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Local Well-Posedness of a Coupled Camassa-Holm System in Critical Spaces

Xingxing Liu

Abstract. In this paper, we consider the local well-posedness of a coupled Camassa-Holm system with initial data in Besov spaces $B^s_{2,1}$ with critical index $s = \frac{3}{2}$.

Keywords. Coupled Camassa-Holm system, Besov spaces, well-posedness, critical index

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1. Introduction

In recent years, there are many integrable multi-component generalizations of the Camassa-Holm equation [2,3,10]. The most popular one among them is the following two-component Camassa-Holm shallow water system [4,5,18,23]:

$$\begin{cases} m_t + 2mu_x + um_x + \sigma\rho\rho_x &= 0, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x &= 0, \quad t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1}$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$. The system (1) appears originally in [22] and its mathematical properties, such as the local well-posedness in Sobolev spaces and in Besov spaces, the global existence results and the blow-up phenomena of strong solutions, the existence of global weak solutions, the orbital stability of the smooth solitary waves and so on, have been investigated extensively in many works [5, 8, 13–16, 20, 21, 26] and references therein. The great interest in the system (1) lies in the fact that Constantin and Ivanov [5] has given the hydrodynamical derivation of the system (1) as a valid approximation to the governing equations for water waves in the shallow water regime without vorticity. Hence, it has a physical interpretation. The system (1) is also integrable and has a bi-Hamiltonian structure [9, 17]. However, in contrast to the Camassa-Holm equation, the system (1) does not have the peaked solitons (peakons) in the form of a superposition of multi-peakons.

X. X. Liu: Department of Mathematics, China University of Mining and Technology, Xuzhou, China; liuxingxing123456@163.com

In the present paper, we consider a coupled Camassa-Holm system with peakons introduced in [11]:

$$\begin{cases} m_t = 2mu_x + m_x u + (mv)_x + nv_x, & t > 0,\ x \in \mathbb{R}, \\ n_t = 2nv_x + n_x v + (nu)_x + mu_x, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{2}$$

with the initial data $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$, where $m = u - u_{xx}$ and $n = v - v_{xx}$. Fu and Qu [11] have shown that the system (2) has peakons in the form of a superposition of multi-peakons. The well-posedness of the system (2) on the line $\mathbb{R}$, and on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ by applying Kato's semigroup theory [19] have been studied in [11, 12] with initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}), s > \frac{3}{2}$ and $(u_0, v_0) \in H^s(\mathbb{S}) \times H^s(\mathbb{S}), s > \frac{3}{2}$, respectively. The Littlewood-Paley decomposition and the nonhomogeneous Besov spaces have been used to study the Cauchy problem of the system (2) in [24] with initial data $(u_0, v_0) \in B^s_{p,r} \times B^s_{p,r}, 1 \le p, r \le \infty, s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$.

For convenience, we rewrite the system (2) with the Fourier integral operator $P_1(D) = (1 - \partial_x^2)^{-1}$ and $P_2(D) = \partial_x(1 - \partial_x^2)^{-1}$ in the following form:

$$\begin{cases} u_t - (u+v)u_x = P_1(D)(uv_x) + P_2(D)(u^2 + \frac{1}{2}u_x^2 \\ \qquad\qquad +u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2), & t > 0,\ x \in \mathbb{R}, \\ v_t - (u+v)v_x = P_1(D)(u_x v) + P_2(D)(v^2 + \frac{1}{2}v_x^2 \\ \qquad\qquad +u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

Following the method in [6, 7, 25], the purpose of our paper is to discuss the local well-posedness of the system (3) in Besov spaces $B^s_{2,1}$ with the critical index $s = \frac{3}{2}$. More precisely, we state our main theorem as follows:

Theorem 1.1. *Given $(u_0, v_0) \in B^{\frac{3}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1}$, then there exists a maximal $T > 0$, and a unique solution (u, v) to the system* (3) *such that*

$$(u, v) \in C\Big([0, T]; B^{\frac{3}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1}\Big) \cap C^1\Big([0, T]; B^{\frac{1}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}\Big).$$

In addition, the solution depends continuously on the initial data, i.e., the mapping $\Psi\colon (u_0, v_0) \mapsto (u, v)$ is continuous from $B^{\frac{3}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1}$ into $C\Big([0, T]; B^{\frac{3}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1}\Big)$ $\cap C^1\Big([0, T]; B^{\frac{1}{2}}_{2,1} \times B^{\frac{1}{2}}_{2,1}\Big)$. Moreover, the system (3) *is not locally well-posed in $B^{\frac{3}{2}}_{2,\infty} \times B^{\frac{3}{2}}_{2,\infty}$.*

Remark 1.2. By Fourier-Plancherel formula, we find that the Besov space $B^s_{2,2}(\mathbb{R})$ coincides with the Sobolev space $H^s(\mathbb{R})$. Thus, we have the chain of continuous embedding for any $s' < \frac{3}{2} < s$: $H^s \hookrightarrow B^{\frac{3}{2}}_{2,1} \hookrightarrow H^{\frac{3}{2}} \hookrightarrow B^{\frac{3}{2}}_{2,\infty} \hookrightarrow H^{s'}$, which combining with Theorem 1.1 show us some more information about local well-posedness on the critical value $\frac{3}{2}$ for the system (3). However, we still have not answered the question of the well-posedness in the intermediate spaces $B^{\frac{3}{2}}_{2,r}, 1 < r < \infty$.

The remainder of the paper is dedicated to the proof of Theorem 1.1. In Section 2, we divide our proof into four subsections to pursue our goal of Theorem 1.1. In Section 3, we enclose the Littlewood-Paley decomposition, Besov spaces and the transport equation theory as an Appendix for completeness.

Notation. Let $C > 0$ be a generic constant and $\bar{\mathbb{N}} \triangleq \mathbb{N} \cup \{\infty\}$. For simplicity, we denote the function space $X \times X$ as $(X)^2$. In the sequel, Lip denotes the space of bounded Lipschitz functions.

2. Proof of Theorem 1.1

In this section, we will complete the proof of Theorem 1.1. We break it into the following four subsections.

2.1. Existence of the solution. In this subsection, we will show the existence of the solution for the system (3) with initial data $(u_0, v_0) \in (B^{\frac{3}{2}}_{2,1})^2$.

By a standard iterative process and Lemma 3.5, starting from $(u^0, v^0) \triangleq (0, 0)$, we define by induction by a sequence of smooth functions $(u^n, v^n)_{n\in\mathbb{N}}$ solving the following linear transport equations:

$$\begin{cases} \big(\partial_t - (u^n + v^n)\partial_x\big)u^{n+1} &= g_1^n(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ \big(\partial_t - (u^n + v^n)\partial_x\big)v^{n+1} &= g_2^n(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u^{n+1}\big|_{t=0} \triangleq u_0^{n+1}(x) &= S_{n+1}u_0, \quad x \in \mathbb{R}, \\ v^{n+1}\big|_{t=0} \triangleq v_0^{n+1}(x) &= S_{n+1}v_0, \quad x \in \mathbb{R}, \end{cases} \tag{4}$$

where

$$\begin{aligned} &g_1^n(t,x) \\ &\triangleq g_{11}^n(t,x) + g_{12}^n(t,x) \\ &= P_1(D)(u^n\partial_x v^n) + P_2(D)\Big((u^n)^2 + \frac{1}{2}(\partial_x u^n)^2 + \partial_x u^n \partial_x v^n + \frac{1}{2}(v^n)^2 - \frac{1}{2}(\partial_x v^n)^2\Big), \\ &g_2^n(t,x) \\ &\triangleq g_{21}^n(t,x) + g_{22}^n(t,x) \\ &= P_1(D)(\partial_x u^n v^n) + P_2(D)\Big((v^n)^2 + \frac{1}{2}(\partial_x v^n)^2 + \partial_x u^n \partial_x v^n + \frac{1}{2}(u^n)^2 - \frac{1}{2}(\partial_x u^n)^2\Big). \end{aligned}$$

Note that $P_1(D) \in Op(S^{-2})$ and $P_2(D) \in Op(S^{-1})$. According to Proposition 3.3 (vi) and the fact $B^{\frac{1}{2}}_{2,1}$ is an algebra, we have

$$\|g_{11}^n(t,x)\|_{B^{\frac{3}{2}}_{2,1}} \le C\|u^n\|_{B^{\frac{1}{2}}_{2,1}} \|v^n\|_{B^{\frac{3}{2}}_{2,1}} \le C\|u^n\|_{B^{\frac{3}{2}}_{2,1}} \|v^n\|_{B^{\frac{3}{2}}_{2,1}},$$

and

$$\|g_{12}^n(t,x)\|_{B^{\frac{3}{2}}_{2,1}} \leq C\big(\|u^n\|^2_{B^{\frac{3}{2}}_{2,1}} + \|u^n\|_{B^{\frac{3}{2}}_{2,1}}\|v^n\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n\|^2_{B^{\frac{3}{2}}_{2,1}}\big).$$

Thus, we can deduce that

$$\|g_1^n(t,x)\|_{B^{\frac{3}{2}}_{2,1}} \leq C\big(\|u^n\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n\|_{B^{\frac{3}{2}}_{2,1}}\big)^2. \tag{5}$$

Similarly,

$$\|g_2^n(t,x)\|_{B^{\frac{3}{2}}_{2,1}} \leq C\big(\|u^n\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n\|_{B^{\frac{3}{2}}_{2,1}}\big)^2. \tag{6}$$

By Lemma 3.4 (i) and (5)–(6), we find that

$$\begin{aligned}
&\|u^{n+1}\|_{B^{\frac{3}{2}}_{2,1}}+\|v^{n+1}\|_{B^{\frac{3}{2}}_{2,1}} \\
&\leq \big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)\exp\Big\{C\int_0^t\big(\|u^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}}+\|v^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau\Big\} \\
&\quad +C\int_0^t\big(\|u^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}}+\|v^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\big)^2\exp\Big\{C\int_\tau^t\big(\|u^n(\tau')\|_{B^{\frac{3}{2}}_{2,1}}+\|v^n(\tau')\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau'\Big\}d\tau.
\end{aligned} \tag{7}$$

Let us fix a $T>0$ such that $2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)T < 1$ and for all $t \in [0,T]$, assume that

$$\|u^n(t)\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n(t)\|_{B^{\frac{3}{2}}_{2,1}} \leq \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)t}. \tag{8}$$

From (8), we obtain

$$\begin{aligned}
C\int_\tau^t \big(\|u^n(\tau')\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n(\tau')\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau' &\leq C\int_\tau^t \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)\tau'}d\tau' \\
&= \ln\frac{\sqrt{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)\tau}}{\sqrt{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)t}}.
\end{aligned}$$

Inserting the above inequality into (7) yields

$$
\begin{aligned}
&\|u^{n+1}(t)\|_{B^{\frac{3}{2}}_{2,1}} + \|v^{n+1}(t)\|_{B^{\frac{3}{2}}_{2,1}} \\
&\leq \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{\sqrt{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)t}} \\
&\quad + \frac{C}{\sqrt{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)t}} \int_0^t \frac{\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)^2}{\Big(1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)\tau\Big)^{\frac{3}{2}}} d\tau \\
&= \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{1-2C\big(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}\big)t}.
\end{aligned}
$$

Thus, $(u^n, v^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];(B^{\frac{3}{2}}_{2,1})^2)$. Using the equations (4), the fact $B^{\frac{1}{2}}_{2,1}$ is an algebra and Proposition 3.3 (vi), we find that $(\partial_t u^n, \partial_t v^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];(B^{\frac{1}{2}}_{2,1})^2)$. Therefore, we gather that the sequence $(u^n, v^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];(B^{\frac{3}{2}}_{2,1})^2)$ $\cap C^1([0,T];(B^{\frac{1}{2}}_{2,1})^2)$.

From Proposition 3.3 (ii), the Arzela-Ascoli theorem and Cantor's diagonal process enable us to get that, up to an extraction, $(u^n, v^n)_{n\in\mathbb{N}}$ tends to a limit (u,v) in $C([0,T];(B^{\frac{1}{2}}_{2,1})^2_{loc})$. Since $(u^n, v^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];(B^{\frac{3}{2}}_{2,1})^2)$, we infer that, according to Proposition 3.3 (iv), $(u,v) \in L^\infty(0,T;(B^{\frac{3}{2}}_{2,1})^2)$. Then by Proposition 3.3 (v), we can prove that $(u^n, v^n)_{n\in\mathbb{N}}$ tends to (u,v) in $C([0,T];(B^{s}_{2,1})^2_{loc})$ for all $s < \frac{3}{2}$. Thus, it is easy to pass to the limit in the equations (4) and to conclude that (u,v) indeed solves the system (3) in the sense of distributions.

Thanks to $(u,v) \in L^\infty(0,T;(B^{\frac{3}{2}}_{2,1})^2)$, the system (3) and Lemma 3.4 (iv), we have $(u,v) \in C([0,T];(B^{\frac{3}{2}}_{2,1})^2)$. Using the system (3) again, we get $(\partial_t u, \partial_t v) \in C([0,T];(B^{\frac{1}{2}}_{2,1})^2)$. Hence, we prove that the system (3) has a solution $(u,v) \in C([0,T];(B^{\frac{3}{2}}_{2,1})^2) \cap C^1([0,T];(B^{\frac{1}{2}}_{2,1})^2)$.

2.2. Uniqueness of the solution. This subsection is devoted to give a priori estimate, which implies the uniqueness of the solution for the system (3).

Suppose that $(u^1,v^1),(u^2,v^2) \in L^\infty(0,T;(B^{\frac{3}{2}}_{2,\infty} \cap Lip)^2) \cap C([0,T];(B^{\frac{1}{2}}_{2,\infty})^2)$

are two given solutions to the system (3) with the initial data $(u_0^1, v_0^1), (u_0^2, v_0^2) \in (B^{\frac{3}{2}}_{2,\infty} \cap Lip)^2$, respectively. Denote $u^{12} \triangleq u^2 - u^1$ and $v^{12} \triangleq v^2 - v^1$. Assume that there exists a positive constant C such that for some $T^* \leq T$,

$$\sup_{t\in[0,T^*]} \Big(e^{-C\int_0^t U(\tau)d\tau} \big(\|u^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} \big) \Big) \leq 1. \tag{9}$$

We will prove, for all $t \in [0, T^*]$,

$$\begin{aligned} &\frac{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{e} \\ &\leq e^{C\int_0^t U(\tau)d\tau} \left(\frac{\|u_0^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v_0^{12}\|_{B^{\frac{1}{2}}_{2,\infty}}}{e} \right)^{\exp(-C\int_0^t \Phi(V(\tau))d\tau)}, \end{aligned} \tag{10}$$

where the function

$$\begin{aligned} \Phi(x) &\triangleq x \ln(e + Mx), \\ M &\triangleq \sup_{t\in[0,T]} \left(\frac{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{\|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}} + \frac{\|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}} + 2 \right), \\ U(t) &\triangleq \|\partial_x u^1(t)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} + \|\partial_x v^1(t)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}, \\ V(t) &\triangleq \|u^1\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|u^2\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v^1\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v^2\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip}. \end{aligned}$$

Indeed, (u^{12}, v^{12}) solves the following transport equations:

$$\begin{cases} \partial_t u^{12} - (u^1 + v^1)\partial_x u^{12} &= (u^{12} + v^{12})\partial_x u^2 + f_1(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ \partial_t v^{12} - (u^1 + v^1)\partial_x v^{12} &= (u^{12} + v^{12})\partial_x v^2 + f_2(t,x), \quad t > 0,\ x \in \mathbb{R}, \\ u^{12}\big|_{t=0} \triangleq u_0^{12}(x) &= u_0^2 - u_0^1, \quad x \in \mathbb{R}, \\ v^{12}\big|_{t=0} \triangleq v_0^{12}(x) &= v_0^2 - v_0^1, \quad x \in \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} f_1(t,x) &\triangleq f_{11}(t,x) + f_{12}(t,x) \\ &= P_1(D)(u^{12}\partial_x v^2 + u^1\partial_x v^{12}) + P_2(D)\Big(u^{12}(u^1 + u^2) + \frac{1}{2}\partial_x u^{12}\partial_x(u^1 + u^2) \\ &\quad + \partial_x u^{12}\partial_x v^2 + \partial_x u^1 \partial_x v^{12} + \frac{1}{2} v^{12}(v^1 + v^2) - \frac{1}{2}\partial_x v^{12}\partial_x(v^1 + v^2) \Big), \\ f_2(t,x) &\triangleq f_{21}(t,x) + f_{22}(t,x) \\ &= P_1(D)(v^{12}\partial_x u^2 + v^1\partial_x u^{12}) + P_2(D)\Big(v^{12}(v^1 + v^2) + \frac{1}{2}\partial_x v^{12}\partial_x(v^1 + v^2) \\ &\quad + \partial_x u^{12}\partial_x v^2 + \partial_x u^1 \partial_x v^{12} + \frac{1}{2} u^{12}(u^1 + u^2) - \frac{1}{2}\partial_x u^{12}\partial_x(u^1 + u^2) \Big). \end{aligned}$$

Note that $P_1(D) \in Op(S^{-2})$ and $P_2(D) \in Op(S^{-1})$. Using Proposition 3.3 (vi) and (vii) with $p = 2$, we obtain

$$\begin{aligned}\|f_{11}(t,x)\|_{B^{\frac{1}{2}}_{2,\infty}} &\leq C\|u^{12}\partial_x v^2 + u^1\partial_x v^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} \\ &\leq C\left(\|u^{12}\|_{B^{\frac{1}{2}}_{2,1}}\|v^2\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} + \|v^{12}\|_{B^{\frac{1}{2}}_{2,1}}\|u^1\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip}\right),\end{aligned} \tag{11}$$

and

$$\|f_{12}(t,x)\|_{B^{\frac{1}{2}}_{2,\infty}} \leq C\left(\|u^{12}\|_{B^{\frac{1}{2}}_{2,1}} + \|v^{12}\|_{B^{\frac{1}{2}}_{2,1}}\right)V(t). \tag{12}$$

Since,

$$\begin{aligned}\|(u^{12}+v^{12})\partial_x u^2\|_{B^{\frac{1}{2}}_{2,\infty}} &\leq \|u^{12}+v^{12}\|_{L^\infty}\|\partial_x u^2\|_{B^{\frac{1}{2}}_{2,\infty}} \\ &\leq C\left(\|u^{12}\|_{B^{\frac{1}{2}}_{2,1}} + \|v^{12}\|_{B^{\frac{1}{2}}_{2,1}}\right)\|u^2\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip}.\end{aligned} \tag{13}$$

Thus, by (11)–(13) and Lemma 3.4 (i), we derive that

$$\|u^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} \leq e^{C\int_0^t U(\tau)d\tau}\left(\|u_0^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} + C\int_0^t e^{-C\int_0^\tau U(\tau')d\tau'}\left(\|u^{12}\|_{B^{\frac{1}{2}}_{2,1}} + \|v^{12}\|_{B^{\frac{1}{2}}_{2,1}}\right)V(\tau)d\tau\right).$$

Similarly,

$$\|v^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} \leq e^{C\int_0^t U(\tau)d\tau}\left(\|v_0^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} + C\int_0^t e^{-C\int_0^\tau U(\tau')d\tau'}\left(\|u^{12}\|_{B^{\frac{1}{2}}_{2,1}} + \|v^{12}\|_{B^{\frac{1}{2}}_{2,1}}\right)V(\tau)d\tau\right).$$

Thus,

$$\begin{aligned}&e^{-C\int_0^t U(\tau)d\tau}\left(\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}\right) \\ &\leq \|u_0^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v_0^{12}\|_{B^{\frac{1}{2}}_{2,\infty}} + C\int_0^t e^{-C\int_0^\tau U(\tau')d\tau'}\left(\|u^{12}(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|v^{12}(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\right)V(\tau)d\tau.\end{aligned} \tag{14}$$

On the other hand, applying Proposition 3.3 (viii) with $p = 2$, we get

$$\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,1}} \leq C\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \ln\left(e + \frac{\|u^{12}\|_{B^{\frac{3}{2}}_{2,\infty}} \frac{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}}{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}\right),$$

and

$$\|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,1}} \leq C\|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \ln\left(e + \frac{\|v^{12}(t)\|_{B^{\frac{3}{2}}_{2,\infty}} \frac{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}{\|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}}{\|u^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|v^{12}(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}\right).$$

Substituting the above two inequalities into (14), we find that

$$\begin{aligned}
&e^{-C\int_0^t U(\tau)d\tau}\left(\|u^{12}(t)\|_{B^{\frac12}_{2,\infty}}+\|v^{12}(t)\|_{B^{\frac12}_{2,\infty}}\right) \qquad (15)\\
&\le \|u_0^{12}\|_{B^{\frac12}_{2,\infty}}+\|v_0^{12}\|_{B^{\frac12}_{2,\infty}}+C\int_0^t e^{-C\int_0^\tau U(\tau')d\tau'}\left(\|u^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}+\|v^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}\right)\\
&\quad\times\ln\left(e+\frac{M\left(\|u^{12}(\tau)\|_{B^{\frac32}_{2,\infty}}+\|v^{12}(\tau)\|_{B^{\frac32}_{2,\infty}}\right)}{\|u^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}+\|v^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}}\right)V(\tau)d\tau\\
&\le \|u_0^{12}\|_{B^{\frac12}_{2,\infty}}+\|v_0^{12}\|_{B^{\frac12}_{2,\infty}}+C\int_0^t e^{-C\int_0^\tau U(\tau')d\tau'}\left(\|u^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}+\|v^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}\right)\\
&\quad\times\ln\left(e+\frac{MV(\tau)}{e^{-C\int_0^\tau U(\tau')d\tau'}\left(\|u^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}+\|v^{12}(\tau)\|_{B^{\frac12}_{2,\infty}}\right)}\right)V(\tau)d\tau.
\end{aligned}$$

Denote $W(t)\triangleq e^{-C\int_0^t U(\tau)d\tau}\big(\|u^{12}(t)\|_{B^{\frac12}_{2,\infty}}+\|v^{12}(t)\|_{B^{\frac12}_{2,\infty}}\big)$. In view of (9) and the fact that $\ln(e+\frac{\alpha}{x})\le\ln(e+\alpha)(1-\ln x)$ for all $x\in(0,1]$, $\alpha>0$, we can rewrite (15) as

$$W(t)\le W(0)+C\int_0^t V(\tau)\ln\left(e+MV(\tau)\right)\left(1-\ln W(\tau)\right)W(\tau)d\tau. \qquad (16)$$

Applying Lemma 3.7 with $\mu(r)=r(1-\ln r)$ to (16) under the hypothesis (9), we obtain

$$\frac{W(t)}{e}\le\left(\frac{W(0)}{e}\right)^{\exp(-C\int_0^t V(\tau)\ln(e+MV(\tau))d\tau)}\le\left(\frac{W(0)}{e}\right)^{\exp(-C\int_0^t \Phi(V(\tau))d\tau)},$$

which proves (10). In particular, if $\|u_0^{12}\|_{B^{\frac12}_{2,\infty}}+\|v_0^{12}\|_{B^{\frac12}_{2,\infty}}\le e^{1-\exp(C\int_0^T\Phi(V(\tau))d\tau)}$, then (10) is true on $[0,T]$, since the above inequality implies that (9) holds with $T^*=T$.

2.3. Continuity with respect to initial data. In this subsection, we will prove the continuity with respect to initial data. We divide the proof into three steps.

First step: Continuity in $C([0,T];(B^{\frac12}_{2,1})^2)$. For a fixed $(u_0,v_0)\in(B^{\frac32}_{2,1})^2$ and a $\delta>0$. We claim that there exist $T,A>0$ such that for any $(u_0',v_0')\in(B^{\frac32}_{2,1})^2$ with $\|(u_0'-u_0,v_0'-v_0)\|_{(B^{\frac32}_{2,1})^2}\le\delta$, the solution $(u',v')=\Psi(u_0',v_0')$ of the system (3) with (u_0', v_0') belongs to $C([0,T];(B^{\frac{3}{2}}_{2,1})^2) \cap C^1([0,T];(B^{\frac{1}{2}}_{2,1})^2)$ and satisfies $\|(u',v')\|_{L^\infty(0,T;(B^{\frac{3}{2}}_{2,1})^2)} \le A$.

Indeed, if we take

$$T = \frac{1}{4C\left(\|(u_0,v_0)\|_{(B^{\frac{3}{2}}_{2,1})^2} + \delta\right)} \quad \text{and} \quad A \triangleq 2\left(\|(u_0,v_0)\|_{(B^{\frac{3}{2}}_{2,1})^2} + \delta\right),$$

then we can deduce the claim from (8) in Subsection 2.1. Thus, combining the a priori estimate in Subsection 2.2 with the above claim, we have

$$\frac{\|\Psi(u_0',v_0') - \Psi(u_0,v_0)\|_{L^\infty(0,T;(B^{\frac{1}{2}}_{2,\infty})^2)}}{e} \le e^{CAT}\left(\frac{\|(u_0'-u_0, v_0'-v_0)\|_{(B^{\frac{1}{2}}_{2,\infty})^2}}{e}\right)^{\exp(-CAT\ln(e+MA))},$$

provided that $\|(u_0'-u_0,v_0'-v_0)\|_{(B^{\frac{1}{2}}_{2,\infty})^2} \le e^{1-\exp(CAT\ln(e+MA))}$. Interpolating with the uniform bounds in $C([0,T];(B^{\frac{3}{2}}_{2,1})^2)$, we gather that the map $\Psi : (B^{\frac{3}{2}}_{2,1})^2 \mapsto C([0,T];(B^{\frac{1}{2}}_{2,1})^2)$ is continuous.

Second step: Continuity in $C([0,T];(B^{\frac{3}{2}}_{2,1})^2)$. Let $(u^n,v^n)_{n\in\bar{\mathbb{N}}}$ be the solution of the system (3) corresponding to datum (u_0^n, v_0^n). Assume that $(u_0^\infty, v_0^\infty) \in (B^{\frac{3}{2}}_{2,1})^2$ and $(u_0^n,v_0^n) \to (u_0^\infty, v_0^\infty)$ in $(B^{\frac{3}{2}}_{2,1})^2$, as $n\to\infty$. To pursue our goal, according to the first step, it suffices to prove that $(w^n,z^n) \triangleq (\partial_x u^n, \partial_x v^n) \to (w^\infty, z^\infty) \triangleq (\partial_x u^\infty, \partial_x v^\infty)$ in $C([0,T];(B^{\frac{1}{2}}_{2,1})^2)$ as $n\to\infty$.

Indeed, $(w^n,z^n)_{n\in\bar{\mathbb{N}}}$ solves the following transport equations:

$$\begin{cases} \partial_t w^n - (u^n+v^n)\partial_x w^n = F^n(t,x), & t>0,\ x\in\mathbb{R}, \\ \partial_t z^n - (u^n+v^n)\partial_x z^n = G^n(t,x), & t>0,\ x\in\mathbb{R}, \\ w^n\big|_{t=0} \triangleq \partial_x u_0^n(x), & x\in\mathbb{R}, \\ z^n\big|_{t=0} \triangleq \partial_x v_0^n(x), & x\in\mathbb{R}, \end{cases}$$

where

$$\begin{aligned} F^n(t,x) &\triangleq \frac{1}{2}(\partial_x u^n)^2 - (u^n)^2 - \frac{1}{2}(v^n)^2 + \frac{1}{2}(\partial_x v^n)^2 + P_2(D)(u^n\partial_x v^n) \\ &\quad + P_1(D)\big((u^n)^2 + \frac{1}{2}(\partial_x u^n)^2 + \partial_x u^n \partial_x v^n + \frac{1}{2}(v^n)^2 - \frac{1}{2}(\partial_x v^n)^2\big), \\ G^n(t,x) &\triangleq \frac{1}{2}(\partial_x v^n)^2 - (v^n)^2 - \frac{1}{2}(u^n)^2 + \frac{1}{2}(\partial_x u^n)^2 + P_2(D)(\partial_x u^n v^n) \\ &\quad + P_1(D)\big((v^n)^2 + \frac{1}{2}(\partial_x v^n)^2 + \partial_x u^n \partial_x v^n + \frac{1}{2}(u^n)^2 - \frac{1}{2}(\partial_x u^n)^2\big). \end{aligned}$$

Following [19], we decompose $w^n = w_1^n + w_2^n$ and $z^n = z_1^n + z_2^n$ for $n \in \mathbb{N}$ with

$$\begin{cases} \partial_t w_1^n - (u^n + v^n)\partial_x w_1^n &= (F^n - F^\infty)(t,x), \quad t>0,\ x\in\mathbb{R},\\ \partial_t z_1^n - (u^n + v^n)\partial_x z_1^n &= (G^n - G^\infty)(t,x), \quad t>0,\ x\in\mathbb{R},\\ w_1^n\big|_{t=0} &\triangleq \big(\partial_x u_0^n - \partial_x u_0^\infty\big)(x), \quad x\in\mathbb{R},\\ z_1^n\big|_{t=0} &\triangleq \big(\partial_x v_0^n - \partial_x v_0^\infty\big)(x), \quad x\in\mathbb{R}, \end{cases} \tag{17}$$

and

$$\begin{cases} \partial_t w_2^n - (u^n + v^n)\partial_x w_2^n &= F^\infty(t,\ x), \quad t>0,\ x\in\mathbb{R},\\ \partial_t z_2^n - (u^n + v^n)\partial_x z_2^n &= G^\infty(t,x), \quad t>0,\ x\in\mathbb{R},\\ w_2^n\big|_{t=0} &\triangleq \partial_x u_0^\infty(x), \quad x\in\mathbb{R},\\ z_2^n\big|_{t=0} &\triangleq \partial_x v_0^\infty(x), \quad x\in\mathbb{R}, \end{cases} \tag{18}$$

According to the claim of the first step, using the similar way exhibited for (5), we have $(F^n)_{n\in\bar{\mathbb{N}}}$ and $(G^n)_{n\in\bar{\mathbb{N}}}$ are uniformly bounded in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Moreover, one can easily get

$$\begin{aligned} &\|F^n - F^\infty\|_{B^{\frac{1}{2}}_{2,1}}, \|G^n - G^\infty\|_{B^{\frac{1}{2}}_{2,1}}\\ &\le C\big(\|u^n\|_{B^{\frac{3}{2}}_{2,1}} + \|u^\infty\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n\|_{B^{\frac{3}{2}}_{2,1}} + \|v^\infty\|_{B^{\frac{3}{2}}_{2,1}}\big) \times \big(\|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}}\\ &\quad + \|\partial_x u^n - \partial_x u^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|\partial_x v^n - \partial_x v^\infty\|_{B^{\frac{1}{2}}_{2,1}}\big). \end{aligned}$$

Thus, applying Lemma 3.4 (i) to the equations of (17), we have

$$\begin{aligned} &\|w_1^n(t)\|_{B^{\frac{1}{2}}_{2,1}} + \|z_1^n(t)\|_{B^{\frac{1}{2}}_{2,1}} \\ &\le \exp\left\{C\int_0^t \big(\|u^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}} + \|v^n(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\big)d\tau\right\} \Big(\|\partial_x u_0^n - \partial_x u_0^\infty\|_{B^{\frac{1}{2}}_{2,1}}\\ &\quad + \|\partial_x v_0^n - \partial_x v_0^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \big(\|(F^n - F^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|(G^n - G^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\Big)\\ &\le CAe^{CAT}\Big(\|\partial_x u_0^n - \partial_x u_0^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|\partial_x v_0^n - \partial_x v_0^\infty\|_{B^{\frac{1}{2}}_{2,1}}\\ &\quad + \int_0^t \big(\|(u^n - u^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|(v^n - v^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\\ &\quad + \int_0^t \big(\|(\partial_x u^n - \partial_x u^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|(\partial_x v^n - \partial_x v^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\big)d\tau\Big). \end{aligned} \tag{19}$$

On the other hand, since $(u^n, v^n)_{n\in\bar{\mathbb{N}}}$ is uniformly bounded in $C([0,T]; (B^{\frac{3}{2}}_{2,1})^2)$ and $(u^n, v^n) \to (u^\infty, v^\infty)$ in $(B^{\frac{1}{2}}_{2,1})^2$, as $n\to\infty$, we then derive (w_2^n, z_2^n) tends to $(w^\infty, z^\infty) = (\partial_x u^\infty, \partial_x v^\infty)$ by applying Lemma 3.6 to the equations of (18). Thus, combining the above result of convergence and the results of the first step with (19), for sufficiently small $\varepsilon > 0$ and large enough $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\|\partial_x u^n - \partial_x u^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|\partial_x v^n - \partial_x v^\infty\|_{B^{\frac{1}{2}}_{2,1}} \\
&\leq \|w_1^n\|_{B^{\frac{1}{2}}_{2,1}} + \|w_2^n - w^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|z_1^n\|_{B^{\frac{1}{2}}_{2,1}} + \|z_2^n - z^\infty\|_{B^{\frac{1}{2}}_{2,1}} \\
&\leq \varepsilon + CAe^{CAT}\Big(\varepsilon T + \|\partial_x u_0^n - \partial_x u_0^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|\partial_x v_0^n - \partial_x v_0^\infty\|_{B^{\frac{1}{2}}_{2,1}} \\
&\quad + \int_0^t \big(\|(\partial_x u^n - \partial_x u^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}} + \|(\partial_x v^n - \partial_x v^\infty)(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\big) d\tau\Big).
\end{aligned}$$

Therefore, applying Gronwall's lemma to the above inequality, we then get the desired result.

Third step: Continuity in $C^1([0,T];(B^{\frac{1}{2}}_{2,1})^2)$. Combining the claim of the first step with the system (3), according to the second step, we reach this step.

2.4. Counterexamples. In this subsection, inspired by [7], we will show that local well-posedness of the system (3) in $(B^{\frac{3}{2}}_{2,\infty})^2$ fails.

Example 2.1. For $v \equiv 0$, the system (3) becomes the Camassa-Holm equation. Therefore, as Proposition 4 in [7], there exists a global solution $u \in L^\infty(\mathbb{R}^+; B^{\frac{3}{2}}_{2,\infty})$ and $v = 0$ to the system (3) such that for any positive T and ε, there exists a solution $u' \in L^\infty(0,T; B^{\frac{3}{2}}_{2,\infty})$ and $v = 0$ with

$$\|u(0) - u'(0)\|_{B^{\frac{3}{2}}_{2,\infty}} \leq \varepsilon \quad \text{and} \quad \|u - u'\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty})} \geq 1.$$

Example 2.2. For $u \equiv v$, the system (3) reduces to the scalar Camassa-Holm equation:

$$u_t - u_{txx} = 6uu_x - 4u_x u_{xx} - 2uu_{xxx}. \tag{20}$$

One can easily check that the equation (20) has the same form of peakon $u_c(t,x) \triangleq ce^{-|x-ct|}$ as the Camassa-Holm equation. Thus, we can follow the method in [7] to show that the system (3) is not locally well-posed in $(B^{\frac{3}{2}}_{2,\infty})^2$. Then there exists a global solution $(u,u) \in L^\infty(\mathbb{R}^+; (B^{\frac{3}{2}}_{2,\infty})^2)$ to the system (3) such that for any $T > 0$ and $\varepsilon > 0$, there exists a solution $(u', u') \in L^\infty(0,T;(B^{\frac{3}{2}}_{2,\infty})^2)$ with

$$\|u(0) - u'(0)\|_{B^{\frac{3}{2}}_{2,\infty}} \leq \varepsilon \quad \text{and} \quad \|u - u'\|_{L^\infty(0,T;B^{\frac{3}{2}}_{2,\infty})} \geq 1.$$

3. Appendix

In this section, we will enclose some basic theory of the Littlewood-Paley decomposition and the transport equation theory on Besov spaces for completeness.

Proposition 3.1 ([1, 6, 7] (Littlewood-Paley decomposition)). *Let* $\mathcal{B} \triangleq \{\xi \in \mathbb{R}, |\xi| \leq \frac{4}{3}\}$ *and* $\mathcal{C} \triangleq \{\xi \in \mathbb{R}, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. *Then there exist* $\psi(\xi) \in C_c^\infty(\mathcal{B})$ *and* $\varphi(\xi) \in C_c^\infty(\mathcal{C})$ *such that*

$$\psi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall\ \xi \in \mathbb{R}.$$

and

$$\begin{aligned} Supp\varphi(2^{-q}\cdot) \cap Supp\varphi(2^{-q'}\cdot) = \emptyset, \quad & if\ |q - q'| \geq 2, \\ Supp\psi(\cdot) \cap Supp\varphi(2^{-q}\cdot) = \emptyset, \quad & if\ q \geq 1. \end{aligned}$$

Then for all $u \in \mathcal{S}'$ *(*$\mathcal{S}'$ *denotes the tempered distribution spaces), we can define the nonhomogeneous Littlewood-Paley decomposition of a distribution* u.

$$u = \sum_{q\in\mathbb{Z}} \Delta_q u,$$

where the localization operators are defined as follows:

$$\Delta_q u \triangleq 0, \quad for\ q \leq -2, \quad \Delta_{-1} u \triangleq \psi(D)u = \mathcal{F}^{-1}(\psi \mathcal{F} u),$$

and

$$\Delta_q u \triangleq \varphi(2^{-q}D)u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u), \quad for\ q \geq 0.$$

Furthermore, we can define the low frequency cut-off operator S_q *as follows:*

$$S_q u \triangleq \sum_{i=-1}^{q-1} \Delta_i u = \psi(2^{-q}D)u = \mathcal{F}^{-1}(\psi(2^{-q}\xi)\mathcal{F}u).$$

Definition 3.2 ([1, 6, 7] (Besov spaces)). Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r} \triangleq \{u \in \mathcal{S}'(\mathbb{R});\ \|u\|_{B^s_{p,r}} < \infty\},$$

where

$$\|u\|_{B^s_{p,r}} \triangleq \begin{cases} (\sum\limits_{q\in\mathbb{Z}} 2^{qsr}\|\Delta_q u\|^r_{L^p})^{\frac{1}{r}}, & r < \infty, \\ \sup\limits_{q\in\mathbb{Z}} 2^{qs}\|\Delta_q u\|_{L^p}, & r = \infty. \end{cases}$$

If $s=\infty$, $B^{\infty}_{p,r} \triangleq \bigcap_{s\in\mathbb{R}} B^s_{p,r}$. In particular, as $p=2$, the norm $\|u\|_{B^s_{2,r}}$ is equivalent to the following norm

$$|u|_{B^s_{2,r}} \triangleq \left(\left(\int_{-1}^{1}(1+\xi^2)^s|\hat{u}(\xi)|^2d\xi\right)^{\frac{r}{2}} + \sum_{q\in\mathbb{N}}\left(\int_{2^q\le|\xi|\le 2^{q+1}}(1+\xi^2)^s|\hat{u}(\xi)|^2d\xi\right)^{\frac{r}{2}}\right)^{\frac{1}{r}}$$

with an obvious modification if $r=+\infty$.

Next, we list the following useful properties for Besov spaces.

Proposition 3.3 ([1,6,7]). *Let $s\in\mathbb{R}$, $1\le p,r,p_i,r_i\le\infty, i=1,2$. Then*

(i) *Density: if $1\le p,r<\infty$, then $\mathcal{C}^\infty_c$ is dense in $B^s_{p,r}$.*

(ii) *Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$, for $p_1\le p_2$ and $r_1\le r_2$,*

$$B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1} \text{ locally compact}, \quad \text{for } s_1<s_2.$$

(iii) *Algebraic properties: if $s>0$, $B^s_{p,r}\cap L^\infty$ is an algebra. Furthermore, $B^s_{p,r}$ is an algebra, provided that $s>\frac{1}{p}$ or $s\ge\frac{1}{p}$ and $r=1$.*

(iv) *Fatou lemma: if $\{u^{(n)}\}_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and tends to u in $\mathcal{S}'$, then $u\in B^s_{p,r}$. Moreover,*

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty}\|u^{(n)}\|_{B^s_{p,r}}.$$

(v) *Complex interpolation: if $u\in B^{s_1}_{p,r}\cap B^{s_2}_{p,r}$, then for all $\theta\in[0,1]$, we have $u\in B^{\theta s_1+(1-\theta)s_2}_{p,r}$. Moreover,*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|u\|^{\theta}_{B^{s_1}_{p,r}}\|u\|^{(1-\theta)}_{B^{s_2}_{p,r}}.$$

(vi) *Action of Fourier multipliers on Besov spaces: let $m\in\mathbb{R}$ and f be a S^m-multiplier (i.e., $f:\mathbb{R}\to\mathbb{R}$ is a smooth function and satisfies that for each multi-index α, there exists a constant C_α such that $|\partial^\alpha f(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|}$, for all $\xi\in\mathbb{R}$.) Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

(vii) *The paraproduct is continuous from $B^{-\frac{1}{p}}_{p,1}\times(B^{\frac{1}{p}}_{p,\infty}\cap L^\infty)$ to $B^{-\frac{1}{p}}_{p,\infty}$, i.e.,*

$$\|uv\|_{B^{-\frac{1}{p}}_{p,\infty}} \le C\|u\|_{B^{-\frac{1}{p}}_{p,1}}\|v\|_{B^{\frac{1}{p}}_{p,\infty}\cap L^\infty}.$$

(viii) *A logarithmic interpolation inequality:*

$$\|u\|_{B^{\frac{1}{p}}_{p,1}} \le C\|u\|_{B^{\frac{1}{p}}_{p,\infty}}\ln\left(e+\frac{\|u\|_{B^{1+\frac{1}{p}}_{p,\infty}}}{\|u\|_{B^{\frac{1}{p}}_{p,\infty}}}\right).$$

Now we state the following transport equation theory that is crucial to our purpose.

Lemma 3.4 ([1, 6] (A priori estimate)). *Let* $1 \le p, r \le +\infty$ *and* $s > -\min\{\frac{1}{p}, 1-\frac{1}{p}\}$. *Assume that* v *be a function such that* $\partial_x v$ *belongs to* $L^1([0,T]; B^{s-1}_{p,r})$ *if* $s > 1+\frac{1}{p}$ *or to* $L^1([0,T]; B^{\frac{1}{p}}_{p,r} \cap L^\infty)$ *otherwise. Suppose also that* $f_0 \in B^s_{p,r}$, $F \in L^1([0,T]; B^s_{p,r})$, *and that* $f \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ *be the solution of the one-dimensional transport equation*

$$\begin{cases} \partial_t f + v \cdot \partial_x f &= F, \\ f\big|_{t=0} &= f_0. \end{cases} \tag{21}$$

Then there exists a constant C *depending only on* s, p, r *such that the following statements hold for* $t \in [0,T]$

(i) *If* $r = 1$ *or* $s \neq 1+\frac{1}{p}$,

$$\|f(t)\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}} d\tau,$$

or

$$\|f(t)\|_{B^s_{p,r}} \le e^{CV(t)}(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} d\tau),$$

where

$$V(t) = \begin{cases} \int_0^t \|\partial_x v(\tau,\cdot)\|_{(B^{\frac{1}{p}}_{p,r}\cap L^\infty)} d\tau, & s < 1+\frac{1}{p}, \\ \int_0^t \|\partial_x v(\tau,\cdot)\|_{B^{s-1}_{p,r}} d\tau, & s > 1+\frac{1}{p}. \end{cases}$$

(ii) *If* $s \le 1+\frac{1}{p}$, *and* $\partial_x f_0, \partial_x f \in L^\infty([0,T]\times\mathbb{R})$ *and* $\partial_x F \in L^1([0,T]; L^\infty)$, *then*

$$\begin{aligned}&\|f(t)\|_{B^s_{p,r}} + \|\partial_x f(t)\|_{L^\infty} \\ &\le e^{CV(t)}(\|f_0\|_{B^s_{p,r}} + \|\partial_x f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)}(\|F(\tau)\|_{B^s_{p,r}} + \|\partial_x F(\tau)\|_{L^\infty}) d\tau),\end{aligned}$$

where $V(t) = \int_0^t \|\partial_x v(\tau,\cdot)\|_{(B^{\frac{1}{p}}_{p,r}\cap L^\infty)} d\tau$.

(iii) *If* $f = v$, *then for all* $s > 0$, *the estimate in* (i) *holds with* $V(t) = \int_0^t \|\partial_x v(\tau,\cdot)\|_{L^\infty} d\tau$.

(iv) *If* $r < \infty$, *then* $f \in C([0,T]; B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0,T]; B^{s'}_{p,1})$ *for all* $s' < s$.

Lemma 3.5 ([6] (Existence and uniqueness))**.** *Let p, r, s, f_0 and F be as in the statement of Lemma 3.4. Suppose that $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ for some $\rho > 1, M > 0$ and $\partial_x v \in L^1([0,T]; B^{\frac{1}{p}}_{p,\infty} \cap L^\infty)$ if $s < 1 + \frac{1}{p}$, and $\partial_x v \in L^1([0,T]; B^{s-1}_{p,r})$ if $s > 1 + \frac{1}{p}$ or $s = 1 + \frac{1}{p}$ and $r = 1$. Then the transport equation (21) has a unique solution $f \in L^\infty([0,T]; B^s_{p,r}) \bigcap (\bigcap_{s'<s} C([0,T]; B^{s'}_{p,1}))$ and the corresponding inequalities in Lemma 3.4 hold true. Moreover, if $r < \infty$, then $f \in C([0,T]; B^s_{p,r})$.*

Lemma 3.6 ([7])**.** *Assume that $(v^n)_{n\in\bar{\mathbb{N}}}$ be a sequence of functions belonging to $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Assume that v^n is the solution to*

$$\begin{cases} \partial_t v^n + a^n \partial_x v^n &= f, \\ \qquad v^n\big|_{t=0} &= v_0, \end{cases}$$

with $v_0 \in B^{\frac{1}{2}}_{2,1}, f \in L^1(0,T; B^{\frac{1}{2}}_{2,1})$ and that, for some $\alpha \in L^1(0,T)$,

$$\sup_{n\in\mathbb{N}} \|\partial_x a^n(t)\|_{B^{\frac{1}{2}}_{2,1}} \le \alpha(t).$$

If in addition $a^n \to a^\infty$ in $L^1(0,T; B^{\frac{1}{2}}_{2,1})$, then $v^n \to v^\infty$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$.

Lemma 3.7 ([1] (Osgood lemma))**.** *Let ρ be a measurable function from $[t_0, T]$ to $[0, a]$, γ a locally integrable function from $[t_0, T]$ to $\mathbb{R}^+$, and μ a continuous and nondecreasing function from $(0, a]$ to $\mathbb{R}^+$. Assume that, for some positive real number c, the function ρ satisfies*

$$\rho(t) \le c + \int_{t_0}^t \gamma(t')\mu(\rho(t'))dt' \quad \text{for a.e. } t \in [t_0, T].$$

Then we have, for a.e. $t \in [t_0, T]$,

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \le \int_{t_0}^t \gamma(t')dt' \quad \text{for} \quad \mathcal{M}(x) = \int_x^a \frac{dr}{\mu(r)}.$$

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