, Y_1)_{\theta,q}$ *if and only if*

$$\theta \notin \cup_{i=1,\ldots,n} [\alpha_0(\psi_i), \beta_0(\psi_i)].$$

Furthermore, if $\theta \notin \cup_{i=1,\ldots,n} [\alpha_0(\psi_i), \beta_0(\psi_i)]$ *and the number of intervals* $[\alpha_0(\psi_i), \beta_0(\psi_i)]$ *that lie on the right of* θ *(i.e.* $\theta < \alpha_0(\psi_i)$*) is equal to* k*, then the space* $(X_0, X_1)_{\theta,q}$ *is a closed subspace of codimension* k *in* $(Y_0, Y_1)_{\theta,q}$*.*

To prove the theorem we need the following lemma.

Lemma 2.5. *Suppose that the conditions of Theorem* 2.4 *are satisfied. Then the couple* (X_0, X_1) *and the couples* $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_i), X_1)$*,* $i = 1, \ldots, n$ *are regular.*

Proof. Let us first note that without loss of generality we can change the norm in Y_0 to an equivalent norm, so we can assume that

$$\left\| x + \sum_{i=1}^n \lambda_i e_i \right\|_{Y_0} = \|x\|_{X_0} + \sum_{i=1}^n |\lambda_i|. \tag{2.3}$$

It is clear from the condition c) above that to prove regularity of the couple (X_0, X_1) it is sufficient to prove that $X_0 \cap X_1$ is dense in X_0. Let $x \in X_0$.

Since $e_1, \ldots, e_n \in Y_0 \cap Y_1$ and $Y_1 = X_1$, we have that $Y_0 \cap Y_1 = (X_0 \cap X_1) \oplus \operatorname{span}\{e_1, \ldots, e_n\}$. Then the regularity of the couple (Y_0, Y_1) implies that for any $x \in X_0$ there exists a sequence

$$y_k = x_k + \sum_{i=1}^{n} \lambda_k^i e_i, \quad x_k \in X_0 \cap X_1, \quad k = 1, 2, \ldots$$

such that $\|x - y_k\|_{Y_0} \to 0$ for $k \to \infty$. Using (2.3) we obtain that

$$\|x - y_k\|_{Y_0} = \|x - x_k\|_{X_0} + \sum_{i=1}^{n} \left|\lambda_k^i\right|,$$

where $x_k \in X_0 \cap X_1$. Then for each $x \in X_0$ there exists a sequence $\{x_k\} \subset X_0 \cap X_1$ such that $\|x - x_k\|_{X_0} \to 0$ for $k \to \infty$, i.e. the couple (X_0, X_1) is regular.

Moreover, as $\operatorname{span}(e_1, \ldots, e_i) \subset X_1$ we have

$$(X_0 \oplus \operatorname{span}(e_1, \ldots, e_i)) \cap X_1 = (X_0 \cap X_1) \oplus \operatorname{span}\{e_1, \ldots, e_i\}.$$

Then from the regularity of the couple (X_0, X_1) it follows that the couple $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_i), X_1)$ is also regular. □

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Since the couple $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_i), X_1)$ is regular, we can consider its dual, which can be written as $(X_0^* \oplus \operatorname{span}(\psi_1, \ldots, \psi_i), X_1^*)$, where as before

$$X_0^* = \{y_* \in Y_0^* : y_*(e_i) = 0, \; i = 1, \ldots, n\}.$$

Moreover, we have

$$\psi_i \in X_0^* \oplus \operatorname{span}(\psi_1, \ldots, \psi_i) = (X_0 \oplus \operatorname{span}(e_1, \ldots, e_i))^*$$

and the kernel of the element ψ_i on the space $X_0 \oplus \operatorname{span}(e_1, \ldots, e_i)$ coincides with

$$X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i-1}).$$

Thus all the conditions of the Ivanov-Kalton theorem are fulfilled for the subcouple $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i-1}), X_1)$ of the couple $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_i), X_1)$, and therefore the space

$$(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i-1}), X_1)_{\theta,q}$$

is a closed subspace of

$$(X_0 \oplus \operatorname{span}(e_1, \ldots, e_i), X_1)_{\theta,q}$$

if and only if $\theta \notin [\alpha_0(\psi_i), \beta_0(\psi_i)]$, where the indices $\alpha_0(\psi_i)$, $\beta_0(\psi_i)$ $(i = 1, \dots, n)$ are defined by the formulas (2.1), (2.2). Moreover, the space

$$(X_0 \oplus \operatorname{span}(e_1, \dots, e_{i-1}), X_1)_{\theta,q}$$

is a closed subspace of codimension one in

$$(X_0 \oplus \operatorname{span}(e_1, \dots, e_i), X_1)_{\theta,q}$$

if $\theta < \alpha_0(\psi_i)$ and it coincides with $(X_0 \oplus \operatorname{span}(e_1, \dots, e_i), X_1)_{\theta,q}$ if $\theta > \beta_0(\psi_i)$.

Suppose $\theta \notin \cup_{i=1,\dots,n} [\alpha_0(\psi_i), \beta_0(\psi_i)]$. Let k be the number of intervals $[\alpha_0(\psi_i), \beta_0(\psi_i)]$ that lie on the right of θ, i.e. $\theta < \alpha_0(\psi_i)$. Then among the embedding operators

$$I_i : (X_0 \oplus \operatorname{span}(e_1, \dots, e_{i-1}), X_1)_{\theta,q} \hookrightarrow (X_0 \oplus \operatorname{span}(e_1, \dots, e_i), X_1)_{\theta,q}$$

there are exactly k operators whose images are closed subspaces of codimension one and $n - k$ operators that are isomorphisms. Thus when

$$\theta \notin \cup_{i=1,\dots,n} [\alpha_0(\psi_i), \beta_0(\psi_i)] ,$$

the image of $(X_0, X_1)_{\theta,q}$ in $(X_0 \oplus \operatorname{span}(e_1, \dots, e_n), X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ is a closed subspace of codimension k.

To prove the theorem we only need to show that in the case when

$$\theta \in \cup_{i=1,\dots,n} [\alpha_0(\psi_i), \beta_0(\psi_i)] ,$$

the space $(X_0, X_1)_{\theta,q}$ is not a closed subspace of $(Y_0, Y_1)_{\theta,q}$. Let

$$i_* = \min \{ i : \theta \in [\alpha_0(\psi_i), \beta_0(\psi_i)] \} .$$

If $i_* > 1$, then $\theta \in [\alpha_0(\psi_{i_*}), \beta_0(\psi_{i_*})]$ and $\theta \notin [\alpha_0(\psi_i), \beta_0(\psi_i)]$ for all $i < i_*$. Therefore, for $i = 1, .., i_* - 1$ the images of embedding operators I_i are closed subspaces of codimension one or zero. As above, the space $(X_0, X_1)_{\theta,q}$ is a closed subspace of a finite codimension in $(X_0 \oplus \operatorname{span}(e_1, \dots, e_{i_*-1}), X_1)_{\theta,q}$, i.e.

$$(X_0 \oplus \operatorname{span}(e_1, \dots, e_{i_*-1}), X_1)_{\theta,q} = (X_0, X_1)_{\theta,q} \oplus M, \tag{2.4}$$

where M is a finite dimensional space. Since $\theta \in [\alpha_0(\psi_{i_*}), \beta_0(\psi_{i_*})]$ we have that

$$(X_0 \oplus \operatorname{span}(e_1, \dots, e_{i_*-1}), X_1)_{\theta,q}$$

is not a closed subspace of $(X_0 \oplus \operatorname{span}(e_1, \dots, e_{i_*}), X_1)_{\theta,q}$ and from (2.4) we see that

$$(X_0, X_1)_{\theta,q} \oplus M$$

is not a closed subspace of $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i_*}), X_1)_{\theta,q}$. Hence $(X_0, X_1)_{\theta,q}$ is not a closed subspace of $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i_*}), X_1)_{\theta,q}$. Indeed, if $(X_0, X_1)_{\theta,q}$ were a closed subspace of the space $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i_*}), X_1)_{\theta,q}$ then the space $(X_0, X_1)_{\theta,q} \oplus M$ would also be a closed subspace of $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i_*}), X_1)_{\theta,q}$ (as a sum of a closed subspace and a finite dimensional space, see [8, Lemma 1.9 in Chapter 3]), which is not true. Hence $(X_0, X_1)_{\theta,q}$ is not a closed subspace of $(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i_*}), X_1)_{\theta,q}$.

Now, from the continuity of the embedding

$$(X_0 \oplus \operatorname{span}(e_1, \ldots, e_{i_*}), X_1)_{\theta,q} \hookrightarrow (Y_0, Y_1)_{\theta,q}$$

it follows that $(X_0, X_1)_{\theta,q}$ is not a closed subspace of $(Y_0, Y_1)_{\theta,q}$.

If $i_* = 1$, the Ivanov-Kalton theorem implies that $(X_0, X_1)_{\theta,q}$ is not closed in $(X_0 \oplus \operatorname{span}(e_1), X_1)_{\theta,q}$ and therefore $(X_0, X_1)_{\theta,q}$ is not a closed subspace of $(Y_0, Y_1)_{\theta,q}$. □

Remark 2.6. Under the conditions of Theorem 2.4 we have that $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ if and only if there are no intervals $[\alpha_0(\psi_i), \beta_0(\psi_i)]$ that lie on the right of θ, i.e.

$$\theta > \max_{1 \le i \le n} \beta_0(\psi_i). \tag{2.5}$$

Since the set of parameters (θ, q) for which we have $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ is independent of the basis $\psi_1, \ldots, \psi_n$ in $X_0^\perp$, then $\max_{1 \le i \le n} \beta_0(\psi_i)$ does not depend on the basis $\psi_1, \ldots, \psi_n$ in $X_0^\perp$.

In fact, in Section 4 we will *obtain an expression for* the right-hand side of (2.5) that does not depend on the basis in $X_0^\perp$.

3. Interpolation of subcouples

In this section we will give necessary and sufficient conditions for the equality $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ without the restrictive assumptions on the couples $\vec{X}$, $\vec{Y}$ and the parameter q that we have in the previous section.

Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be two Banach couples and let $T : \vec{X} \to \vec{Y}$ be a bounded linear operator. By $\ker T$ we will denote the kernel of T on the sum $X_0 + X_1$, i.e.

$$\ker T = \{x \in X_0 + X_1 : Tx = 0\}.$$

Let us fix $\theta \in (0, 1)$ and consider two special subspaces of $\ker T$:

$$V^0_{\theta,\infty} = \left\{ x \in \ker T : \sup_{0<t\le 1} \frac{K(t, x; \vec{X})}{t^\theta} < \infty \right\},$$

$$V^1_{\theta,\infty} = \left\{ x \in \ker T : \sup_{1\le t} \frac{K(t, x; \vec{X})}{t^\theta} < \infty \right\}.$$

Our main tool is the following result (see [1, p. 209]).

Theorem 3.1. *Let* $\theta \in (0,1), q \in [1,\infty]$ *and let* $T : \vec{X} \to \vec{Y}$ *be a bounded linear operator. Suppose that the restrictions* $T : X_i \longrightarrow Y_i$ *have bounded inverses* $(i = 0,1)$*. Then* $T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ *is invertible if and only if*

$$\ker T = V^0_{\theta,\infty} \oplus V^1_{\theta,\infty}$$

and there are positive constants γ, ε *such that for all* $0 < s < t$ *and all* $x \in V^0_{\theta,\infty}$ *we have*

$$K(s,x;\vec{X}) \leq \gamma \left(\frac{s}{t}\right)^{\theta+\varepsilon} K(t,x;\vec{X}) \tag{3.1}$$

and for all $0 < s < t$ *and all* $x \in V^1_{\theta,\infty}$ *we have*

$$K(s,x;\vec{X}) \geq \gamma \left(\frac{s}{t}\right)^{\theta-\varepsilon} K(t,x;\vec{X}). \tag{3.2}$$

Remark 3.2. In [1, p. 209], this result was formulated in terms of *some* indices, here we use an equivalent (γ,ε) formulation that is more suitable for our purposes.

In the next theorem, $\vec{Y} = (Y_0, Y_1)$ is a Banach couple and X_j is a closed complemented subspace of Y_j $(j = 0,1)$, say $Y_j = X_j \oplus M_j$, $j = 0,1$.

Theorem 3.3. *Let* $\theta \in (0,1)$ *and* $q \in [1,\infty]$*. Then* $(X_0,X_1)_{\theta,q} = (Y_0,Y_1)_{\theta,q}$ *if and only if the following condition holds: there exist positive constants* γ, ε *such that whenever* $u \in M_0$, $v \in M_1$ *and* $u + v \in X_0 + X_1$*, then for any* $0 < s < t$ *we have*

$$\frac{K(s,v;\vec{X}) + s\,\|v\|_{Y_1}}{s^{\theta+\varepsilon}} \leq \gamma \frac{K(t,v;\vec{X}) + t\,\|v\|_{Y_1}}{t^{\theta+\varepsilon}}$$

and

$$\frac{K(t,u;\vec{X}) + \|u\|_{Y_0}}{t^{\theta-\varepsilon}} \leq \gamma \frac{K(s,u;\vec{X}) + \|u\|_{Y_0}}{s^{\theta-\varepsilon}}.$$

Proof. Let $A_0 = X_0 \times M_0 \times \{0\}$ and $A_1 = X_1 \times \{0\} \times M_1$ with the respective norms $\|(x,u,0)\|_{A_0} = \|x\|_{Y_0} + \|u\|_{Y_0}$ and $\|(x,0,v)\|_{A_1} = \|x\|_{Y_1} + \|v\|_{Y_1}$. Clearly, $\vec{A} = (A_0, A_1)$ is a Banach couple. Consider the operator $T : \vec{A} \longrightarrow \vec{Y}$ defined by $T(x,u,v) = x + u + v$. It is not difficult to verify that T is a bounded linear operator from $\vec{A}$ to $\vec{Y}$ and $T : A_0 \longrightarrow Y_0$, $T : A_1 \longrightarrow Y_1$ are invertible.

For any $w = (x,u,v) \in A_0 + A_1$, we have

$$K(t,w;\vec{A}) = K(t,x;\vec{X}) + \|u\|_{Y_0} + t\|v\|_{Y_1}.$$

Hence

$$(A_0,A_1)_{\theta,q} = (X_0,X_1)_{\theta,q} \times \{0\} \times \{0\}$$

and therefore the equality $(X_0,X_1)_{\theta,q} = (Y_0,Y_1)_{\theta,q}$ is equivalent to the invertibility of $T : (A_0,A_1)_{\theta,q} \longrightarrow (Y_0,Y_1)_{\theta,q}$. This allows us to use Theorem 3.1.

Note that if $w = (x, u, v)$ belongs to $\ker T$ then $x = -u - v$. By definition of $V^0_{\theta,\infty}$, if $w \in V^0_{\theta,\infty}$ we obtain that

$$\sup_{0<t\leq 1} \frac{K(t, u + v; \vec{X}) + \|u\|_{Y_0} + t\|v\|_{Y_1}}{t^\theta} < \infty.$$

This implies that $\sup_{0<t\leq 1} t^{-\theta}\|u\|_{Y_0} < \infty$ and then $u = 0$. Hence $V^0_{\theta,\infty}$ consists of all the vectors $w = (-v, 0, v)$ with $v \in (X_0 + X_1) \cap M_1$ such that

$$\sup_{0<t\leq 1} \frac{K(t, v; \vec{X}) + t\|v\|_{Y_1}}{t^\theta} < \infty.$$

The last condition is equivalent to

$$\sup_{0<t\leq 1} \frac{K(t, v; \vec{X})}{t^\theta} < \infty. \tag{3.3}$$

Similarly, $V^1_{\theta,\infty}$ is formed by all the vectors $w = (-u, u, 0)$ with $u \in (X_0 + X_1) \cap M_0$ such that

$$\sup_{t\geq 1} \frac{K(t, u; \vec{X})}{t^\theta} < \infty. \tag{3.4}$$

From the shape of the vectors in $V^0_{\theta,\infty}$ and $V^1_{\theta,\infty}$, it is clear that $V^0_{\theta,\infty} \cap V^1_{\theta,\infty} = \{0\}$. Moreover, any $w = (-u-v, u, v)$ in $\ker T$ can be decomposed as $w = (-v, 0, v) + (-u, u, 0)$. Hence we have $\ker T = V^0_{\theta,\infty} \oplus V^1_{\theta,\infty}$, provided that (3.3) and (3.4) hold whenever $u \in M_0$, $v \in M_1$ and $u + v \in X_0 + X_1$.

The inequality (3.1) now reads

$$\frac{K(s, v; \vec{X}) + s\|v\|_{Y_1}}{s^{\theta+\varepsilon}} \leq \gamma \frac{K(t, v; \vec{X}) + t\|v\|_{Y_1}}{t^{\theta+\varepsilon}}, \quad 0 < s < t, \tag{3.5}$$

for all $v \in (X_0 + X_1) \cap M_1$. Taking $t = 1$ in (3.5), we obtain the inequality (3.3). Analogously, (3.2) implies

$$\frac{K(t, u; \vec{X}) + \|u\|_{Y_0}}{t^{\theta-\varepsilon}} \leq \gamma \frac{K(s, u; \vec{X}) + \|u\|_{Y_0}}{s^{\theta-\varepsilon}}, \quad 0 < s < t \tag{3.6}$$

for all $u \in (X_0 + X_1) \cap M_0$ and choosing $s = 1$ yields (3.4).

In conclusion, applying Theorem 3.1 to the operator T we derive that the necessary and sufficient condition for the equality $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ is that there exist constants $\gamma, \varepsilon > 0$ such that (3.5) and (3.6) hold whenever $u \in M_0$, $v \in M_1$ and $u + v \in X_0 + X_1$. This completes the proof. □

Since the conditions stated in Theorem 3.3 do not depend on q, as a consequence we obtain the following result.

Corollary 3.4. *Let $\vec{Y} = (Y_0, Y_1)$ be a Banach couple, let X_j be a closed complemented subspace of Y_j ($j = 0, 1$) and $\theta \in (0, 1)$. If there is $q_0 \in [1, \infty]$ such that $(X_0, X_1)_{\theta,q_0} = (Y_0, Y_1)_{\theta,q_0}$, then $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ for any $q \in [1, \infty]$.*

In the rest of this section we work with the couple $(L^2(\Omega), W^{1,2}(\Omega))$. Here Ω is a bounded connected open domain in $\mathbb{R}^n$ with a $\mathcal{C}^\infty$ boundary and $W^{1,2}(\Omega)$ is the Sobolev space defined by the norm

$$\|f\|_{W^{1,2}(\Omega)} = \left(\sum_{i=1}^{n} \| \frac{\partial f}{\partial x_i} \|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

where the derivatives $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, n$, are considered in the sense of distributions. Let $\mathcal{C}_0^\infty(\Omega)$ be the set of $\mathcal{C}^\infty$ functions with compact support in Ω. We consider the space $W_0^{1,2}(\Omega)$, which is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. This space plays a very important role in the theory of PDE (see, for example, [7]). In particular, $W_0^{1,2}(\Omega)$ is the kernel of the trace operator.

It is known (see [7, Corollary 7.3.1 and Lemma 7.3.1 (p. 171)]) that

$$W^{1,2}(\Omega) = W_0^{1,2}(\Omega) \oplus W,$$

where W is the space of weakly harmonic functions, i.e. such functions v from $W^{1,2}(\Omega)$ that for any function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int_\Omega \left(\frac{\partial v}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \cdots + \frac{\partial v}{\partial x_n} \frac{\partial \varphi}{\partial x_n} \right) dx = 0.$$

Clearly, the space of weakly harmonic functions is a closed subspace of $W^{1,2}(\Omega)$. It is also known (see [9, Theorems 1.11.6 (p. 64) and 1.11.1 (p. 55)]) that

$$(L^2(\Omega), W^{1,2}(\Omega))_{[\theta]} = (L^2(\Omega), W_0^{1,2}(\Omega))_{[\theta]} \quad \text{if and only if} \quad 0 < \theta < \frac{1}{2}.$$

Since $(L^2(\Omega), W^{1,2}(\Omega))$ and $(L^2(\Omega), W_0^{1,2}(\Omega))$ are couples of Hilbert spaces, the complex method of interpolation produces the same space as the real method with the parameter $q = 2$ (see [13, p. 143]). Therefore, we obtain that

$$(L^2(\Omega), W^{1,2}(\Omega))_{\theta,2} = (L^2(\Omega), W_0^{1,2}(\Omega))_{\theta,2} \quad \text{if and only if} \quad 0 < \theta < \frac{1}{2}.$$

The following result is a consequence of Corollary 3.4.

Corollary 3.5. *For any $0 < \theta < \frac{1}{2}$ and $1 \le q \le \infty$, we have that*

$$(L^2(\Omega), W^{1,2}(\Omega))_{\theta,q} = (L^2(\Omega), W_0^{1,2}(\Omega))_{\theta,q}.$$

Remark 3.6. According to [3, Theorem 3.4.2] or [13, Theorem 1.6.2], for $0 < \theta < 1$ and $1 \le q < \infty$ we have that $(Y_0, Y_1)_{\theta,q} = (Y_0, Y_1^\circ)_{\theta,q}$, where Y_1° is the closure of $Y_0 \cap Y_1$ into Y_1. Corollary 3.5 shows that equality $(Y_0, Y_1)_{\theta,q} = (Y_0, X_1)_{\theta,q}$ may also hold for *small* subspaces X_1 of Y_1, i.e. subspaces having infinite codimension.

4. Duality

Below we will show how the equality $(X_0, X_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ (see Theorem 3.3) can be characterized in terms of some indices defined with respect to the dual couple (Y_0^*, Y_1^*). This result establishes connections between Theorem 3.3 and the generalization of the Ivanov-Kalton theorem (Theorem 2.4).

Let us assume that (Y_0, Y_1) is a regular Banach couple. Then we can consider the dual Banach couple (Y_0^*, Y_1^*) and the annihilator of X_0 in Y_0^*, $X_0^\perp = \{\psi \in Y_0^* : \psi(X_0) = 0\}$. Let $\beta_0(X_0^\perp) = \beta_0(X_0^\perp; Y_0^*, Y_1^*)$ be the infimum of all those $\delta \in [0, 1]$ such that there is a constant $\gamma = \gamma(\delta) > 0$ for which

$$K(t, \psi; Y_0^*, Y_1^*) \leq \gamma \left(\frac{t}{s}\right)^\delta K(s, \psi; Y_0^*, Y_1^*)$$

for all $\psi \in X_0^\perp$ and all $0 < s < t \leq 1$. Clearly, $0 \leq \beta_0(X_0^\perp) \leq 1$.

Next, we show that if $X_1 = Y_1$, then the equality between the interpolation spaces generated by the subcouple (X_0, X_1) and by the couple (Y_0, Y_1) can be characterized using the index $\beta_0(X_0^\perp)$. We will need the following lemma.

Lemma 4.1. *Let (Y_0, Y_1) be a regular Banach couple and let X_0 be a closed complemented subspace of $Y_0, Y_0 = X_0 \oplus M_0$. Assume also that $M_0 \subset X_0 + Y_1$. Then*

$$K(t, \psi; Y_0^*, Y_1^*) \approx \sup_{u \in M_0} \frac{|\langle \psi, u \rangle|}{\|u\|_{Y_0} + K(t^{-1}, u; X_0, Y_1)}, \quad \psi \in X_0^\perp.$$

Proof. Let $\psi \in X_0^\perp$. From the duality of K- and J-functionals (see [3, Section 3.7 (p. 53)] or [4, Proposition 3.1.21 (p. 304)]), we have

$$K(t, \psi; Y_0^*, Y_1^*) = \sup_{y \in Y_0 \cap Y_1} \frac{| < \psi, y > |}{J(t^{-1}, y; Y_0, Y_1)} \approx \sup_{y \in Y_0 \cap Y_1} \frac{| < \psi, y > |}{\|y\|_{Y_0} + t^{-1}\|y\|_{Y_1}}.$$

Using the fact that $Y_0 = X_0 \oplus M_0$ and $M_0 \subset X_0 + Y_1$, we derive

$$\begin{aligned} K(t, \psi; Y_0^*, Y_1^*) &\approx \sup_{x \in X_0, u \in M_0, x + u \in Y_1} \frac{| < \psi, u > |}{\|u\|_{Y_0} + \| - x\|_{X_0} + t^{-1}\|x + u\|_{Y_1}} \\ &\approx \sup_{u \in M_0} \frac{| < \psi, u > |}{\|u\|_{Y_0} + K(t^{-1}, u; X_0, Y_1)}, \end{aligned}$$

where the constants of equivalence do not depend on ψ or t. □

Remark 4.2. From the assumptions $Y_0 = X_0 \oplus M_0$ and $M_0 \subset X_0 + Y_1$ we have $Y_0 \subseteq X_0 + Y_1$. Moreover, this embedding is continuous, i.e. there exists a constant $C > 0$ such that

$$\|u\|_{X_0 + Y_1} \leq C \|u\|_{Y_0}, \quad u \in Y_0. \tag{4.1}$$

Indeed, from the continuity of the embeddings $Y_0 \hookrightarrow Y_0+Y_1, X_0+Y_1 \hookrightarrow Y_0+Y_1$ it follows that for any sequence $\{u_n\} \subset Y_0$ such that $u_n \to u$ in Y_0 and $u_n \to v$ in X_0+Y_1 we have $u=v$. Hence, using the closed graph theorem we immediately have the continuity of the embedding operator $i : Y_0 \to X_0+Y_1$.

Theorem 4.3. *Let (Y_0, Y_1) be a regular Banach couple and let X_0 be a closed complemented subspace of $Y_0, Y_0 = X_0 \oplus M_0$, with $M_0 \subset X_0+Y_{1.}$. Let $0<\theta<1$, $1 \le q \le \infty$. Then*

$$\theta > \beta_0(X_0^\perp)$$

is the necessary and sufficient condition for the equality $(X_0, Y_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$.

Proof. Applying Theorem 3.3 with $M_1 = \{0\}$ and using $M_0 \subset X_0+Y_1$, we obtain that the equality $(X_0, Y_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ holds if and only if there exist constants $\gamma, \varepsilon > 0$ such that

$$\frac{K(t, u; X_0, Y_1) + \|u\|_{Y_0}}{t^{\theta-\varepsilon}} \le \gamma \frac{K(s, u; X_0, Y_1) + \|u\|_{Y_0}}{s^{\theta-\varepsilon}} \tag{4.2}$$

for any $u \in M_0$ and all positive $s<t$. From (4.1) it follows that for $\tau \le 1$ we have

$$K(\tau, u; X_0, Y_1) + \|u\|_{Y_0} \le C \|u\|_{Y_0}$$

with $C>0$ independent of u. Hence it is sufficient to verify the inequality (4.2) for $1 \le s < t$. This inequality can be written as

$$K(t, u; X_0, Y_1) + \|u\|_{Y_0} \le \gamma \left(\frac{t}{s}\right)^{\theta-\varepsilon} (K(s, u; X_0, Y_1) + \|u\|_{Y_0}) \tag{4.3}$$

for any $u \in M_0$ and all $1 \le s < t$.

Denote by $\|\cdot\|_t$ the norm in M_0 given by

$$\|u\|_t = K(t, u; X_0, Y_1) + \|u\|_{Y_0}$$

and write $M_{0,t}^*$ for the dual space of $(M_0, \|\cdot\|_t)$. The inequality (4.3) means that

$$\|u\|_t \le \gamma \left(\frac{t}{s}\right)^{\theta-\varepsilon} \|u\|_s, \quad u \in M_0, \quad 1 \le s < t. \tag{4.4}$$

As $\|\cdot\|_t \ge \|\cdot\|_{Y_0}$ on M_0, we see that all functionals $\psi \in Y_0^*$ can be considered as elements of $M_{0,t}^*$ and therefore $\|\psi\|_{M_{0,t}^*}$ makes sense. We claim that (4.4) is equivalent to

$$\|\psi\|_{M_{0,s}^*} \le \gamma \left(\frac{t}{s}\right)^{\theta-\varepsilon} \|\psi\|_{M_{0,t}^*}, \quad \psi \in X_0^\perp, \quad 1 \le s < t. \tag{4.5}$$

Indeed, if (4.4) holds then the embedding $(M_0, \|\cdot\|_s) \hookrightarrow (M_0, \|\cdot\|_t)$ has the norm less than or equal to $\gamma\left(\frac{t}{s}\right)^{\theta-\varepsilon}$. Hence the factorization

$$(M_0, \|\cdot\|_s) \hookrightarrow (M_0, \|\cdot\|_t) \xrightarrow{\psi} \mathbb{K}$$

yields (4.5). Conversely, given any $u \in M_0$, we can find $\varphi \in M_{0,t}^*$ such that

$$\|u\|_t =< \varphi, u > \quad \text{and} \quad \|\varphi\|_{M_{0,t}^*} = 1.$$

Using (4.1), it is not difficult to verify that $\|\cdot\|_t$ is equivalent to $\|\cdot\|_{Y_0}$ on M_0. Therefore, φ is bounded on M_0 with the norm $\|\cdot\|_{Y_0}$. Let $\psi = \varphi \circ P$ where $P : Y_0 \longrightarrow M_0$ is the projection. Then $\psi \in X_0^\perp$, with $\|u\|_t =< \psi, u >$ and $\|\psi\|_{M_{0,t}^*} = 1$. Hence

$$\|u\|_t = |< \psi, u >| \le \|\psi\|_{M_{0,s}^*}\|u\|_s \le \gamma\left(\frac{t}{s}\right)^{\theta-\varepsilon}\|u\|_s,$$

where we used (4.5) in the last inequality.

Consequently, the equality $(X_0, Y_1)_{\theta,q} = (Y_0, Y_1)_{\theta,q}$ holds if and only if there are constants $\gamma, \varepsilon > 0$ such that the inequality

$$\sup_{u\in M_0} \frac{|< \psi, u >|}{K(s, u; X_0, Y_1) + \|u\|_{Y_0}} \le \gamma\left(\frac{t}{s}\right)^{\theta-\varepsilon} \sup_{u\in M_0} \frac{|< \psi, u >|}{K(t, u; X_0, Y_1) + \|u\|_{Y_0}}$$

is valid for any $\psi \in X_0^\perp$ and $1 \le s < t$. By Lemma 4.1, the last inequality can be rewritten (perhaps with a new constant γ) as

$$\frac{K(t, \psi; Y_0^*, Y_1^*)}{t^{\theta-\varepsilon}} \le \gamma \frac{K(s, \psi; Y_0^*, Y_1^*)}{s^{\theta-\varepsilon}}, \quad \psi \in X_0^\perp, \quad 0 < s < t \le 1. \qquad (4.6)$$

Finally, (4.6) means that $\beta_0(X_0^\perp) < \theta$. This completes the proof. □

Remark 4.4. If we compare Theorem 4.3 with Theorem 2.4, we can see that under the conditions of Theorem 2.4 for any bases $\psi_1, \dots, \psi_n$ in $X_0^\perp$ we have

$$\beta_0(X_0^\perp; Y_0^*, Y_1^*) = \max_{1\le i\le n} \beta_0(\psi_i),$$

i.e. we obtain the desired "invariant" description of the quantity from the right-hand side (see Remark 2.6).

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