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Existence of Homoclinic Orbits of Superquadratic Second-Order Hamiltonian Systems

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Abstract. Using critical point theory, we study the existence of homoclinic orbits for the second-order Hamiltonian system

$$\ddot{z} - K_z(t,z) + V_z(t,z) = h(t),$$

where $V(t,z)$ depends periodically on t and is superquadratic.

Keywords. Homoclinic orbit, Hamiltonian system, critical point theory
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1. Introduction

1.1. Research background. The purpose of this paper is to study the existence of homoclinic orbits for the superquadratic second-order Hamiltonian system

$$\ddot{z} - K_z(t,z) + V_z(t,z) = h(t) \tag{1}$$

where $t \in \mathbf{R}$, $z \in \mathbf{R}^n$, $K, V \in C^1(\mathbf{R}\times\mathbf{R}^n, \mathbf{R})$ is $T-$periodic in t, and $h : \mathbf{R} \to \mathbf{R}^n$ is a continuous and bounded function.

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. For second order Hamiltonian systems we refer the reader to [2,7,8,10-13] and for first order [1,3-5, 9, 14-17].

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We note that many results were obtained under the Ambrosetti-Rabinowitz growth condition, that is, there is a $\mu > 2$ such that

$$0 < \mu V(t,z) \le (z, V_z(t,z)) \quad \text{whenever } z \ne 0. \tag{$\mathbf{AR}$}$$

It is easy to see that (**AR**) does not include some superquadratic nonlinearities like

$$V(t,z) = |z|^2(\ln(1+|z|^p))^q, \quad p,q > 1. \tag{2}$$

In this paper, we study the homoclinic solutions of (1) under some superquadratic condition which covers a case like (2).

We suppose that V, K and h in (1) satisfy the following assumptions:

(H_1) There are a continuous T-periodic function $k(t)$ and two constants k_1, $k_2 > 0$ such that for all $(t,z) \in \mathbf{R} \times \mathbf{R}^n$

$$k_1|z|^2 \le k(t)|z|^2 \le K(t,z) \le k_2|z|^2$$
$$\text{and} \quad \frac{1}{2}(z, K_z(t,z)) \le K(t,z) \le (z, K_z(t,z)).$$

Here and in the sequel, $(\cdot,\cdot) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ denotes the standard inner product in $\mathbf{R}^n$ and $|\cdot|$ the induced norm.

(H_2) $V(t,z) \ge 0$, for all $(t,z) \in [0,T] \times \mathbf{R}^n$.

(H_3) $V(t,z) = o(|z|^2)$ as $|z| \to 0$ uniformly in t.

(H_4) $\frac{V(t,z)}{|z|^2} \to +\infty$ as $|z| \to +\infty$ uniformly in t.

(H_5) $(z, V_z(t,z)) - 2V(t,z) \ge 0$ for all z, t.

(H_6) There exist positive constants $1 < \lambda \le \beta$, c_1 and c_2 such that

$$(z, V_z(t,z)) - 2V(t;z) \ge c_1|z|^\beta, \quad \forall |z| \ge 1, \ \forall t \in [0,T] \tag{3}$$
$$\text{and} \quad |V_z(t,z)| \le c_2|z|^\lambda, \quad \forall |z| \ge 1, \ \forall t \in [0,T]. \tag{4}$$

(H_7) Set $\overline{M} := \sup\{V(t,z) : t \in [0,T], |z| = 1\}$, $a_1 := \min\{1, 2k_1\}$, $a_2 := \max\{1, 2k_2\}$. There exist positive constants η, c_3 and c_4 such that

$$\int_{\mathbf{R}} |h(t)|dt \le c_3, \quad \left(\int_{\mathbf{R}} |h(t)|^2 dt\right)^{\frac{1}{2}} \le \frac{\eta}{2\varrho},$$
$$a_1 - 2\overline{M} - \eta > 0, \quad c_4 = \max_{t \in \mathbf{R}} |h(t)| < c_1,$$

where ϱ is a positive constant which will be defined in Proposition 1.3 later.

As usual, a solution $z(t)$ of (1) is said to be homoclinic (to 0) if $z(t) \to 0$ as $t \to \pm\infty$. In addition, if $z(t) \not\equiv 0$ then $z(t)$ is called a nontrivial homoclinic solution.

Theorem 1.1. *Suppose* $V \in C^1([0,T] \times \mathbf{R}^n, \mathbf{R})$ *is* T*-periodic in* t *and satisfies* (H_1)–(H_7)*. Then system* (1) *possesses a nontrivial homoclinic solution* $z \in W^{1,2}(\mathbf{R}, \mathbf{R}^n)$ *such that* $\dot{z}(t) \to 0$ *as* $t \to \pm\infty$*.*

This paper is largely motivated by the work of Rabinowitz [12] in which the existence of nontrivial homoclinic solutions for the second order Hamiltonian system

$$\ddot{q} + V_q(t,q) = 0$$

was proved.

1.2. Variational structure. For each $k \in \mathbf{N}$, let $E_k := W^{1,2}_{2kT}(\mathbf{R}, \mathbf{R}^n)$, the Hilbert space of $2kT$-periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^n$ under the norm

$$\|z\|^2_{E_k} := \int_{-kT}^{kT} [|\dot{z}(t)|^2 + |z(t)|^2]dt, \quad z \in E_k.$$

Furthermore, let $L^\infty_{[-kT,kT]}(\mathbf{R}, \mathbf{R}^n)$ denote a space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbf{R}$ into $\mathbf{R}^n$ equipped with the norm

$$\|z\|_{L^\infty_{[-kT,kT]}} := \operatorname{ess\,sup}\{|z(t)| : t \in [-kT, kT]\}.$$

As in [10], a homoclinic solution of (1) will be obtained as a limit, as $k \to \pm\infty$, of a certain sequence of functions $z_k \in E_k$. We consider a sequence of systems of differential equations

$$\ddot{z}(t) - K_z(t,z) + V_z(t,z) = h_k(t), \tag{5}$$

where for each $k \in \mathbf{N}$, $h_k : \mathbf{R} \to \mathbf{R}^n$ is a $2kT$-periodic extension of the restriction of h to the interval $[-kT, kT]$ and z_k, a $2kT$-periodic solution of (5), will be obtained via a linking theorem.

Let

$$\phi_k(z) = \left(\int_{-kT}^{kT} [|\dot{z}(t)|^2 + 2K(t, z(t))]dt \right)^{\frac{1}{2}}. \tag{6}$$

We have from (H_1) that

$$a_1 \|z\|^2_{E_k} \le \phi^2_k(z) \le a_2 \|z\|^2_{E_k}. \tag{7}$$

Let

$$Az = -\ddot{z} + k(t)z, \quad z \in E_k, \tag{8}$$

$$< Az, y > = \int_{-kT}^{kT} (-\ddot{z} + k(t)z, y)dt, \quad \forall z, y \in E_k \tag{9}$$

and

$$\begin{aligned} I_k(z) = &\int_{-kT}^{kT}\left[\frac{1}{2}|\dot{z}(t)|^2 + K(t,z(t))\right]dt - \int_{-kT}^{kT} V(t,z(t))dt \\ &+\int_{-kT}^{kT}(h_k(t),z(t))dt \\ =&\frac{1}{2} < Az, z > + \int_{-kT}^{kT}\left[K(t,z(t)) - \frac{k(t)}{2}z^2(t)\right]dt - \int_{-kT}^{kT} V(t,z(t))dt \\ &+\int_{-kT}^{kT}(h_k(t),z(t))dt. \end{aligned} \tag{10}$$

Then A has a sequence of eigenvalues

$$0 < \xi_k^1 \le \xi_k^2 \le \cdots \le \xi_k^m \cdots$$

with $\xi_k^m \to \infty$ as $m \to \infty$. Let φ_k^j be the eigenvector of A corresponding to ξ_k^j, $j = 1, 2, \ldots, m, \ldots$. Set

$E_k^0 = ker(A)$, $E_k^- =$ *negative eigenspace* of A, $E_k^+ =$ *positive eigenspace* of A.

It is easy to see that $E_k^- = \{0\}$ and $E_k = E_k^0 \oplus E_k^+$.

Lemma 1.2 ([11]). *Let E be a real Hilbert space with $E = E^{(1)} \oplus E^{(2)}$ and $E^{(1)} = (E^{(2)})^\perp$. Suppose $I \in C^1(E, \mathbf{R})$ satisfies the* (**PS**) *condition*[1], *and*

(C$_1$) *$I(u) = \frac{1}{2}(Lu, u) + b(u)$, where $Lu = L_1P_1u + L_2P_2u$, $L_i : E^{(i)} \mapsto E^{(i)}$ is bounded and selfadjoint, P_i is the projector of E onto $E^{(i)}$, i=1,2,*

(C$_2$) *b' is compact, and*

(C$_3$) *there exists a subspace $\widetilde{E} \subset E$ and sets $S \subset E$, $Q \subset \widetilde{E}$ and constants $\alpha > \omega$ such that*

- (i) *$S \subset E^{(1)}$ and $I|_S \ge \alpha$,*
- (ii) *Q is bounded and $I|_{\partial Q} \le \omega$,*
- (iii) *S and ∂Q link.*

Then I possesses a critical value $c \ge \alpha$ given by

$$c = \inf_{g\in\Gamma} \sup_{u\in Q} I(g(1,u)),$$

where

$$\Gamma \equiv \{g \in C([0,1] \times E, E)|\ g \text{ satisfies } (\Gamma_1)\text{–}(\Gamma_3)\},$$

(Γ_1) *$g(0,u) = u$,*

(Γ_2) *$g(t,u) = u$ for $u \in \partial Q$,*

(Γ_3) *$g(t,u) = e^{\theta(t,u)L}u + \chi(t,u)$, where $\theta(t,u) \in C([0,1]\times E, \mathbf{R})$ and χ is compact.*

[1]Condition (**PS**) (see [8, p. 1171]): Let E be a real Banach space, $I \in C_1(E, R)$, i.e. I is a continuously Fréchet-differentiable functional defined on E. I is said to be satisfying (**PS**) if any sequence $x(t) \subset E$ for which $I(x(t))$ is bounded and $I'(x(t)) \to 0$, as $t \to \infty$, possesses a convergent subsequence in E.

1.3. Proof of the main result. The following result of Rabinowitz [12] will be used.

Proposition 1.3. *There is a positive constant* ϱ *such that for each* $k \in \mathbf{N}$ *and* $z \in E_k$ *the following inequality holds:*

$$\|z\|_{L^\infty_{[-kT,kT]}} \leq \varrho \|z\|_{E_k}. \tag{11}$$

Lemma 1.4. *Under the conditions of Theorem* 1.1*,* I_k *satisfies the* (**PS**) *condition.*

Proof. Assume that $\{z_{k_n}\}_{n\in\mathbf{N}}$ in E_k is a sequence such that $\{I_k(z_{k_n})\}_{n\in\mathbf{N}}$ is bounded and $I_k'(z_{k_n}) \to 0$ as $n \to +\infty$. Then there exists a constant $d_1 > 0$ such that

$$|I_k(z_{k_n})| \leq d_1, \quad I_k'(z_{k_n}) \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

We first prove that $\{z_{k_n}\}_{n\in\mathbf{N}}$ is bounded. Let $z_{k_n} = z_{k_n}^0 + z_{k_n}^+ \in E_k^0 \oplus E_k^+$. From ($H_1$), ($H_5$), (3) of ($H_6$) and ($H_7$) we have that

$$\begin{aligned} 2d_1 &\geq 2I_k(z_{k_n}) - < I_k'(z_{k_n}), z_{k_n} > \\ &= \left(\int_{|z_{k_n}|\geq 1} + \int_{|z_{k_n}|<1}\right) [(z_{k_n}, V_{z_{k_n}}(t, z_{k_n})) - 2V(t, z_{k_n}) + (h_k(t), z_{k_n})]dt \\ &\geq \int_{|z_{k_n}|\geq 1} [(z_{k_n}, V_{z_{k_n}}(t, z_{k_n})) - 2V(t, z_{k_n}) + (h_k(t), z_{k_n})]dt \\ &\quad - \int_{|z_{k_n}|<1} |h_k(t)||z_{k_n}|dt \\ &\geq \int_{|z_{k_n}|\geq 1} (c_1 - \|h_k(t)\|_{L^\infty_{[-kT,kT]}})|z_{k_n}|^\beta dt - \int_{-kT}^{kT} |h_k(t)|dt. \end{aligned} \tag{13}$$

This implies

$$\int_{|z_{k_n}|\geq 1} |z_{k_n}|^\beta dt \leq \frac{(2d_1 + c_3)}{(c_1 - c_4)} = \widetilde{M}_0. \tag{14}$$

On the other hand, using a well known fact in [10, p. 378], we have

$$\int_{|z|<1} V(t, z(t))dt \leq \int_{|z|<1} V\left(t, \frac{z(t)}{|z(t)|}\right)|z(t)|^2 dt \leq \overline{M}\int_{-kT}^{kT} |z(t)|^2 dt \leq \overline{M}\|z\|_{E_k}^2 \tag{15}$$

for $z \in E_k$. From (H_2), (H_5), (4) of (H_6), (7), (11), and (15) (keeping in mind that $a_1 - 2\overline{M} > 0$) we have

$$
\begin{aligned}
\frac{a_1}{2}\|z_{k_n}\|^2_{E_k} &\le \int_{-kT}^{kT}\left[\frac{1}{2}|\dot z_{k_n}(t)|^2 + K(t,z_{k_n}(t))\right]dt\\
&= I_k(z_{k_n}) + \int_{-kT}^{kT} V(t,z_{k_n}(t))dt - \int_{-kT}^{kT}(h_k(t),z_{k_n}(t))dt\\
&= I_k(z_{k_n}) + \left(\int_{|z_{k_n}|\ge 1} + \int_{|z_{k_n}|<1}\right)[V(t,z_{k_n}) - (h_k(t),z_{k_n})]dt\\
&\le d_1 + \frac{1}{2}\int_{|z_{k_n}|\ge 1}(z_{k_n}, V_{z_{k_n}}(t,z_{k_n}))dt + \overline{M}\|z_{k_n}\|^2_{E_k}\\
&\quad + \left(\int_{-kT}^{kT}|h_k(t)|^2dt\right)^{\frac{1}{2}}\left(\int_{-kT}^{kT}|z_{k_n}|^2dt\right)^{\frac{1}{2}}dt \qquad (16)\\
&\le d_1 + \frac{\|z_{k_n}\|_{L^\infty_{[-kT,kT]}}}{2}\int_{|z_{k_n}|\ge 1}|V_{z_{k_n}}(t,z_{k_n})|dt + \overline{M}\|z_{k_n}\|^2_{E_k}\\
&\quad + \left(\int_{-kT}^{kT}|h_k(t)|^2dt\right)^{\frac{1}{2}}\left(\int_{-kT}^{kT}|z_{k_n}|^2dt\right)^{\frac{1}{2}}dt\\
&\le d_1 + \frac{\varrho}{2}\|z_{k_n}\|_{E_k}c_2\int_{|z_{k_n}|\ge 1}|z_{k_n}|^\lambda dt + \overline{M}\|z_{k_n}\|^2_{E_k}\\
&\quad + \|h_k(t)\|_{L^2_{[0,2kT]}}\|z_{k_n}\|_{E_k}.
\end{aligned}
$$

Since $\lambda \le \beta$, we have from (14) and (16) that

$$
\left(\frac{a_1}{2} - \overline{M}\right)\|z_{k_n}\|^2_{E_k} - \left(\frac{\eta}{2\varrho} + \frac{c_2\varrho\widetilde{M_0}}{2}\right)\|z_{k_n}\|_{E_k} \le d_1. \qquad (17)
$$

Now (17) guarantees that $\{\|z_{k_n}\|_{E_k}\}_{n\in\mathbf{N}}$ is bounded. Going if necessary to a subsequence, we can assume that there exists $z \in E_k$ such that $z_{k_n} \rightharpoonup z$, as $n \to +\infty$, in E_k, which implies $z_{k_n} \to z$ uniformly on $[-kT,kT]$. Hence $(I_k'(z_{k_n}) - I_k'(z))(z_{k_n} - z) \to 0$ and $\|z_{k_n} - z\|_{L^2[-kT,kT]} \to 0$. Set

$$
\Phi = \int_{-kT}^{kT}(V_{z_{k_n}}(t,z_{k_n}) - V_z(t,z), z_{k_n} - z)dt - \int_{-kT}^{kT}(K_{z_{k_n}}(t,z_{k_n}) - K_z(t,z), z_{k_n} - z)dt.
$$

It is easy to check that $\Phi \to 0$ as $n \to +\infty$. Moreover, an easy computation shows that

$$
(I_k'(z_{k_n}) - I_k'(z))(z_{k_n} - z) = \|\dot z_{k_n} - \dot z\|_{L^2_{[0,2kT]}} - \Phi,
$$

and so $\|\dot z_{k_n} - \dot z\|_{L^2_{[0,2kT]}} \to 0$. Consequently, $\|z_{k_n} - z\|_{E_k} \to 0$. □

Lemma 1.5. *If H, K and h satisfy* (H_1)–(H_7)*, then for every $k \in \mathbf{N}$ the system* (5) *possesses a $2kT$-periodic solution.*

Proof. The proof will be divided into three steps.

Step 1. Assume that $0 < \|z\|_{L^\infty_{[0,2kT]}} \le 1$ for $z \in E_k^{(1)} = E_k^+$. From (7), (10), (11), and (15) we have that

$$\begin{aligned} I_k(z) &= \int_{-kT}^{kT}\Big[\frac{1}{2}|\dot{z}(t)|^2 + K(t,z(t))\Big]dt - \int_{-kT}^{kT} V(t,z(t))dt + \int_{-kT}^{kT}(h_k(t),z(t))dt \\ &\ge \frac{a_1}{2}\|z\|_{E_k}^2 - \overline{M}\|z\|_{E_k}^2 - \frac{\eta}{2\varrho}\|z\|_{E_k} \\ &= \frac{1}{2}(a_1 - 2\overline{M} - \eta)\|z\|_{E_k}^2 + \frac{\eta}{2}\|z\|_{E_k}^2 - \frac{\eta}{2\varrho}\|z\|_{E_k}. \end{aligned} \tag{18}$$

Note from (H_7) that $a_1 - 2\overline{M} - \eta > 0$. Set

$$\rho = \frac{1}{\varrho}, \quad \alpha = \frac{a_1 - 2\overline{M} - \eta}{2\varrho^2}.$$

Let B_ρ denote the open ball in E_k with radius ρ about 0 and let ∂B_ρ denote its boundary. Let $S_k = \partial B_\rho \cap E_k^+$. If $z \in S_k$ then $||z||_{E_k} = \frac{1}{\varrho}$ (note $||z||_{L^\infty_{[0,2kT]}} \le 1$ from (11)) and so (18) gives

$$I_k(z) \ge \alpha, \quad z \in S_k.$$

Then (C_3)(i) of Lemma 1.2 holds.

Step 2. Choose $e \in E_k^+$ with $\|e\|_{E_k} = 1$. Let $\widetilde{E}_k = span\{e\} \oplus E_k^0$ and $\Theta_k = \{z \in \widetilde{E}_k : \|z\|_{E_k} = 1\}$. Note that $dim(\widetilde{E}_k) < +\infty$.

The argument in [6] guarantees that there exists $\varepsilon_k^1 > 0$ such that, $\forall u \in \Theta_k$,

$$meas\{t \in [0,2kT] : |u(t)| \ge \varepsilon_k^1\} \ge \varepsilon_k^1. \tag{19}$$

For $z = z^+ + z^0 \in \Theta_k$, let $\Omega_k^z = \{t \in [0,2kT] : |z(t)| \ge \varepsilon_k^1\}$. By ($H_4$), for $M_k^* = \frac{a_2}{(\varepsilon_k^1)^3} > 0$, there exists L_k such that

$$V(t,z) \ge M_k^*|z|^2, \quad \forall |z| \ge L_k, \quad \text{uniformly in } t. \tag{20}$$

Let $\gamma_k \ge \max\left\{\frac{2\eta}{\varrho a_2}, \frac{L_k}{\varepsilon_k^1}\right\}$. For $\gamma \ge \gamma_k$, from (19) and (20) we have that

$$V(t,\gamma z) \ge M_k^*|\gamma z|^2 \ge M_k^*\gamma^2(\varepsilon_k^1)^2, \quad \forall t \in \Omega_k^z. \tag{21}$$

From (H_1) (or (7)) and (21), we have for $z = z^+ + z^0 \in \Theta_k$ that

$$\begin{aligned} I_k(\gamma z) &= \int_{-kT}^{kT}\left[\frac{\gamma^2}{2}|\dot{z}(t)|^2 + K(t,\gamma z)\right]dt - \int_{-kT}^{kT} V(t,\gamma z)dt + \int_{-kT}^{kT}(h_k(t),\gamma z)dt \\ &\le \frac{a_2}{2}\gamma^2 - \int_{\Omega_k^z} V(t,\gamma z)dt + \gamma\|h_k(t)\|_{L^2[0,2kT]} \\ &\le \frac{a_2}{2}\gamma^2 - M_k^*\gamma^2(\varepsilon_k^1)^3 + \gamma\frac{\eta}{2\varrho} \\ &\le 0. \end{aligned} \tag{22}$$

Therefore

$$I_k(\gamma z) \le 0, \quad \text{for any } z \in \Theta_k \text{ and } \gamma \ge \gamma_k. \tag{23}$$

Let $Q_k = \{\gamma e : 0 \le \gamma \le 2\gamma_k\} \oplus \{z \in E_k^0 : \|z\|_{E_k} \le 2\gamma_k\}$. It is easy to see that $\|z\|_{E_k} = 2\gamma_k$ or $\|z\|_{E_k} = 0$ for $\forall z \in \partial Q_k$. From ($H_2$) and (23) we have $I_k|_{\partial Q_k} \le 0$, i.e., I_k satisfies (C_2)(ii) of Lemma 1.2.

Step 3. (C_3)(iii) (i.e., S_k links ∂Q_k) holds from the definition of S_k and Q_k and [11, p. 32]. Thus (C_3)(iii) holds.

Note that (C_1) and (C_2) of Lemma 1.2 are true. Now from Lemma 1.2, I_k possesses a critical value c_k given by

$$c_k = \inf_{g_k \in \Upsilon_k} \sup_{u_k \in Q_k} I_k(g_k(1,u_k)), \tag{24}$$

where Υ_k satisfies (Γ_1)–(Γ_3). Hence, for every $k \in \mathbf{N}$, there is $z_k^* \in E_k$ such that

$$I_k(z_k^*) = c_k, \quad I_k'(z_k^*) = 0. \tag{25}$$

The function z_k^* is a desired classical $2kT$-periodic solution of (5). Since $c_k \ge \alpha = \frac{a_1 - 2\overline{M} - \eta}{2\varrho^2} > 0$, z_k^* is a nontrivial solution. □

Lemma 1.6. *Let $\{z_k^*\}_{k\in\mathbf{N}}$ be the sequence given by Lemma 1.5. There exists a $z_0^* \in C^1(\mathbf{R},\mathbf{R}^n)$ such that $z_k^* \to z_0^*$ in $C_{loc}^1(\mathbf{R},\mathbf{R}^n)$ as $k \to +\infty$.*

Proof. The first step in the proof is to show that the sequences $\{c_k\}_{k\in\mathbf{N}}$ and $\{\|z_k^*\|_{E_k}\}_{k\in\mathbf{N}}$ are bounded. There exists $\widehat{z}_1^* \in E_1$ with $\widehat{z}_1^*(\pm T) = 0$ such that

$$c_1 \le I_1(\widehat{z}_1^*) = \inf_{g_1\in\Upsilon_1} \sup_{u_1\in Q_1, u_1(\pm T)=0} I_1(g_1(1,u_1)). \tag{26}$$

For every $k \in \mathbf{N}$, let

$$\widehat{z}_k^*(t) = \begin{cases} \widehat{z}_1^*(t) & \text{for } |t| \le T \\ 0 & \text{for } T < |t| \le kT \end{cases} \tag{27}$$

and $\widetilde{g}_k : [0,1] \times E_k \to E_k$ be a curve given by $\widetilde{g}_k(t,z) \equiv z$, where $z \in E_k$. Then $\widetilde{g}_k \in \Upsilon_k$ and $I_k(\widetilde{g}_k(1,\widehat{z}_k^*)) = I_1(\widetilde{g}_1(1,z_1^*)) = I_1(z_1^*)$ for all $k \in \mathbf{N}$. Therefore, from (24), (26) and (27),

$$c_k \le I_k(\widetilde{g}_k(1,\widehat{z}_k^*)) = I_1(\widetilde{g}_1(1,z_1^*)) = I_1(z_1^*) \equiv M_0. \tag{28}$$

From (H_5) and (3) of (H_6) we have that

$$\begin{aligned}
2M_0 &\geq 2I_k(z_k^*) - < I_k^{'}(z_k^*), z_k^* > \\
&\geq \int_{-kT}^{kT} [(z_k^*, V_{z_k^*}(t, z_k^*)) - 2V(t, z_k^*)]dt + \int_{-kT}^{kT} (h_k(t), z_k^*)dt \\
&= \left(\int_{|z_k^*| \geq 1} + \int_{|z_k^*| < 1} \right) [(z_k^*, V_{z_k^*}(t, z_k^*)) - 2V(t, z_k^*) + (h_k(t), z_k^*)]dt \\
&\geq \int_{|z_k^*| \geq 1} [(z_k^*, V_{z_k^*}(t, z_k^*)) - 2V(t, z_k^*) + (h_k(t), z_k^*)]dt - \int_{|z_k^*| < 1} |h_k(t)||z_k^*|dt \\
&\geq \int_{|z_k^*| \geq 1} (c_1 - c_4)|z_k^*|^\beta dt - \int_{-kT}^{kT} |h_k(t)|dt.
\end{aligned} \tag{29}$$

This implies

$$\int_{|z_k^*| \geq 1} |z_k^*|^\beta dt \leq \frac{(2M_0 + c_3)}{(c_1 - c_4)} = \widetilde{M}_0^*. \tag{30}$$

On the other hand, from (H_2), (H_5), (4) of (H_6), (H_7), (7) and (11) (keeping in mind $a_1 - 2\overline{M} > 0$) we have

$$\begin{aligned}
\frac{a_1}{2} \|z_k^*\|_{E_k}^2 &\leq \int_{-kT}^{kT} \left[\frac{1}{2} |\dot{z}_k^*|^2 + K(t, z_k^*) \right] dt \\
&= I_k(z_k^*) + \int_{-kT}^{kT} V(t, z_k^*)dt - \int_{-kT}^{kT} (h_k(t), z_k^*)dt \\
&= I_k(z_k^*) + \left(\int_{|z_k^*| \geq 1} + \int_{|z_k^*| < 1} \right) [V(t, z_k^*) - (h_k(t), z_k^*)]dt \\
&\leq M_0 + \frac{1}{2} \int_{|z_k^*| \geq 1} (z_k^*, V_{z_k^*}(t, z_k^*))dt + \overline{M} \, \|z_k^*\|_{E_k}^2 \\
&\quad + \left(\int_{-kT}^{kT} |h_k(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-kT}^{kT} |z_k^*|^2 dt \right)^{\frac{1}{2}} dt \\
&\leq M_0 + \frac{\|z_k^*\|_{L^\infty_{[-kT,kT]}}}{2} \int_{|z_k^*| \geq 1} |V_{z_k^*}(t, z_k^*)|dt + \overline{M} \, \|z_k^*\|_{E_k}^2 \\
&\quad + \left(\int_{-kT}^{kT} |h_k(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-kT}^{kT} |z_k^*|^2 dt \right)^{\frac{1}{2}} dt \\
&\leq M_0 + \frac{\varrho}{2} \|z_k^*\|_{E_k} c_2 \int_{|z_k^*| \geq 1} |z_k^*|^\lambda dt + \overline{M} \|z_k^*\|_{E_k}^2 \\
&\quad + \|h_k(t)\|_{L^2[0,2kT]} \|z_k^*\|_{E_k}.
\end{aligned} \tag{31}$$

Since $\lambda \le \beta$, we have from (30) and (31) that

$$\left(\frac{a_1}{2} - \overline{M}\right) \|z_k^*\|_{E_k}^2 - \left(\frac{\eta}{2\varrho} + \frac{c_2 \varrho \widetilde{M_0^*}}{2}\right) \|z_k^*\|_{E_k} \le M_0. \tag{32}$$

Now (32) guarantees that $\{\|z_k^*\|_{E_k}\}_{k\in\mathbf{N}}$ is bounded. Therefore, there exists a constant $M_1 > 0$ such that

$$\|z_k^*\|_{E_k} \le M_1. \tag{33}$$

We now show that for a large enough k,

$$\|z_k^*\|_{L^\infty_{[-kT,kT]}} \le M_2. \tag{34}$$

From (11) and (33), there exists a positive constant $M_2 = \varrho M_1$ such that (34) holds. Since z_k^* satisfies (5), we have if $t \in [-kT, kT]$

$$|\ddot{z}_k^*| \ \ \le |h_k(t)| + |K_{z_k^*}(t, z_k^*)| + |V_{z_k^*}(t, z_k^*)|. \tag{35}$$

Therefore, (H_1), (H_3), (H_7), (34), and (35) imply that there is $\widetilde{M_3} > 0$ independent of k such that

$$\|\ddot{z}_k^*\|_{L^\infty_{[0,2kT]}} \le \widetilde{M_3}. \tag{36}$$

By (34) and (36), we have

$$|\dot{z}_k^*(t)| = \left|\int_{\gamma_k}^t \ddot{z}_k^*(s)ds + \dot{z}_k^*(\tau_k)\right| \le \int_{t-1}^t |\ddot{z}_k^*(s)|ds + |z_k^*(t) - z_k^*(t-1)| \le \widetilde{M_3} + 2M_2.$$

Thus for every $k \in \mathbf{N}$ we have

$$\|\dot{z}_k^*\|_{L^\infty_{[0,2kT]}} \le \widetilde{M_4}. \tag{37}$$

Let $k \in \mathbf{N}$ and $t, t_0 \in \mathbf{R}$, then

$$|z_k^*(t) - z_k^*(t_0)| = \left|\int_{t_0}^t \dot{z}_k^*(s)ds\right| \le \int_{t_0}^t |\dot{z}_k^*(s)|ds \le \widetilde{M_4}(t - t_0).$$

and

$$|\dot{z}_k^*(t) - \dot{z}_k^*(t_0)| = \left|\int_{t_0}^t \ddot{z}_k^*(s)ds\right| \le \int_{t_0}^t |\ddot{z}_k^*(s)|ds \le \widetilde{M_3}(t - t_0).$$

Since both $\{z_k^*\}_{k\in\mathbf{N}}$ and $\{\dot{z}_k^*\}_{k\in\mathbf{N}}$ are bounded in $L^\infty_{[-kT,kT]}(\mathbf{R}, \mathbf{R}^{2n})$ and equicontinuous, we obtain that the sequence $\{z_k^*\}_{k\in\mathbf{N}}$ converges to a certain $z_0^* \in C^1(\mathbf{R}, \mathbf{R}^n)$ by using the Arzelà-Ascoli theorem. □

Lemma 1.7. *The function z_0^* determined by Lemma* 1.5 *is the desired homoclinic solution of* (1.1).

The following result of Izydorek and Janczewska [10] will be used.

Proposition 1.8. *Let $z : \mathbf{R} \to \mathbf{R}^n$ be a continuous mapping such that $\dot{z} \in L^2_{loc}(\mathbf{R}, \mathbf{R}^n)$. For every $t \in \mathbf{R}$ the following inequality holds:*

$$|z(t)| \leq \sqrt{2} \left[\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|z(s)|^2 + |\dot{z}(s)|^2) ds \right]^{\frac{1}{2}}. \tag{38}$$

Proof of Lemma 1.7. The proof will be divided into four steps.

Step 1. We prove that $z_0^*(t) \to 0$, as $t \to \pm\infty$. Note we have

$$\begin{aligned}\int_{-\infty}^{+\infty} (|z_0^*(t)|^2 + |\dot{z}_0^*(t)|^2) dt &= \lim_{j \to +\infty} \int_{-jT}^{jT} (|z_0^*(t)|^2 + |\dot{z}_0^*(t)|^2) dt \\ &= \lim_{j \to +\infty} \lim_{k \to +\infty} \int_{-jT}^{jT} (|z_k^*(t)|^2 + |\dot{z}_k^*(t)|^2) dt.\end{aligned}$$

Clearly, by (32), for every $j \in \mathbf{N}$ there exists $n_j \in \mathbf{N}$ such that for all $k \geq n_j$ we have

$$\int_{-jT}^{jT} (|z_{n_k}^*(t)|^2 + |\dot{z}_{n_k}^*(t)|^2) dt \leq \|z_{n_k}^*\|^2_{E_{n_k}} \leq M_1^2,$$

Letting $k \to +\infty$, we get $\int_{-jT}^{jT} (|z_0^*(t)|^2 + |\dot{z}_0^*(t)|^2) dt \leq M_1^2$, and now, letting $j \to +\infty$, we have $\int_{-\infty}^{+\infty} (|z_0^*(t)|^2 + |\dot{z}_0^*(t)|^2) dt \leq M_1^2$, and so

$$\int_{|t| \geq m} (|z_0^*(t)|^2 + |\dot{z}_0^*(t)|^2) dt \to 0, \quad \text{as } m \to +\infty. \tag{39}$$

Then (39) shows that our claim holds.

Step 2. We now show that $\dot{z}_0^*(t) \to 0$, as $t \to \pm\infty$. Note that from (38) we get

$$\begin{aligned}|\dot{z}_0^*(t)|^2 &\leq 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{z}_0^*(s)|^2 ds + 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{z}_0^*(s)|^2 ds \\ &\leq 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|z_0^*(s)|^2 + |\dot{z}_0^*(s)|^2) ds + 2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{z}_0^*(s)|^2 ds.\end{aligned} \tag{40}$$

Since we have (39) and (40) it suffices to prove that

$$\int_{m}^{m+1} |\ddot{z}_0^*(t)|^2 dt \to 0, \quad \text{as } m \to +\infty. \tag{41}$$

By (5) we obtain

$$\begin{aligned}\int_m^{m+1}|\ddot{z}_0^*(t)|^2dt = &\int_m^{m+1}(|K_{z_0^*}(t,Z_0^*(t)) - V_{z_0^*}(t,z_0^*(t))|^2dt\\ &-2\int_m^{m+1}(V_{z_0^*}(t,z_0^*(t)),h(t))dt\\ &+\int_m^{m+1}|h(t)|^2)dt + 2\int_m^{m+1}(K_{z_0^*}(t,z_0^*(t)),h(t))dt.\end{aligned}$$

Since $V(t,0)=0, K_z(t,0)=0$ for all $t\in \mathbf{R}$, $z_0^*(t)\to 0$, as $t\to\pm\infty$ and $\int_m^{m+1}|h(t)|^2dt\to 0$, as $m\to\pm\infty$, (41) follows.

Step 3. We show that $z_0^*\not\equiv 0$ when $h(t)\equiv 0$. Now, up to a subsequence, we have either

$$\int_{-\infty}^{+\infty}|z_0^*(t)|^2dt = \lim_{j\to+\infty}\int_{-jT}^{jT}|z_0^*(t)|^2dt = \lim_{j\to+\infty}\lim_{k\to+\infty}\int_{-jT}^{jT}|z_{n_k}^*(t)|^2dt = 0. \quad (42)$$

or there exist $\alpha>0$ such that

$$\int_{-\infty}^{+\infty}|z_0^*(t)|^2dt \ge \alpha > 0. \quad (43)$$

In the first case we shall say that z_0^* is vanishing, in the second we shall say that z_0^* is nonvanishing.

By assumptions (H_3)–(H_5), for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|V_{z_{n_k}^*}(t,z_{n_k}^*)| \le \varepsilon|z_{n_k}^*| + C_\varepsilon|z_{n_k}^*|^\lambda. \quad (44)$$

Hence, from (44) there exists a positive constant $\widetilde{\gamma}$ such that

$$\int_{-kT}^{kT}|(z_{n_k}^*)^+||V_{z_{n_k}^*}(t,z_{n_k}^*)|dt \le \widetilde{\gamma}(\varepsilon\|z_{n_k}^*\|_{E_k}\|(z_{n_k}^*)^+\|_{E_k} + C_\varepsilon\|z_{n_k}^*\|_{E_k}^\lambda\|(z_{n_k}^*)^+\|_{E_k}. \quad (45)$$

Arguing indirectly, we suppose $\{z_{n_k}^*\}_{k=1}^\infty$ is vanishing. From (42) and (44) we have that

$$\lim_{k\to\infty}\int_{-kT}^{kT}((z_{n_k}^*)^+, V_{z_{n_k}^*}(t,z_{n_k}^*))dt = \lim_{k\to\infty}\int_{-kT}^{kT}V(t,z_{n_k}^*)dt = 0. \quad (46)$$

Since $< I_k^{'}(z_{n_k}^*), (z_{n_k}^*)^\pm >= 0$, for some positive constant $\widetilde{C}$, we obtain using (42) and (45) that

$$
\begin{aligned}
&\xi_1\|(z^*_{n_k})^+\|^2_{E_k}\\
&\le < A(z^*_{n_k})^+,(z^*_{n_k})^+ >\\
&\le -\int_{-kT}^{kT}[(K_{z^*_{n_k}}(t,z^*_{n_k})-k(t)(z^*_{n_k})^+,(z^*_{n_k})^+)]dt+\int_{-kT}^{kT}((z^*_{n_k})^+,V_{z^*_{n_k}}(t,z^*_{n_k}))dt\\
&\le \int_{-kT}^{kT}((z^*_{n_k})^+,V_{z^*_{n_k}}(t,z^*_{n_k}))dt\\
&\le \frac{\xi_1}{4}\|z^*_{n_k}\|^2_{E_k}+\widetilde{C}\|z^*_{n_k}\|^{\lambda+1}_{E_k},
\end{aligned}
\tag{47}
$$

where ξ_1 is the smallest positive eigenvalue of operator A.

On the other hand, note that $dim(E^0_k) < +\infty$, there exist two positive constants b_1 and b_2 such that

$$b_1|(z^*_{n_k})^0|^2_2 \le \|(z^*_{n_k})^0\|^2_{E_k} \le b_2|(z^*_{n_k})^0|^2_2. \tag{48}$$

From (42) and (48) we have that

$$\xi_1\|(z^*_{n_k})^0\|^2_{E_k} \le b_\varepsilon\|z^*_{n_k}\|^2_{E_k}, \tag{49}$$

where $0 < b_\varepsilon \le \frac{\xi_1}{4}$.

Hence from (47) and (49) we have that

$$\xi_1\|z^*_{n_k}\|^2_{E_k} \le \frac{\xi_1}{2}\|z^*_{n_k}\|^2_{E_k}+\widetilde{C}\|z^*_{n_k}\|^{\lambda+1}_{E_k},$$

and $\|z^*_{n_k}\|_{E_k} \ge \widetilde{c}$ for some $\widetilde{c} > 0$.

On the other hand, we have from (42), (46)–(48) we have that $\|(z^*_{n_k})^+\|^2_{E_k} \to 0$ and $\|(z^*_{n_k})^0\|^2_{E_k} \to 0$ as $k \to \infty$. This means that $\|z^*_{n_k}\|_{E_k} \to 0$ as $k \to \infty$, which leads to a contradiction. Hence $\{z^*_{n_k}\}$ is nonvanishing, so (43) holds, and this shows that our claim holds.

Step 4. We show that $z^*_0(t)$ is a nontrivial homoclinic solution of (1). According to Step 3, $z^*_0(t) \not\equiv 0$, so it suffices to prove for any $\varphi \in C^\infty_0(\mathbf{R},\mathbf{R}^n)$

$$\int_{-\infty}^{+\infty}((\ddot{z}^*_0(t)-K_{z^*_0}(t,z^*_0)+V_{z^*_0}(t,z^*_0)-h(t)),\varphi(t))dt=0. \tag{50}$$

By Step 1, we can choose k_0 such that $supp\varphi \subseteq [-k_iT, k_iT]$ for all $k_i \ge k_0$, and we have for $k_i \ge k_0$

$$\int_{-\infty}^{+\infty}((\ddot{z}^*_{k_i}(t)-K_{z^*_{k_i}}(t,z^*_{k_i})+V_{z^*_{k_i}}(t,z^*_0)-h_i(t)),\varphi(t))dt=0. \tag{51}$$

By (39) and (51), letting $k_i \to \infty$ we get (50), which shows $z^*_0(t)$ is a nontrivial homoclinic solution of (1). □

Proof of Theorem 1.1. The result follows from Lemma 1.7. □

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