Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 34 (2015), 17–26
DOI: 10.4171/ZAA/1526


# On a Class of Second-Order Differential Inclusions on the Positive Half-Line

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**Abstract.** Consider in a real Hilbert space $H$ the differential equation (inclusion) (E): $p(t)u''(t)+q(t)u'(t)\in Au(t)+f(t)$ a.e. in $(0,\infty)$, with the condition (B): $u(0)=x\in\overline{D(A)}$, where $A: D(A)\subset H\to H$ is a (possibly set-valued) maximal monotone operator whose range contains 0; $p,q\in L^\infty(0,\infty)$, such that $\operatorname{ess\,inf}\ p>0$, $\frac{q}{p}$ is differentiable a.e., and $\operatorname{ess\,inf}\ \big[(\frac{q}{p})^2+2(\frac{q}{p})'\big]>0$. We prove existence of a unique (weak or strong) solution $u$ to (E), (B), satisfying $a^{\frac{1}{2}}u\in L^\infty(0,\infty;H)$ and $t^{\frac{1}{2}}a^{\frac{1}{2}}u'\in L^2(0,\infty;H)$, where $a(t)=\exp\big(\int_0^t\frac{q}{p}\,d\tau\big)$, showing in particular the behavior of $u$ as $t\to\infty$.

**Keywords.** Strong solution, weak solution, existence, uniqueness, asymptotic behavior

**Mathematics Subject Classification (2010).** 34G20, 34G25, 47J35

## 1. Introduction

Let $H$ be a real Hilbert space with the inner product $(\cdot,\cdot)$ and the induced norm $\|x\|=(x,x)^{\frac{1}{2}}$. Consider the following second-order, non-homogeneous, differential equation (inclusion)

$$p(t)u''(t)+q(t)u'(t)\in Au(t)+f(t)\qquad \text{for a.a. } t\in\mathbb{R}_+:=[0,\infty), \tag{E}$$

with the condition

$$u(0)=x\in\overline{D(A)}, \tag{B}$$

where

(H1) $A\colon D(A)\subset H\to H$ is a (possibly set-valued) maximal monotone operator, such that $[0,0]\in\operatorname{Graph}(A)$;

(H2) $p,q\in L^\infty(\mathbb{R}_+):=L^\infty(\mathbb{R}_+;\mathbb{R})$, such that $\operatorname{ess\,inf} p>0$, $\frac{q}{p}$ is differentiable a.e., and $\operatorname{ess\,inf}\ \big[(\frac{q}{p})^2+2(\frac{q}{p})'\big]>0$;

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and $f$ is a given $H$-valued function whose (required) properties will be specified later. In fact, one can assume the more general condition that the range $R(A)$ of $A$ contains 0. Indeed, this case reduces to $[0,0] \in \text{Graph}\,(\tilde{A})$, where $\tilde{A}$ is obtained from $A$ by shifting its domain.

For information on monotone operators we refer the reader to [5, 7, 12].

V. Barbu [3, 4] (see also [5, Chapter V]) established the existence of a unique bounded solution to (E), (B) in the particular case $p \equiv 1$, $q \equiv 0$ and $f \equiv 0$. Subsequently the existence and uniqueness of bounded solutions in the homogeneous case ($f \equiv 0$) has been further investigated by H. Brezis [6], N. Pavel [14], L. Véron [17, 18], and by E. I. Poffald and S. Reich [15, 16] when $A$ is an $m$-accretive operator in a Banach space. The non-homogeneous case has received less attention from this point of view. Bruck [8] proved that if (E), (B), with $p \equiv 1$, $q \equiv 0$ and $f \in L^2_{loc}(0,\infty;H)$, has a bounded solution $u \in C([0,\infty);H) \cap W^{2,2}_{loc}((0,\infty);H)$ for some $x \in \overline{D(A)}$, then (E), (B) has a unique bounded solution $u \in C([0,\infty);H) \cap W^{2,2}_{loc}((0,\infty);H)$ for every $x \in \overline{D(A)}$. This result has been extended to the case when $A$ is an $m$-accretive operator in a Banach space by Poffald and Reich [15, 16]. Note that in these papers the existence for some $x \in \overline{D(A)}$ was hypothesized to derive existence for all $x \in \overline{D(A)}$. Recall also that Bruck [9] established the existence of a bounded solution on $\mathbb{R}$ of equation (E) (implying that all solutions of (E) are bounded on $[0,\infty)$), in the case $p \equiv 1$, $q \equiv 0$ and $f \in L^\infty(\mathbb{R};H)$, under the restrictive condition that $A$ is coercive. We also mention the relatively recent article by Apreutesei [2] addressing the case of smooth coefficients $p$, $q$, with $p(t) \ge p_0 > 0$, $q(t) \ge q_0 > 0$, and $x \in D(A)$.

In a recent paper [13] we established the existence of a unique bounded solution of equation (E), subject to (B), for all $x \in \overline{D(A)}$, under the same mild conditions (H1) and (H2) above, with one exception: instead of the condition on $\frac{q}{p}$ specified above, we assumed there $q^+ \in L^1(0,\infty;H)$. Our present alternative condition on $\frac{q}{p}$ ensures the existence of a unique (weak or strong) solution to (E), (B) satisfying $a^{\frac{1}{2}}u \in L^\infty(0,\infty;H)$ and $t^{\frac{1}{2}}a^{\frac{1}{2}}u' \in L^2(0,\infty;H)$, where $a(t) = \exp\left(\int_0^t \frac{q}{p}\,d\tau\right)$. So, in addition to existence and uniqueness, we get information about the asymptotic behavior of $u$ as $t \to \infty$. If in particular $q(t) \ge q_0 > 0$, then $\|u(t)\|$ decays exponentially to zero as $t \to \infty$.

The new framework requires separate analysis. However, some steps in our proofs are similar to those developed in [13]. In such cases, the reader will be referred to that paper.

It is worth pointing out that this paper covers in particular the case $q(t) < 0$ which allows using our existence theory to approximate the solutions of some parabolic and hyperbolic problems by the method of artificial viscosity, introduced by J. L. Lions [11]. See [13] for details. Note that the case $q \equiv 0$ was covered in [10, 13].

## 2. Results

Let us first recall the concepts of strong and weak solution for equation (E) (respectively, equation (E) plus condition (B)). These concepts have been introduced in [10, 13].

For an interval $J \subset \mathbb{R}$, open or not, denote by $L^p_{loc}(J; H)$ (resp. $W^{k,p}_{loc}(J; H)$) the space of all $H$-valued functions defined on $J$, whose restrictions to compact intervals $[a, b] \subset J$ belong to $L^p(a, b; H)$ (respectively, to $W^{k,p}(a, b; H)$).

**Definition 2.1.** Let $f \in L^2_{loc}([0, \infty); H)$ and let $x \in \overline{D(A)}$. A $H$-valued function $u = u(t)$ is said to be a *strong solution* of equation (E) (respectively, of equation (E) plus condition (B)) if $u \in C([0, \infty); H) \cap W^{2,2}_{loc}((0, \infty); H)$ and $u(t)$ satisfies equation (E) for a.a. $t > 0$ (and, in addition, $u(0) = x$, respectively).

Denote $Y = L^1(0, \infty; H; t\sqrt{a(t)}dt)$, where $a(t) = \exp\left(\int_0^t \frac{q(\tau)}{p(\tau)}\, d\tau\, d\tau\right)$. Obviously, $Y$ is real Banach space with respect to the norm

$$\|f\|_Y = \int_0^\infty \|f(t)\| t\sqrt{a(t)}\, dt.$$

If $f \in Y$ we cannot expect in general existence of strong solutions for (E), so we need the following definition of a weaker concept:

**Definition 2.2.** Let $f \in Y$ and let $x \in \overline{D(A)}$. A $H$-valued function $u = u(t)$ is said to be a *weak solution* of equation (E) (respectively, of equation (E) plus condition (B)) if there exist sequences $u_n \in C([0, \infty); H) \cap W^{2,2}_{loc}((0, \infty); H)$ and $f_n \in Y \cap L^2_{loc}([0, \infty); H)$, such that:

(i) $f_n$ converges to $f$ in $Y$;
(ii) $u_n(t)$ satisfies equation (E) with $f = f_n$ for a.a. $t > 0$ and all $n \in \mathbb{N}$;
(iii) $u_n$ converges uniformly to $u$ on any compact interval $[0, T]$ (and, in addition, $u(0) = x$, respectively).

Note that the couple (E), (B) is an incomplete problem. We need an additional condition to obtain a complete problem. In this paper we consider the following condition

$$\sup_{t \geq 0} a(t)\|u(t)\|^2 < \infty. \tag{C}$$

Obviously, if $\frac{q}{p} \in L^1(\mathbb{R}_+)$ (which is equivalent to $q \in L^1(\mathbb{R}_+)$ if $p \in L^\infty(\mathbb{R}_+)$ and $\operatorname{ess\,inf}\ p > 0$), then (C) becomes $\sup_{t\geq 0} \|u(t)\| < \infty$.

Before stating the first main result of the paper, let us recall two lemmas from [13]:

**Lemma 2.3.** *Let $A$ satisfy* (H1), $p, q \in L^\infty(0,T)$, *with* $\operatorname{ess\,inf} p > 0$, *and let* $f \in L^2(0,T;H)$, *where $T$ is a given positive number. Then, for all $x, y \in D(A)$, there exists a unique* $u = u(t) \in W^{2,2}(0,T;H)$ *satisfying*

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad \text{for a.a. } t \in (0,T), \tag{1}$$

$$u(0) = x, \quad u(T) = y. \tag{2}$$

**Lemma 2.4.** *Assume that $A$ satisfies* (H1), $p, q \in L^\infty(0,T)$, *with* $\operatorname{ess\,inf} p > 0$, $f \in L^2(0,T;H)$, *and* $x, y \in D(A)$. *For $\lambda > 0$ denote by $A_\lambda$ the Yosida approximation of $A$ and by $u_\lambda$ the unique solution of*

$$p(t)u_\lambda''(t) + q(t)u_\lambda'(t) = A_\lambda u_\lambda(t) + f(t) \quad \text{for a.a. } t \in (0,T),$$

$$u_\lambda(0) = x, \quad u_\lambda(T) = y$$

*(which exists by Lemma* 2.3*). Then, $u_\lambda \to u$ in $C([0,T];H)$ as $\lambda \to 0^+$, where $u$ is the solution of problem* (1), (2). *Moreover, $u_\lambda' \to u'$ in $C([0,T];H)$ and $u_\lambda'' \to u''$ weakly in $L^2(0,T;H)$, as $\lambda \to 0^+$.*

**Theorem 2.5.** *Assume* (H1) *and* (H2) *hold. If $x \in \overline{D(A)}$ and $f \in Y \cap L^2_{loc}([0,\infty);H)$, then there exists a unique strong solution $u$ of* (E), (B), (C), *such that $t^{\frac{1}{2}}a^{\frac{1}{2}}u' \in L^2(\mathbb{R}_+;H)$ and $t^{\frac{3}{2}}u'' \in L^2_{loc}([0,\infty);H)$. If in addition $x \in D(A)$, then $u \in W^{2,2}_{loc}([0,\infty);H)$.*

*Proof.* Assume in a first stage that $x \in D(A)$ (and $f \in Y \cap L^2_{loc}([0,\infty);H)$, as hypothesized). For each $\lambda > 0$ and $n \in \mathbb{N}$, denote by $u_{n\lambda}, u_n$ the solutions of the following problems

$$pu_{n\lambda}'' + qu_{n\lambda}' = A_\lambda u_{n\lambda} + f \quad \text{a.e. in } (o,n), \tag{3}$$

$$u_{n\lambda}(0) = x, \quad u_{n\lambda}(n) = 0, \tag{4}$$

and

$$pu_n'' + qu_n' \in Au_n + f \quad \text{a.e. in } (o,n), \tag{5}$$

$$u_n(0) = x, \quad u_n(n) = 0. \tag{6}$$

Lemma 2.3 ensures the existence and uniqueness of $u_{n\lambda}, u_n \in W^{2,2}(0,n;H)$. By Lemma 2.4, $u_{n\lambda} \to u_n$, $u_{n\lambda}' \to u_n'$ in $C([0,n];H)$, as $\lambda \to 0^+$, and $u_{n\lambda}'' \to u_n''$ weakly in $L^2(0,n;H)$, as $\lambda \to 0^+$. Note that equations (3), (5) can be equivalently expressed as follows (see [1])

$$\left(au_{n\lambda}'\right)' = b(A_\lambda u_{n\lambda} + f) \quad \text{a.e. in } (0,n), \tag{7}$$

and, respectively,

$$\left(au_n'\right)' \in b(Au_n + f) \quad \text{a.e. in } (0,n), \tag{8}$$

where $b(t) = \frac{a(t)}{p(t)}$. Recall that $a(t) = \exp\big(\int_0^t \frac{q}{p}\,d\tau\big)$. We have for a.a. $t \in (0,n)$

$$
\begin{aligned}
&\frac{d^2}{dt^2}\left[a\|u_n\|^2\right]\\
&= \frac{d}{dt}\Big[a\frac{q}{p}\|u_n\|^2 + 2(au_n', u_n)\Big]\\
&= a\Big[\Big(\frac{q}{p}\Big)^2 + \Big(\frac{q}{p}\Big)'\Big]\|u_n\|^2 + 2a\frac{q}{p}(u_n', u_n) + 2a\|u_n'\|^2 + 2\Big(\big(au_n'\big)', u_n\Big)\\
&\geq a\Big(\Big[\Big(\frac{q}{p}\Big)^2 + \Big(\frac{q}{p}\Big)'\Big]\|u_n\|^2 + 2\frac{q}{p}(u_n', u_n) + 2\|u_n'\|^2\Big) - 2b\|u_n\|\cdot\|f\|. \qquad (9)
\end{aligned}
$$

The last inequality follows from (8) and the monotonicity of $A$. Taking into account the condition on $\frac{q}{p}$ (see (H2)) we derive from (9)

$$\frac{d^2}{dt^2}\left[a\|u_n\|^2\right] \geq -2b\|f\|\cdot\|u_n\|. \qquad (10)$$

Integration of (10) over $[\tau, n]$ leads to $\frac{d}{d\tau}\big(a(\tau)\|u_n(\tau)\|^2\big) \leq 2\int_\tau^n b\|f\|\cdot\|u_n\|\,ds$. A new integration, this time over $[0,t]$, yields

$$
\begin{aligned}
a(t)\|u_n(t)\|^2 &\leq \|x\|^2 + 2\int_0^t d\tau \int_\tau^n b\|f\|\cdot\|u_n\|\,ds\\
&\leq \|x\|^2 + 2\int_0^n d\tau \int_\tau^n b\|f\|\cdot\|u_n\|\,ds\\
&= \|x\|^2 + 2\int_0^n \tau b\|f\|\cdot\|u_n\|\,d\tau, \quad 0 \leq t \leq n. \qquad (11)
\end{aligned}
$$

Denoting $M_n = \sup_{0\leq t\leq n}\sqrt{a(t)}\|u_n(t)\|$, from (11) we derive

$$M_n^2 \leq \|x\|^2 + 2M_n\int_0^n \tau\frac{\sqrt{a}}{p}\|f\|\,d\tau \leq \|x\|^2 + 2\frac{M_n}{p_0}\|f\|_Y,$$

where $p_0 = \operatorname{ess\,inf} p$. Therefore,

$$M_n \leq \frac{1}{p_0}\|f\|_Y + \sqrt{\frac{1}{p_0^2}\|f\|_Y^2 + \|x\|^2} =: E = E(x,f).$$

Thus,

$$\sup_{0\leq t\leq n} a(t)\|u_n(t)\|^2 \leq E^2. \qquad (12)$$

Similarly,

$$\sup_{0\leq t\leq n} a(t)\|u_{n\lambda}(t)\|^2 \leq E^2.$$

Now, let $0 < R < m < n$, with $m, n \in \mathbb{N}$. Denote

$$g(t) = a(t)\|u_n(t) - u_m(t)\|^2, \quad 0 \le t \le m.$$

We have

$$\begin{aligned} g'(t) &= a\frac{q}{p}\|u_n - u_m\|^2 + 2\big(a(u_n' - u_m'), u_n - u_m\big), \\ g''(t) &= a\frac{q^2}{p^2}\|u_n - u_m\|^2 + a\Big(\frac{q}{p}\Big)'\|u_n - u_m\|^2 + 2a\frac{q}{p}(u_n' - u_m', u_n - u_m) \\ &\quad + 2\Big(\big(a(u_n' - u_m')\big)', u_n - u_m\Big) + 2a\|u_n' - u_m'\|^2. \end{aligned}$$

Therefore,

$$g''(t) \ge a\Big(\Big[\frac{q^2}{p^2} + \Big(\frac{q}{p}\Big)'\Big]\|u_n - u_m\|^2 + 2\frac{q}{p}(u_n' - u_m', u_n - u_m) + 2\|u_n' - u_m'\|^2\Big). \quad (13)$$

Denoting $\alpha := \operatorname{ess\,inf}\,[(\frac{q}{p})^2 + 2(\frac{q}{p})'] > 0$, and observing that

$$\Big(\frac{q}{p}\Big)^2 + \Big(\frac{q}{p}\Big)' \ge \frac{1}{2}\Big(\frac{q}{p}\Big)^2 + \frac{\alpha}{2},$$

from (13) we derive

$$\begin{aligned} g''(t) &\ge \frac{a}{2}\Big(\Big(\frac{q^2}{p^2} + \alpha\Big)\|u_n - u_m\|^2 + 4\frac{q}{p}(u_n' - u_m', u_n - u_m) + 4\|u_n' - u_m'\|^2\Big) \\ &\ge \beta a\|u_n' - u_m'\|^2, \end{aligned} \quad (14)$$

for a.a. $t \in (0, m)$, where $\beta$ is a small positive number. We multiply (14) by $(m - t)$ and then integrate the resulting inequality over $[0, m]$:

$$\begin{aligned} \beta\int_0^m (m - t)a\|u_n' - u_m'\|^2 dt &\le (m - t)g'(t)|_0^m + \int_0^m g'(t)\,dt \\ &= g(m) \\ &= a(m)\|u_n(m)\|^2 \\ &\le E^2. \end{aligned}$$

We have used (12). It follows that $\beta(m - R)\int_0^R a\|u_n' - u_m'\|^2 dt \le E^2$, which shows that $(u_n')$ is a Cauchy (hence convergent) sequence in $L^2(0, R; H)$. Therefore, since $u_n(t) - u_m(t) = \int_0^t (u_n - u_m)'(s)\,ds$, $u_n$ converges in $C([0, R]; H)$ to some $u \in C([0, R]; H)$, and so $u_n' \to u'$ in $L^2(0, R; H)$. In particular, $u(0) = x$. Obviously, since $R > 0$ was arbitrarily chosen, $u$ can be extended to $[0, \infty)$, such that $u \in C([0, \infty); H) \cap W_{loc}^{1,2}([0, \infty); H)$, and $u$ satisfies (cf. (12))

$$\sup_{t \ge 0} a(t)\|u(t)\|^2 \le E^2 < \infty. \quad (15)$$

By arguments similar to those used in [13], we deduce that $u_n''$ is bounded in $L^2(0, \frac{R}{2}; H)$, hence weakly convergent to $u''$ in this space, and finally that $u$ is a strong solution of equation (E).

Now, assume that $x \in \overline{D(A)}$ and $f \in Y \cap L^2_{loc}([0,\infty); H)$. Let $x_k \in D(A)$, $\|x_k - x\| \to 0$. Denote by $u_k$ the strong solution of equation (E) satisfying $u_k(0) = x_k$, and $\sqrt{a}\|u_k\| \in L^\infty(\mathbb{R}_+)$. Existence of $u_k$ is ensured by the first part of the proof. In fact, according to (15),

$$\sup_{t\geq 0} \sqrt{a(t)}\|u_k(t)\| \leq E(x_k, f) \leq E_0 < \infty. \tag{16}$$

Denote by $u_{kn}$, $u_{kn\lambda}$ the corresponding approximations of $u_k$ and $u_{kn}$ (as defined above, see problems (5), (6) and (3), (4)). We see that for a.a. $t \in (0, n)$

$$\frac{1}{2}\frac{d}{dt}\left( a\frac{d}{dt}\|u_{kn} - u_{jn}\|^2 \right) \geq \ a\|u'_{kn} - u'_{jn}\|^2,$$

so the function $t \to a(t)\frac{d}{dt}\|u_{kn}(t) - u_{jn}(t)\|^2$ is nondecreasing on $[0, n]$. Since it is equal to zero at $t = n$, it follows that it is non-positive in $[0, n]$. Then the function $t \to \|u_{kn}(t) - u_{jn}(t)\|$ is nonincreasing on $[0, n]$. In particular,

$$\|u_{kn}(t) - u_{jn}(t)\| \leq \|x_k - x_j\| \quad \forall t \in [0, n].$$

Therefore, according to the first part of the proof, we have

$$\|u_k(t) - u_j(t)\| \leq \|x_k - x_j\| \quad \forall t \geq 0.$$

This shows that there exists a function $u \in C([0,\infty); H)$ such that $u_k$ converges to $u$ in $C([0, R]; H)$ for all $R \in (0,\infty)$, so in particular $u(0) = x$. According to (16), we also have $\sqrt{a}\|u\| \in L^\infty(\mathbb{R}_+)$. Now, set

$$h(t) = a(t)\|u_{kn\lambda}(t)\|^2, \quad 0 \leq t \leq n.$$

We have

$$h'(t) = a\frac{q}{p}\|u_{kn\lambda}\|^2 + 2\big(au'_{kn\lambda}, u_{kn\lambda}\big),$$
$$h''(t) = a\left[\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)'\right]\cdot\|u_{kn\lambda}\|^2 + 2a\frac{q}{p}(u'_{kn\lambda}, u_{kn\lambda}) + 2\Big(\big(au'_{kn\lambda}\big)', u_{kn\lambda}\Big) + 2a\|u'_{kn\lambda}\|^2.$$

Therefore,

$$\begin{aligned} h''(t) &\geq a\left(\left[\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)'\right]\|u_{kn\lambda}\|^2 + 2\frac{q}{p}(u'_{kn\lambda}, u_{kn\lambda}) + 2\|u'_{kn\lambda}\|^2\right) - 2b\|f\|\cdot\|u_{kn\lambda}\| \\ &\geq \beta a\|u'_{kn\lambda}\|^2 - 2\frac{E_0}{p_0}\sqrt{a}\|f\|. \end{aligned} \tag{17}$$

Multiply (17) by $t$ and integrate the resulting inequality over $[0,n]$ to obtain

$$\begin{aligned}\beta\int_0^n ta\|u'_{kn\lambda}\|^2 dt &\le 2\frac{E_0}{p_0}\int_0^n t\sqrt{a}\|f\|\,dt+\int_0^n th''(t)\,dt\\ &\le 2\frac{E_0}{p_0}\|f\|_Y+th'(t)|_0^n-\int_0^n h'(t)\,dt\\ &\le 2\frac{E_0}{p_0}\|f\|_Y+\|x_k\|^2\\ &\le K_0<\infty. \end{aligned}\tag{18}$$

According to Lemma 2.4, it follows by (18) that

$$\beta\int_0^n ta\|u'_{kn}\|^2 dt\le K_0. \tag{19}$$

By the first part of the proof, we also have

$$\beta\int_0^\infty ta\|u'_k\|^2 dt\le K_0. \tag{20}$$

In fact, $\sqrt{ta}u'\in L^2(\mathbb{R}_+;H)$ and $\sqrt{ta}u'_k\to\sqrt{ta}u'$ in $L^2(\mathbb{R}_+;H)$. Indeed, denoting

$$r(t)=a(t)\|u_{kn}(t)-u_{jn}(t)\|^2,\quad 0\le t\le n,$$

we derive by a computation similar to that we have used above for $g(t)$

$$r''(t)\ge\beta a(t)\|u'_{kn}(t)-u'_{jn}(t)\|^2\quad\text{for a.a. } t\in(0,n),$$

which implies $\beta\int_0^n ta\|u'_{kn}-u'_{jn}\|^2dt\le tr'(t)|_0^n-\int_0^n r'(t)\,dt=\|x_k-x_j\|^2$. Hence,

$$\beta\int_0^\infty ta\|u'_k-u'_j\|^2dt\le\|x_k-x_j\|^2,$$

which confirms our assertion above.

Next, using the sequence $(u_{kn\lambda})$ (and in particular (18)), we can show by a procedure similar to that used in [13] that $t^{\frac{3}{2}}u''\in L^2_{loc}([0,\infty);H)$ and that $u$ is a strong solution of equation (E). Uniqueness of $u$ follows as in [13], so the proof of the theorem is complete. □

**Theorem 2.6.** *Assume* (H1) *and* (H2) *hold. Then, for each* $x\in\overline{D(A)}$ *and* $f\in Y$*, there exists a unique weak solution* $u$ *of* (E), (B), (C), *and* $\sqrt{ta}u'\in L^2(\mathbb{R}_+;H)$.

*Proof.* Let $x\in\overline{D(A)}$ and let $f_1,f_2\in Y\cap L^2_{loc}([0,\infty);H)$. Denote by $u(t,x,f_i)$, $i=1,2$, the corresponding strong solutions given by Theorem 2.5, and by $u_n(t,x,f_i)$ their approximations ($i=1,2,\dots,n\in\mathbb{N}$), as defined above (see

(5), (6)). Recall that (by the uniqueness property) every strong solution is obtained by the limiting procedure developed in the proof of Theorem 2.5. By a computation involving (H2), similar to that performed above for $g(t)$, we derive the inequality

$$\begin{aligned}&\frac{d^2}{dt^2}\big[a(t)\|u_n(t,x,f_1)-u_n(t,x,f_2)\|^2\big]\\&\geq -2b(t)\|f_1(t)-f_2(t)\|\cdot\|u_n(t,x,f_1)-u_n(t,x,f_2)\|.\end{aligned}\tag{21}$$

Successive integrations of (21), over $[\tau,n]$ and then over $[0,t]$, lead to

$$\begin{aligned}&a(t)\|u_n(t,x,f_1)-u_n(t,x,f_2)\|^2\\&\leq 2\int_0^n d\tau\int_\tau^n b(s)\|f_1(s)-f_2(s)\|\cdot\|u_n(s,x,f_1)-u_n(s,x,f_2)\|\,ds\\&=2\int_0^n \tau b(\tau)\|f_1(\tau)-f_2(\tau)\|\cdot\|u_n(\tau,x,f_1)-u_n(\tau,x,f_2)\|\,d\tau.\end{aligned}\tag{22}$$

Obviously, (22) implies $\sqrt{a(t)}\|u_n(t,x,f_1)-u_n(t,x,f_2)\|\leq\frac{2}{p_0}\|f_1-f_2\|_Y$, for $0\leq t\leq n$, and hence

$$\sqrt{a(t)}\|u(t,x,f_1)-u(t,x,f_2)\|\leq\frac{2}{p_0}\|f_1-f_2\|_Y\quad\forall t\geq 0.\tag{23}$$

From inequality (23) we can derive the existence of a unique weak solution $u(t;x,f)$ for each $(x,f)\in\overline{D(A)}\times Y$. Indeed, $f$ can be approximated (with respect to the norm of $Y$) by a sequence $(f_k)$ of smooth functions with compact support $\subset(0,\infty)$, so it is enough to take in (23) $f_1:=f_k$ and $f_2:=f_j$. So, there exists uniquely $u(\cdot;x,f)\in C([0,\infty);H)$ the uniform limit on compact intervals of $u(\cdot;x,f_k)$ as $k\to\infty$.

Note that (19) holds true for $u_n'(t;x,f_k)$ with another constant $K_0$ (since $E(x,f_k)$ is also bounded), so (20) also holds true for $u'(t;x,f_k)$. Therefore, $\sqrt{tau}'\in L^2(\mathbb{R}_+;H)$ (as the weak limit in $L^2(\mathbb{R}_+;H)$ of the sequence $(\sqrt{tau}_k'))$. This completes the proof of the theorem. □

**Acknowledgement.** Many thanks to an anonymous referee for some useful comments and suggestions.

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Received February 15, 2014; revised July 18, 2014