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Moduli of Smoothness Related to Fractional Riesz-Derivatives

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Abstract. New moduli of smoothness $\omega_{\langle\beta\rangle}(f,\delta)_p$, $0<\beta<1$, related to the Riesz derivative of order β are introduced. Their properties are studied in L_p-spaces of 2π-periodic functions for $0<p\le+\infty$. The modulus $\omega_{\langle\beta\rangle}(f,\delta)_p$ is shown to be equivalent to the polynomial K-functional associated with the corresponding Riesz derivative. As a consequence direct Jackson-type and inverse Bernstein-type estimates are proved, the quality of approximation by methods generated by Riesz kernels is described in terms of the these moduli.
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1. Introduction

For a 2π-periodic function $f(x)$ in the space L_p, $0<p\le+\infty$, equipped with the standard (quasi-) norm $\|\cdot\|_p$ and for $0<\beta<1$ we introduce a modulus of smoothness by

$$\omega_{\langle\beta\rangle}(f,\delta)_p = \sup_{0\le h\le\delta}\left\| \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{+\infty} \frac{f(x+\nu h)-f(x)}{|\,\nu\,|^{\beta+1}}\right\|_p, \quad \delta\ge 0. \tag{1}$$

Since

$$\begin{aligned}\left\| \sum_{\nu\neq 0} \frac{f(x+\nu h)-f(x)}{|\,\nu\,|^{\beta+1}}\right\|_p^{\widetilde{p}} &\le \sum_{\nu\neq 0}\frac{\|\,f(x+\nu h)-f(x)\,\|_p^{\widetilde{p}}}{|\,\nu\,|^{\widetilde{p}(\beta+1)}}\\ &\le 2\left(\sum_{\nu\neq 0}|\,\nu\,|^{-\widetilde{p}(\beta+1)}\right)\|\,f\,\|_p^{\widetilde{p}},\end{aligned}$$

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where $\widetilde{p} = \min(1, p)$, modulus (1) is well-defined for $p > \frac{1}{\beta+1}$ in the sense that

$$\omega_{\langle\beta\rangle}(f, \delta)_p \leq c \| f \|_p < +\infty$$

for $f \in L_p$ and $\delta \geq 0$. Here the positive constant c is independent of f and δ.

In this paper we prove that for $p > \frac{1}{\beta+1}$ modulus (1) is equivalent (up to constants independent of f and δ) to the polynomial K-functional given by

$$K^{(\mathcal{P})}_{\langle\beta\rangle}(f, \delta)_p = \inf_{T \in \mathcal{T}_{\frac{1}{\delta}}} \left\{ \| f - T \|_p + \delta^\beta \| T^{\langle\beta\rangle} \|_p \right\}, \quad f \in L_p, \ \delta > 0, \qquad (2)$$

where $(\cdot)^{\langle\beta\rangle}$ is the Riesz derivative of fractional order β, that is, the linear operator defined on the space of all real-valued trigonometric polynomials $\mathcal{T}$ by

$$(\cdot)^{\langle\beta\rangle} : e^{i\nu x} \longrightarrow |\nu|^\beta e^{i\nu x}, \quad \nu \in \mathbb{Z},$$

and where ($\overline{c}$ is the complex conjugate of c)

$$\mathcal{T}_\sigma = \left\{ T(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx} : c_{-k} = \overline{c_k}, \ |k| \leq \sigma \right\}, \quad \sigma \geq 0,$$

is the space of all real-valued trigonometric polynomials of order σ. Combining this result with the properties of functional (2) established in [8] and [10] (see Section 2 for exact formulations) we immediately deduce fundamental properties of modulus (1), in particular, a quasi-homogeneity property, Jackson- and Bernstein-type estimates. Moreover, the equivalence of (1) and (2) enables us to describe the error of approximation for methods generated by Riesz kernels (see (16) and (17)) by means of smoothness properties of functions expressed in terms of modulus (1). In particular, it applies to approximation in L_p-spaces with values $p < 1$. This follows from Theorem 2.3, (19), proved in [10]. The main results in this respect are established in Theorem 4.1.

Our approach to study modulus (1) is based on a general result concerning the equivalence of the modulus

$$\omega_\theta(f, \delta)_p = \sup_{0 \leq h \leq \delta} \left\| \sum_{\nu=-\infty}^{+\infty} \theta^\wedge(\nu) f(x + \nu h) \right\|_p, \quad \delta \geq 0, \qquad (3)$$

generated by an arbitrary 2π-periodic function θ with $\theta^\wedge(\nu)$, $\nu \in \mathbb{Z}$, as its Fourier coefficients and the generalized polynomial K-functional

$$K^{(\mathcal{P})}_\psi(f, \delta)_p = \inf_{T \in \mathcal{T}_{\frac{1}{\delta}}} \left\{ \| f - T \|_p + \delta^s \| \mathcal{D}(\psi) T \|_p \right\}, \quad f \in L_p, \ \delta > 0, \qquad (4)$$

generated by an appropriate homogeneous function ψ of order $s > 0$ which has been proved in [12] (see Section 2, Theorem 2.1, for exact formulations). In (4) the operator $\mathcal{D}(\psi)$ (*ψ-derivative*) is a linear operator given on $\mathcal{T}$ by

$$\mathcal{D}(\psi) : e^{i\nu x} \longrightarrow \psi(\nu) e^{i\nu x}, \quad \nu \in \mathbb{Z}.$$

Obviously, the functional (2) corresponds to (4) with $\psi(\xi) = |\xi|^\beta$ and modulus (1) is a special case of (3) with θ replaced by

$$\theta_{\langle\beta\rangle}(\xi) = \sum_{\nu\neq 0} |\,\nu\,|^{-\beta-1}\,(\,e^{i\nu\xi} - 1), \tag{5}$$

In order to apply the general theory we have to show that the generators $\psi(\xi) = |\xi|^\beta$ and $\theta_{\langle\beta\rangle}(\xi)$ satisfy the assumptions of Theorem 2.1. This is done in Theorem 3.6 and can be considered as the main contribution in this paper. In contrast to many other constructions as, for instance, to the classical modulus of smoothness of order $k \in \mathbb{N}$ generated by $\theta(\xi) = -(1 - e^{i\xi})^k$, the generator $\theta_{\langle\beta\rangle}$ of modulus (1) is given by its Fourier series (5) and can not be represented explicitely. For this reason the verification of conditions with respect to θ and ψ providing the above mentioned equivalence is a technically complicated problem. Here it will be solved applying Fourier-analytical methods to homogeneous tempered distributions.

On the other hand as in the classical case the Fourier coefficients which appear in the general construction (3) are explicitely known. Thus, the generalized difference

$$\Delta_h^{\langle\beta\rangle} f(x) := \sum_{\nu\neq 0} \frac{f(x+\nu h) - f(x)}{|\,\nu\,|^{\beta+1}}$$

occuring in (1) can be calculated at least approximately.

Let us also have a brief look at the case $\beta \geq 1$. The equivalence of modulus (1) and functional (2) for $\beta = 1$ was shown in [1, 11] for $1 \leq p \leq +\infty$ and in [12] for $p > \frac{1}{2}$ using some specific adapted methods. The following simple observation shows that modulus (1) is not appropriate for arbitrary β. It was proved in [12] that the equivalence of modulus (3) and functional (4) holds if θ and ψ are close to each other near the point 0 in a certain sense (see also Section 2 for details). However, if, for example, $\beta > 2$, then the second derivative of $|\cdot|^\beta$ is equal to 0 at the point 0. On the other hand the second derivative of $\theta_{\langle\beta\rangle}(\cdot)$ does not have this property. For this reason the conditions of the general equivalence theorem are not satisfied. It will be the aim of a forthcoming paper to show that for $\beta > 1$ the first difference $\Delta_{\nu h} f(x) = f(x+\nu h) - f(x)$ in modulus (1) should be replaced by differences of higher order. Let us mention that modulus (1) together with its possible extensions to $\beta > 1$ can be considered as discrete counterparts of the moduli ($r > \beta$)

$$\widetilde{\omega}_\beta(f,\delta)_p = \left\| \int_{|\,u\,|\geq 1} \frac{\Delta_{u\delta}^{2r} f(x)}{|\,u\,|^{\beta+1}}\, du \right\|_p, \quad f \in L_p,\ \delta > 0. \tag{6}$$

Here $\Delta_{u\delta}^{2r}$ denotes the usual difference of order $2r$ with step $u\delta$. The modulus (6) has been introduced and studied in [4]. Clearly, this modulus is well-defined for $1 \le p \le +\infty$ only. For these parameters it is equivalent to the polynomial K-functional given by (2). In this sense our construction (1) seems to be more preferable because it is relevant also for $p < 1$.

The paper is organized as follows. In Section 2 we collect some general results which are needed to study modulus (1). Section 3 deals with the behavior of the generator $\theta_{\langle\beta\rangle}$ near the point 0. More precisely, it will be shown that $\theta_{\langle\beta\rangle}(\cdot)$ is close to the function $|\cdot|^\beta$ in a certain sense. This is the crucial part of the paper. The main properties of modulus (1) and its applications to trigonometric approximation are discussed in Section 4.

2. Auxiliary results

As already mentioned in the Introduction the proofs of our results in this paper are mainly based on three ingredients: an equivalence theorem for general moduli of smoothness (3) and polynomial K-functionals (4), the properties of polynomial K-functionals related to fractional Riesz derivatives, and the theorem on the equivalence of the approximation error of the families of linear polynomial operators generated by Riesz kernels and the polynomial K-functionals (2). These results can be found in [8, 10, 12]. For the convenience of the reader we give here the corresponding concepts and formulations.

2.1. Equivalence of moduli and polynomial K-functionals. Let v and w be continuous functions defined on $\mathbb{R}$. Let $0 < q \le +\infty$ and let η be an infinitely differentiable function with compact support (test function). In the following we shall write $v(\cdot) \overset{(q,\eta)}{\prec} w(\cdot)$, if $\mathcal{F}\left(\frac{\eta v}{w}\right)$ belongs to $L_q(\mathbb{R})$. Here $\mathcal{F}$ stands for the Fourier transform. The notation $v(\cdot) \overset{(q,\eta)}{\asymp} w(\cdot)$ indicates equivalence. It means that $v(\cdot) \overset{(q,\eta)}{\prec} w(\cdot)$ and $w(\cdot) \overset{(q,\eta)}{\prec} v(\cdot)$ simultaneously. Following [12] a couple of test functions $\Lambda_* = (\eta, \zeta)$ is called *a plane periodic resolution of unity* if there exist $0 < \rho < 1$ such that the supports of η and ζ are contained in $[-2\rho, 2\rho]$ and $[\rho, 2\pi - \rho]$, respectively, $\eta(\xi) = 1$ for $|\,\xi\,| \le \rho$, $\zeta(\xi) = 1$ for $2\rho \le \xi \le 2(\pi - \rho)$ and $\eta_*(\xi) + \zeta_*(\xi) = 1$ on $\mathbb{R}$, where g_* denotes the 2π-periodization of g given by

$$g_*(\xi) \;=\; \sum_{j=-\infty}^{+\infty} g(\xi + 2\pi j) \tag{7}$$

Let $s > 0$. By $\mathcal{H}_s$ we denote the class of functions ψ satisfying the following properties:

1) ψ is a complex-valued function defined on $\mathbb{R}$ and it holds $\psi(-\xi) = \overline{\psi(\xi)}$ for $\xi \in \mathbb{R}$.
2) ψ is continuous on $\mathbb{R}$.
3) ψ is infinitely differentiable on $\mathbb{R} \setminus \{0\}$.
4) ψ is homogeneous of order s, i.e. it holds $\psi(t\xi) = t^s \psi(\xi)$ for $t > 0$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.
5) $\psi(\xi) \neq 0$ for $\xi \in \mathbb{R}^d \setminus \{0\}$.

We put

$$\mathcal{H} = \bigcup_{s>0} \mathcal{H}_s. \tag{8}$$

Recall that $K_\psi^{(\mathcal{P})}(f,\delta)_p,\ \delta > 0,\ 0 < p \leq +\infty$, has the meaning of (4).

By $\mathcal{G}$ we denote the class of functions θ satisfying the properties:

1) θ is a complex-valued function defined on $\mathbb{R}$ and it holds $\theta(-\xi) = \overline{\theta(\xi)}$ for $\xi \in \mathbb{R}$.
2) θ is continuous on $\mathbb{R}$ and 2π-periodic.
3) $\{\theta^\wedge(\nu)\}_{\nu\in\mathbb{Z}} \in l_1$.
4) $\theta(0) = 0$.
5) $\theta^\wedge(0) = -1$.

Here

$$\theta^\wedge(\nu) = \frac{1}{2\pi}\int_0^{2\pi} \theta(\xi) e^{-i\nu\xi}\, d\xi, \quad \nu \in \mathbb{Z},$$

stands for the ν-th Fourier coefficent. An important characteristic of the function $\theta \in G$ is the set

$$P_\theta = \big\{p \in (0,+\infty] : \{\,\theta^\wedge(k)\}_{k\in\mathbb{Z}} \in l_p\big\} = \big\{p \in (0,+\infty] : \sigma_p(\theta) < +\infty\big\}, \tag{9}$$

where

$$\sigma_p(\theta) = \big\|\,\{\,\theta^\wedge(k)\,\}_{k\in\mathbb{Z}}\,\big\|_{l_p}. \tag{10}$$

Clearly, we have

$$P_\beta \equiv P_{\theta_{\langle\beta\rangle}} = \left(\frac{1}{\beta+1}, +\infty\right] \tag{11}$$

because of (5).

For shortness we shall write $A \lesssim B$ if the relation $A \leq cB$ holds, where c is a positive constant independent of f (function) and n or δ (variables approximation methods, K-functionals and moduli may depend on). The symbol $\asymp$ indicates equivalence. It means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

Recall that $\omega_\theta(f,\delta)_p$ has the meaning of (3).

Theorem 2.1 ([12])**.** *Let $\theta \in \mathcal{G}$, $p \in P_\theta$ and let $\psi \in \mathcal{H}$. If there exists a plane periodic resolution of unity $\Lambda_* = (\eta, \zeta)$ such that*

$$\psi(\cdot) \overset{(\widetilde{p},\eta)}{\asymp} \theta(\cdot) \quad \textit{and} \quad 1 \overset{(\widetilde{p},\zeta)}{\prec} \theta(\cdot),$$

where $\widetilde{p} = \min(1, p)$, then

$$\omega_\theta(f, \delta)_p \asymp K_\psi^{(\mathcal{P})}(f, \delta)_p, \quad f \in L_p,\ \delta \geq 0. \tag{12}$$

2.2. Polynomial K-functional related to Riesz derivatives. Applying the general results obtained in [8] for generalized polynomial K-functional (4) to the case $\psi(\cdot) = |\cdot|^\beta$ we immediately obtain the following properties of functionals (2). As usual,

$$E_n(f)_p = \inf_{T \in \mathcal{T}_n} \| f - T \|_p,$$

denotes the best approximation of f in L_p by trigonometric polynomials of order $n \in \mathbb{N}_0$.

Theorem 2.2 ([8])**.** *Let $\beta > 0$, $0 < p \leq +\infty$, and let $\widetilde{p} = \min(1, p)$.*

(i) (Jackson type estimate) *There exists a constant $c_1 > 0$ such that*

$$E_n(f)_p \leq c_1 K_{\langle\beta\rangle}^{(\mathcal{P})}(f, (n+1)^{-1})_p \tag{13}$$

for all $f \in L_p$ and $n \in \mathbb{N}_0$.

(ii) (Bernstein type estimate) *There exists a constant $c_2 > 0$ such that*

$$K_{\langle\beta\rangle}^{(\mathcal{P})}(f, \delta)_p \leq c_2 \min(\delta^\beta, 1) \left(\sum_{0 \leq \nu < \frac{1}{\delta}} (\nu+1)^{\beta\widetilde{p}-1} E_\nu(f)_p^{\widetilde{p}} \right)^{\frac{1}{\widetilde{p}}}. \tag{14}$$

for all $f \in L_p$ and $\delta > 0$.

(iii) *There exists a constant $c_2 > 0$ such that*

$$K_{\langle\beta\rangle}^{(\mathcal{P})}(f, t\delta)_p \leq c_3 (t+1)^{\beta + \frac{1}{\widetilde{p}} - 1} K_{\langle\beta\rangle}^{(\mathcal{P})}(f, \delta)_p, \tag{15}$$

for all $f \in L_p$ and δ, $t > 0$.

2.3. Approximation methods generated by Riesz kernels. Next we recall (see e.g. [10]) the Riesz family of linear polynomial operators in L_p-spaces of 2π-periodic functions. Let $0 < p \leq +\infty$, $0 \leq \alpha < \infty$, $0 < \beta < \infty$, and let $n \in \mathbb{N}_0$. We put

$$\mathcal{R}_{n;\lambda}^{(\alpha,\beta)}(f; x) = (2n+1)^{-d} \cdot \sum_{k=0}^{2n} f(t_n^k + \lambda) \cdot R_n^{(\alpha,\beta)}(x - t_n^k - \lambda), \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}, \tag{16}$$

where

$$R_0^{(\alpha,\beta)}(h) = 1, \quad R_n^{(\alpha,\beta)}(h) = \sum_{|k|\le n} \left(1 - \frac{|k|^\beta}{n^\beta}\right)^\alpha \cdot e^{ikh}, \quad n \in \mathbb{N},$$

are the Riesz kernels, and $t_n^k = \frac{2\pi k}{2n+1}$, $k \in \mathbb{Z}$. The corresponding Riesz means which are given by

$$\mathcal{R}_n^{(\alpha,\beta)}(f;x) = (2\pi)^{-1} \int_0^{2\pi} f(x+h) \cdot R_n^{(\alpha,\beta)}(h)\, dh, \quad n \in \mathbb{N}_0, \tag{17}$$

are classical objects of both harmonic analysis and approximation theory. They have been intensively studied by many mathematicians (see, e.g. [2, 5, 13–15]). In particular, the means (17) converge in L_p for all $1 \le p \le +\infty$ independently on β, provided that $\alpha > 0$ ([5]). For further results and more details we refer to [16].

Families of linear polynomial operaters as defined in (16) can be used as universal constructive approximation method for all $0 < p \le +\infty$, in particular, they turn out to be relevant if $0 < p < 1$. For more information we refer to [6]. To this end we put

$$\| g(\cdot,\cdot) \|_{\overline{p}} = \left(\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |g(x,\lambda)|^p \, dx\, d\lambda\right)^{\frac{1}{p}}$$

for a 2π-periodic function g of two variables $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. The following convergence result has been proved in [10]:
Let $\alpha > 0$ and let $\beta \notin \mathbb{E}$, where $\mathbb{E} = \{2k,\ k \in \mathbb{N}_0\}$. Then it holds

$$\| f - \mathcal{R}_{n;\lambda}^{(\alpha,\beta)}(f) \|_{\overline{p}} \to 0 \quad (n \to \infty)$$

for all $f \in L_p$ if, and only if, $\frac{1}{1+\min(\alpha,\beta)} < p \le +\infty$. Let us also mention that according to the general convergence theory described in [6] approximation by the Riesz family (16) and approximation by Riesz means (17) are equivalent in the sence that

$$\| f - \mathcal{R}_{n;\lambda}^{(\alpha,\beta)}(f) \|_{\overline{p}} \asymp \| f - \mathcal{R}_n^{(\alpha,\beta)}(f) \|_p, \quad f \in L_p, \in \mathbb{N}_0. \tag{18}$$

if $1 \le p \le +\infty$.

The quality of approximation for methods generated by Riesz kernels has been characterized in terms of polynomial K-functionals in [10] applying the general equivalence results obtained in [7]. Taking into account (18) this result can be stated as follows.

Theorem 2.3 ([10]). *Let $\alpha > 0$, $\beta \notin \mathbb{E}$. If $p \in \left(\frac{1}{\min(\alpha,\beta)+1}, +\infty\right]$ then*

$$\| f - \mathcal{R}_{n;\lambda}^{(\alpha,\beta)}(f) \|_{\overline{p}} \asymp K_{\langle\beta\rangle}^{(\mathcal{P})}(f,(n+1)^{-1})_p, \quad f \in L_p, \in \mathbb{N}_0. \tag{19}$$

If $1 \le p \le +\infty$ then in (19) the family $\{\mathcal{R}_{n;\lambda}^{(\alpha,\beta)}\}$ can be replaced by the Riesz means $\mathcal{R}_n^{(\alpha,\beta)}$.

3. Asymptotic properties of the generating function

In this section we show that the generating function $\theta_{\langle\beta\rangle}$ of modulus (1) given by (5) and the function $\psi(\cdot) = |\cdot|^{\beta}$ associated with the polynomial K-functional (2) satisfy the assumptions of Theorem 2.1 for a certain range of parameters β and p. This is the crucial part of the paper because it allows to apply the general theory presented in the preceding section to describe the interrelation of approximation by Riesz families and means and smoothness properties of functions characterized by moduli of smoothness related to fractional Riesz derivatives. The result is based on a series of lemmas. Henceforth,

$$\widetilde{\Delta}_h g(x) = g\left(x + \frac{h}{2}\right) - g\left(x - \frac{h}{2}\right)$$

stands for the symmetric difference of g. By C^m, $m \in \mathbb{N}$, we denote the space of 2π-periodic m times continuously differentiable functions.

Lemma 3.1. *Let g be an infinitely differentiable function defined on $\mathbb{R} \setminus \{0\}$. Let $n \geq m$ be a natural number and let $0 < h < n^{-1}$. Then there exist real numbers $c > 0$ and γ_j, $j = n, \ldots, m$, $\gamma_m \neq 0$, such that*

$$\left| g^{(2m)}(x) - \sum_{j=m}^{n} \gamma_j \, \widetilde{\Delta}_h^{2j} g(x) \right| \leq c \max_{|t| \leq nh} \left| g^{(2n+1)}(x+t) \right| \tag{20}$$

for all x, $|x| \geq 1$.

Proof. Let $j \in \{m, \ldots, n\}$ and let $|x| \geq 1$. Then it holds

$$\widetilde{\Delta}_h^{2j} g(x) = \int_{\Omega_j} g^{(2j)}(x + \Lambda_j) \, d\lambda_{(j)}, \tag{21}$$

where

$$\Omega_j \equiv \Omega_j(h) = \left[-\frac{h}{2}, \frac{h}{2}\right]^{2j}, \quad \Lambda_j = \lambda_1 + \cdots + \lambda_{2j}, \quad d\lambda_{(j)} = d\lambda_1 \cdots d\lambda_{2j}.$$

Applying Taylor's formula

$$\eta(x+\tau) = \sum_{\nu=0}^{k} \frac{\eta^{(\nu)}(x)}{\nu!} \tau^{\nu} + \frac{1}{k!} \int_0^{\tau} (\tau - t)^k \, \eta^{(k+1)}(x+t) \, dt$$

to $\eta = g^{(2j)}$, $\tau = \Lambda_j$ and $k = 2(n-j)$ and taking into account that $\int_{\Omega_j} \Lambda_j^r \, d\lambda_{(j)} = 0$ for odd numbers r we obtain

$$\begin{aligned} \widetilde{\Delta}_h^{2j} g(x) &= \sum_{\nu=2j}^{2n} \frac{g^{(\nu)}(x)}{(\nu-2j)!} \int_{\Omega_j} \Lambda_j^{\nu-2j} \, d\lambda_{(j)} + G_j(x) \\ &= \sum_{s=j}^{n} \frac{g^{(2s)}(x)}{(2(s-j))!} \int_{\Omega_j} \Lambda_j^{2(s-j)} \, d\lambda_{(j)} + G_j(x), \end{aligned} \tag{22}$$

from (21). Here,

$$G_j(x) \;=\; \frac{1}{(2(n-j))!}\int_{\Omega_j}\int_0^{\Lambda_j} (\Lambda_j - t)^{2(n-j)}\, g^{(2n+1)}(x+t)\,dt\,d\lambda_{(j)}. \tag{23}$$

Clearly,

$$\begin{aligned} |\,G_j(x)\,| &\le \max_{j=m,\ldots,n} \frac{h^{2j}|\,\Lambda_j\,|^{2(n-j)+1}}{(2(n-j)+1)!} \max_{|\,t\,|\le|\,\Lambda_j\,|} |\,g^{(2n+1)}(x+t)\,| \\ &\le\; h^{2m}(nh)^{2(n-m)+1} \max_{|\,t\,|\le nh} |\,g^{(2n+1)}(x+t)\,|, \quad j=m,\ldots,n. \end{aligned} \tag{24}$$

Formula (22) can be rewritten as

$$\sum_{s=j}^{n} \alpha_{js} g^{(2s)}(x) \;=\; \widetilde{\Delta}_h^{2j}\, g(x) \;-\; G_j(x), \quad j=m,\ldots,n, \tag{25}$$

where

$$\alpha_{js} \;=\; \frac{1}{(2(s-j))!}\int_{\Omega_j} \Lambda_j^{2(s-j)}\, d\lambda_{(j)}, \quad s=j,\ldots,n,\; j=m,\ldots,n.$$

We consider (25) as a system of linear equations with respect to $g^{(2s)}$, $s=m,\ldots,n$, as unknown variables. Because of $\alpha_{mm}\cdot\ldots\cdot\alpha_{nn}\neq 0$ this system has a unique solution. In particular, we have

$$g^{(2m)}(x) \;=\; \sum_{j=m}^{n} \gamma_j\,(\,\tilde{\Delta}_h^{2j}\, g(x) \;-\; G_j(x)\,), \tag{26}$$

where $\gamma_m = \alpha_{mm}^{-1} \neq 0$. Combining (24) and (26) we find

$$\Big|g^{(2m)}(x) - \sum_{j=m}^{n} \gamma_j\, \tilde{\Delta}_h^{2j} g(x)\Big| = \Big|\sum_{j=m}^{n} \gamma_j G_j(x)\Big| \le c \max_{|t|\le nh} \big|g^{(2n+1)}(x+t)\big|. \qquad \square$$

In order to prove the next Lemma we need some Fourier analytic tools. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing infinitely differentiable functions and by $\mathcal{S}'(\mathbb{R})$ its topological dual the space of all tempered distributions. $\mathcal{F}$ stands for the Fourier transform in $\mathcal{S}'(\mathbb{R})$. If $f \in L_1(\mathbb{R}) \subset S'(\mathbb{R})$ is a regular distribution then $\mathcal{F}f$ can be represented pointwise as

$$\mathcal{F}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{-ix\xi}\,d\xi, \quad \xi\in\mathbb{R}.$$

For details and properties let us refer, for example, to the comprehensive book [3].

Lemma 3.2. *Let $m, j \in \mathbb{N}$, $j \geq m$, $0 < \beta < 1$, $\xi \in \mathbb{R}$ and let $0 < h < j^{-1}$. Then the function*

$$\eta(t) \;\equiv\; \eta_{j,\xi}(t) \;\equiv\; \eta_{j,\xi,m,h,\beta}(t) \;=\; \widetilde{\Delta}_h^{2j}\big(|\,\cdot\,|^{2m-\beta-1}\big)(t)(\,e^{i\xi t}-1\,) \tag{27}$$

belongs to $L_1(\mathbb{R})$ and it holds

$$\mathcal{F}\eta(x) \;=\; a\Big(\Big(\sin\frac{h(x-\xi)}{2}\Big)^{2j}|\,x-\xi\,|^{\beta-2m} - \Big(\sin\frac{hx}{2}\Big)^{2j}|\,x\,|^{\beta-2m}\Big), \tag{28}$$

for its Fourier transform, where the real number $a \equiv a(j,m,\beta)$ is independent of x, h and ξ.

Proof. The function η is continuous. Moreover, in view of (21) we get

$$\begin{aligned}|\,\eta(t)\,| \;&\leq\; 2h^{2j}\,\max\nolimits_{|\,\tau\,|\leq jh}\big|\,\big(\,|\,t+\tau|^{2m-\beta-1}\big)^{(2j)}\,\big| \\ &\leq\; 2\prod_{k=1}^{2j}|\,2m-\beta-k\,|\,(\,|\,t\,|-jh\,)^{-(2(j-m)+\beta+1)}\end{aligned}$$

for $|\,t\,| \geq 1$. This implies,

$$|\,\eta(t)\,| \;=\; O\big(|\,t|^{-(\beta+1)}\big), \quad |\,t\,| \to +\infty.$$

Thus, η belongs to $L_1(\mathbb{R})$ and its Fourier transform can be defined pointwise. However, for the purpose of its calculation we consider the Fourier transform in the framework of tempered distributions $\mathcal{S}'(\mathbb{R})$. Clearly,

$$\begin{aligned}\big\langle\,\mathcal{F}\big(\widetilde{\Delta}_h^{2j}\big(|\,\cdot\,|^{2m-\beta-1}\big)\big),\varphi\,\big\rangle \;&=\; (-4)^j\Big\langle\,\Big(\,\sin\frac{h\cdot}{2}\,\Big)^{2j}\mathcal{F}\big(|\,\cdot\,|^{2m-\beta-1}\big),\varphi\,\Big\rangle \\ &=\; (-4)^j\Big\langle\,\mathcal{F}\big(|\,\cdot\,|^{2m-\beta-1}\big),\Big(\sin\frac{h\cdot}{2}\Big)^{2j}\varphi(\cdot)\,\Big\rangle\end{aligned} \tag{29}$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R})$. It follows from [3, Subsection 2.4.3, Formula (2.4.3) (p. 128)], that

$$\begin{aligned}\big\langle\,\mathcal{F}\big(|\,\cdot\,|^{2m-\beta-1}\big),\psi\,\big\rangle \;=\; &\frac{\pi^{\frac{\beta-2m+1}{2}}}{\Gamma\Big(\frac{\beta-2m+1}{2}\Big)}\Big(\,\int_{|\,x|\,\geq 1}|\,x|^{\beta-2m}\psi(x)\,dx \\ &+\sum_{k=0}^{2(m-1)}\frac{1}{(2k)!}\,\frac{2}{2(k-m)+\beta+1}\,\langle\,\delta^{(2k)},\psi\,\rangle \\ &+\int_{|\,x|\,<1}\Big(\psi(x)-\sum_{k=0}^{2(m-1)}\frac{\psi^{(2k)}(0)}{(2k)!}\,x^{2k}\Big)|x|^{\beta-2m}dx\Big)\end{aligned} \tag{30}$$

for $\psi \in \mathcal{S}(\mathbb{R})$. Applying (30) to the function $\psi(x) = \sin^{2j}\left(\frac{hx}{2}\right)\,\varphi(x)$, $\varphi \in \mathcal{S}(\mathbb{R})$, and taking into account that all of its derivatives up to the order $2(m-1)$ are equal to 0 we get

$$\left\langle \mathcal{F}\left(\widetilde{\Delta}_h^{2j}\left(|\cdot|^{2m-\beta-1}\right)\right), \varphi \right\rangle = (-4)^j \frac{\pi^{\frac{\beta-2m+1}{2}}}{\Gamma\left(\frac{\beta-2m+1}{2}\right)} \int_{-\infty}^{+\infty} \left(\sin\frac{hx}{2}\right)^{2j} |x|^{\beta-2m} \varphi(x)\,dx,$$

from (29). Hereby we used the identity

$$\mathcal{F}\left(\widetilde{\Delta}_h^{2j}\left(|\cdot|^{2m-\beta-1}\right)\right)(x) = \left(\sin\frac{hx}{2}\right)^{2j} |x|^{\beta-2m}. \tag{31}$$

Now (28) immediately follows from (31). □

Lemma 3.3. *Let $n \geq m$ be natural numbers, $0 < h < n^{-1}$ and let $0 < \beta < 1$. Then the function*

$$\mathcal{X}(\xi) = \sum_{\nu \neq 0} \mathcal{X}^\wedge(\nu)\left(e^{i\nu\xi} - 1\right), \tag{32}$$

where

$$\mathcal{X}^\wedge(\nu) = \left(|\cdot|^{2m-\beta-1}\right)^{(2m)}(\nu) - \sum_{j=m}^{n} \gamma_j\, \tilde{\Delta}_h^{2j}\left(|\cdot|^{2m-\beta-1}\right)(\nu), \quad \nu \neq 0, \tag{33}$$

and where the numbers γ_j, $j = m, \ldots, n$, have the same meaning as in Lemma 3.1, belongs to the space $C^{2(n-m)+1}$.

Proof. Applying Lemma 3.1 to the function $g(x) = |x|^{2m-\beta-1}$ we obtain

$$|\mathcal{X}^\wedge(\nu)| \leq c \max_{|t|\leq nh} \left| \left(|\nu+t|^{2m-\beta-1}\right)^{(2n+1)} \right| \leq c_1(|\nu| - nh)^{-(2(n-m+1)+\beta)}$$

for $\nu \neq 0$. In particular, we have

$$|\mathcal{X}^\wedge(\nu)| = O\left(|\nu|^{-(2(n-m+1)+\beta)}\right), \quad |\nu| \to +\infty.$$

This implies

$$\left| \sum_{\nu\neq 0} \mathcal{X}^\wedge(\nu)\,(i\nu)^{2(n-m)+1} e^{i\nu\xi} \right| \leq c \sum_{\nu \neq 0} \frac{|\nu|^{2(n-m)+1}}{|\nu|^{2(n-m+1)+\beta}} = c \sum_{\nu\neq 0} \frac{1}{|\nu|^{\beta+1}} < \infty$$

and yields absolute and uniform convergence of the series on the left-hand side. Hence, the function represented by the series on the right-hand side of (32) is $2(n-m)+1$-times differentiable and the derivative $\mathcal{X}^{(2(n-m)+1)}$ is a continuous and periodic function on $\mathbb{R}$. □

Lemma 3.4. *Let* $m, j \in \mathbb{N}$, $j \geq m$, $0 < \beta < 1$ *and let* $h \in \mathbb{R}$. *Then the function* $\Omega_j \equiv \Omega_{j,m,h,\beta}$ *given by*

$$\Omega_j(\xi) = \sum_{k \neq 0} \Big(\sin^{2j} \frac{h(\xi - 2\pi k)}{2} \,|\, \xi - 2\pi k|^{\beta - 2m} - \sin^{2j} \frac{2\pi k h}{2} \,|\, 2\pi k|^{\beta - 2m} \Big) \quad (34)$$

is infinitely differentiable on $(-2\pi, 2\pi)$.

Proof. We observe that

$$\begin{aligned} \big|\, \omega_k^{(s)}(\xi) \big| \; &:= \; \Big\| \sum_{\nu=0}^{s} \Big(\sin^{2j} \frac{h(\xi - 2\pi k)}{2} \Big)^{(s-\nu)} \Big(|\, \xi - 2\pi k|^{\beta - 2m} \Big)^{(\nu)} \Big\| \\ &\leq \; c \sum_{\nu=0}^{s} |\, \xi - 2\pi k|^{\beta - 2m - \nu}, \end{aligned} \quad (35)$$

for $s \in \mathbb{N}$ and $\xi \in (-2\pi, 2\pi)$, where the positive constant c is independent of ξ. Let $0 < \rho < 1$. In view of (35) we obtain

$$\sum_{k \neq 0} \big|\, \omega_k^{(s)}(\xi) \big| \;\leq\; c \sum_{\nu=0}^{s} \sum_{k=0}^{+\infty} (k + \rho)^{\beta - 2m - \nu} \;\leq\; c_1 \Big(\rho^{2m - \beta - s} + \sum_{k=1}^{+\infty} k^{2m - \beta} \Big) \;<\; +\infty$$

for each $\xi \in (-2\pi(1-\rho), 2\pi(1-\rho))$. Hence, the function $\Omega_j(\xi)$ represented by the series on the right-hand side of (34) is s-times differentable on the interval $(-2\pi(1-\rho), 2\pi(1-\rho))$ for each $0 < \rho < 1$ and $s \in \mathbb{N}$. Thus, the function Ω_j is infinitely differentiable on $(-2\pi, 2\pi)$. □

Lemma 3.5. *Let* ψ *be an even function belonging to* C^3 *on* $(-\rho, \rho)$, $\rho > 0$, *satisfying* $\psi(0) = 0$. *Let also* $0 < \beta < 1$.

(i) *It holds*

$$\psi(\cdot) \overset{(p,\eta_1)}{\prec} |\cdot|^{\beta}$$

for all test functions η_1 *with* $supp\, \eta_1 \subset (-\rho, \rho)$.

(ii) *There exists a number* ρ', $0 < \rho' < \rho$ *such that*

$$1 \overset{(p,\eta_2)}{\prec} 1 + |\cdot|^{-\beta}\, \psi(\cdot)$$

for all test functions η_2 *with* $supp\, \eta_2 \subset (-\rho', \rho')$.

Proof. Step 1. We prove (i). Clearly,

$$\psi(\xi) = \frac{\psi''(0)}{2}\, \xi^2 + \zeta(\xi), \quad \xi \in (-\rho, \rho);\; \zeta^{(j)}(0) = 0,\; j = 0, 1, 2, 3. \quad (36)$$

In view of (36) we get for $\xi \to 0$

$$|\,\xi\,|^{-\beta}\zeta(\xi) \;=\; o(|\,\xi\,|^{3-\beta}); \tag{37}$$

$$\begin{aligned}\big(\,\xi\,|^{-\beta}\zeta(\xi)\big)' &= |\,\xi\,|^{-\beta}\zeta'(\xi) \;-\; \beta\,|\,\xi\,|^{-(\beta+1)}\zeta(\xi)\,\mathrm{sgn}\xi \\ &= |\,\xi\,|^{-\beta}o(\xi^2) \;+\; |\,\xi\,|^{-(\beta+1)}o(\xi^3) \\ &= o(|\,\xi\,|^{2-\beta});\end{aligned} \tag{38}$$

$$\begin{aligned}\big(|\,\xi\,|^{-\beta}\zeta(\xi)\big)'' &= |\,\xi\,|^{-\beta}\zeta''(\xi) - 2\beta\,|\,\xi\,|^{-(\beta+1)}\zeta'(|\,\xi\,|) + \beta(\beta+1)|\,\xi\,|^{-(\beta+2)}\zeta(\xi) \\ &= |\,\xi\,|^{-\beta}o(\xi) + |\,\xi\,|^{-(\beta+1)}o(\xi^2) + |\,\xi\,|^{-(\beta+2)}o(\xi^3) \\ &= o(|\,\xi\,|^{1-\beta}).\end{aligned} \tag{39}$$

By (37)–(39) there exist $\lim_{\xi\to 0}\big(\,|\,\xi\,|^{-\beta}\zeta(\xi)\,\big)^{(j)} = 0$, $j = 0,1,2$, and, in particular, the function $|\cdot|^{-\beta}\zeta(\cdot)$ belongs to C^2. Hence, by elementary properties of the Fourier transform

$$|\,\mathcal{F}(\,|\cdot|^{-\beta}\zeta(\cdot)\,\eta_1(\cdot)\,)(x)\,| \le \frac{c}{1+x^2}, \quad x \in \mathbb{R},$$

for any test function η_1, whose support is contained in $(-\rho,\rho)$. Therefore, $\mathcal{F}(\,|\cdot|^{-\beta}\zeta(\cdot)\,\eta_1(\cdot)\,)$ belongs to $L_p(\mathbb{R})$ for $p > \frac{1}{2}$. The function $|\cdot|^{2-\beta}$ is homogeneous of degree $2-\beta$. Therefore, $\mathcal{F}(\,|\cdot|^{2-\beta}\,\eta(\cdot)\,)$ belongs to $L_p(\mathbb{R})$ if, and only if, $p > \frac{1}{3-\beta}$ (see [9, Theorem 4.1]). Taking into account that $\frac{1}{3-\beta} < \frac{1}{2}$ for $0 < \beta < 1$ we obtain the desired relation $\psi(\cdot) \overset{(p,\eta_1)}{\prec} |\cdot|^{\beta}$ because of (36).

Step 2. We prove (ii). Due to (37) the function $\big(1 + |\cdot|^{-\beta}\psi(\cdot)\,\big)^{-1}$ is well-defined on $(-\rho',\rho')$ for some $0 < \rho' \le \rho$. Moreover,

$$\big(1 + |\,\xi\,|^{-\beta}\psi(\xi)\,\big)^{-1} \;=\; 1 - |\,\xi\,|^{-\beta}\psi(\xi) \;+\; \mathcal{X}(\xi) \tag{40}$$

for $0 < |\,\xi\,| < \rho'$, where

$$\mathcal{X}(\xi) \;=\; (g(\xi))^2(\,1 + g(\xi)\,)^{-1}, \quad g(\xi) \;=\; |\,\xi\,|^{-\beta}\psi(\xi). \tag{41}$$

We observe that

$$\mathcal{X}' \;=\; (2+g)(1+g)^{-1}gg',$$

$$\mathcal{X}'' \;=\; (1+g)^{-2}\big(2(g')^2(1+g) + gg''(2+g)\big) - 2(1+g)^{-3}(2+g)g(g')^2$$

and $g^{(j)}(\xi) = O(|\,\xi\,|^{2-j-\beta})$, $\xi \to 0$, $j = 0,1,2$. Taking into account also (36)–(39) we obtain

$$\mathcal{X}^{(j)}(\xi) \;=\; O(|\,\xi\,|^{4-j-2\beta}), \quad \xi \to 0,\; j = 0,1,2. \tag{42}$$

By (42) there exist $\lim_{\xi\to 0}\mathcal{X}^{(j)}(\xi) = 0$, $j = 0,1,2$. In particular, the function $\mathcal{X}$ belongs to C^2 on $(-\rho',\rho')$. Hence, the Fourier transform of $\mathcal{X}\eta_2$ belongs to $L_p(\mathbb{R})$ for $p > \frac{1}{2}$ for any test function η_2, whose support is contained in $(-\rho',\rho')$. Moreover, as a consequence of part (i) the Fourier transform $\mathcal{F}((1 - |\cdot|^{-\beta}\psi(\cdot))\,\eta_2(\cdot))$ belongs to $L_p(\mathbb{R})$ for $p > \frac{1}{2}$. Now, it follows $1 \overset{(p,\eta_2)}{\prec} 1 + |\cdot|^{-\beta}\,\psi(\cdot)$ from (40). □

Theorem 3.6. *Let $0 < \beta < 1$. If $p > \frac{1}{2}$ then there exists a plane plane periodic resolution of unity $\Lambda_* = (\eta, \zeta)$ such that*

(i) $\theta_{\langle\beta\rangle}(\cdot) \overset{(p,\eta)}{\asymp} |\cdot|^{\beta}$;

(ii) $\theta_{\langle\beta\rangle}(\cdot) \overset{(p,\zeta)}{\asymp} 1$.

Proof. Let $n \ge m$ be natural numbers. We use the identity

$$|x|^{-\beta-1} = b\left(|x|^{2m-\beta-1}\right)^{(2m)}, \quad x \neq 0,$$

where $b \equiv b(\beta, m) = \prod_{j=1}^{2m}(2m-\beta-j)^{-1} \neq 0$, to see that

$$\begin{aligned} \theta(\xi) &= \sum_{\nu\neq 0} |\nu|^{-\beta-1}(e^{i\nu\xi}-1) \\ &= b\sum_{\nu\neq 0}\left(|\cdot|^{2m-\beta-1}\right)^{(2m)}(\nu)(e^{i\nu\xi}-1) \\ &= b\Big(\sum_{j=m}^{n}\gamma_j\sum_{\nu\neq 0}\widetilde{\Delta}_h^{2j}\left(|\cdot|^{2m-\beta-1}\right)(\nu)\,(e^{i\nu\xi}-1) + \mathcal{X}(\xi)\Big) \\ &= b\Big(\sum_{j=m}^{n}\gamma_j\sum_{\nu=-\infty}^{+\infty}\eta_{j,\xi}(\nu) + \mathcal{X}(\xi)\Big). \end{aligned} \tag{43}$$

Here $0 < h < n^{-1}$, the numbers γ_j, $j = m, \dots, n$, have the meaning of Lemma 3.1 and the functions $\eta_{j,\xi}$ and $\mathcal{X}$ are given by (27) and (32)–(33), respectively. Applying Poisson's summation formula to the right-hand side of (43) and afterwards Lemma 3.2 (see (28)) we get

$$\begin{aligned} \theta(\xi) &= b\Big(\sum_{j=m}^{n}\gamma_j\sum_{k=-\infty}^{+\infty}\mathcal{F}\eta_{j,\xi}(2\pi k) + \mathcal{X}(\xi)\Big) \\ &= ab\,|\xi|^{\beta-2m}\sum_{j=m}^{n}\gamma_j\Big(\sin\frac{h\xi}{2}\Big)^{2j} + ab\sum_{j=m}^{n}\gamma_j\,\Omega_j(\xi) + b\,\mathcal{X}(\xi) \\ &= ab\gamma_m|\xi|^{\beta}\left(\frac{\sin\frac{h\xi}{2}}{\xi}\right)^{2m}\Big(1 + \gamma_m^{-1}\sum_{\nu=1}^{n-m}\gamma_{m+\nu}\Big(\sin\frac{h\xi}{2}\Big)^{2\nu}\Big) \\ &\quad + ab\sum_{j=m}^{n}\gamma_j\,\Omega_j(\xi) + b\,\mathcal{X}(\xi) \\ &\equiv ab\,\gamma_m|\xi|^{\beta}P(\xi) + Q(\xi) \end{aligned} \tag{44}$$

where the functions Ω_j, $j = m, \dots, n$, are given by (34).

First let us prove (i). Since the function P is analytic and $P(0) = 1 \neq 0$ there exists $0 < \rho < 1$ such that $ab\,\gamma_m|\cdot|^{\beta}P(\cdot) \overset{(q,\eta)}{\asymp} |\cdot|^{\beta}$ for any test function η

with support in $(-\rho,\rho)$ and $q>0$. By Lemma 3.4 the functions Ω_j, $j=m,\dots,n$, are infinitely differentiable on $(-\rho,\rho)$, and in view of Lemma 3.3 the function $\mathcal{X}$ belongs to C^3 on this interval if n is chosen such that $n\ge m+1$. The function Q is even and $Q(0)=0$ in view of (32) and (34). Part (i) of Lemma 3.5 yields $Q(\cdot)\overset{(p,\eta)}{\prec}|\cdot|^\beta$. Thus, $\theta(\cdot)\overset{(p,\eta)}{\prec}|\cdot|^\beta$. In order to prove the converse relation we notice that by (44)

$$|\,\xi\,|^\beta\,(\theta(\xi))^{-1} \;=\; (\,ab\,\gamma_m P(\xi)\,)^{-1}\big(\,1+|\,\xi\,|^{-\beta}Q(\xi)(\,ab\,\gamma_m P(\xi)\,)^{-1}\,\big)^{-1}$$

for $0<|\,\xi\,|<\rho$. Applying part (ii) of Lemma 3.5 to the function $\psi(\cdot)=Q(\cdot)(\,ab\,\gamma_m P(\cdot)\,)^{-1}$ we immediately obtain that $|\cdot|^\beta\overset{(p,\eta)}{\prec}\theta(\cdot)$.

Now we prove part (ii). First we notice that the function θ is infinitely differentiable on $(0,2\pi)$. Indeed, for given $s\in\mathbb{N}$ we choose $n,m\in\mathbb{N}$ from the condition $2(n-m)+1\ge s$. Then the function $\mathcal{X}$ belongs to C^s by Lemma 3.3. Because of Lemma 3.4 the functions Ω_j, $j=m,\dots,n$, are infinitely differentiable on $(0,2\pi)$. Furthermore, the function $|\cdot|^\beta P(\cdot)$ is analytic on $(0,2\pi)$. By means of (44) we conclude that θ belongs to C^s on $(0,2\pi)$ for any $s>0$. Since

$$\theta_{\langle\beta\rangle}(\xi) \;=\; \sum_{\nu\neq 0}|\,\nu\,|^{-\beta-1}(\,e^{i\nu\xi}-1\,) \;=\; -4\sum_{\nu=1}^{+\infty}|\,\nu\,|^{-\beta-1}\,\sin^2\frac{\nu\xi}{2}\;<0$$

for $\xi\neq 2\pi k$, $k\in\mathbb{Z}$, the function $(\theta(\cdot))^{-1}$ is well defined and infinitely differentiable on $(0,2\pi)$ as well. Hence, $\theta(\cdot)\overset{(q,\zeta)}{\asymp}1$ for any $q>0$ and an appropriate function ζ. □

4. Main results

Now we are ready to state the main properties of our new moduli of smoothness introduced in (1) as well as their interrelations with polynomial K-functionals related to fractional Riesz derivatives (2) and approximation by Riesz-means (17) and Riesz families (16), respectively.

Theorem 4.1. *Let* $0<\beta<1$, $\frac{1}{\beta+1}<p\le+\infty$ *and* $\widetilde{p}=\min(1,p)$.

(i) (Equivalence to polynomial K-functionals) *It holds*

$$\omega_{\langle\beta\rangle}(f,\delta)_p \;\asymp\; K^{(\mathcal{P})}_{\langle\beta\rangle}(f,\delta)_p \tag{45}$$

for all $f\in L_p$ *and* $\delta>0$.

(ii) (Jackson-type estimate) *There exists a positve constant* c_1 *such that*

$$E_n(f)_p \;\le\; c_1\omega_{\langle\beta\rangle}(f,(n+1)^{-1})_p \tag{46}$$

for all $f\in L_p$ *and* $n\in\mathbb{N}_0$.

(iii) (Bernstein-type estimate) *There exists a positve constant* c_2 *such that*

$$\omega_{\langle\beta\rangle}(f,\delta)_p \;\le\; c_2 \min(\delta^\beta,1) \left(\sum_{0\le\nu<\frac{1}{\delta}} (\nu+1)^{\beta\widetilde{p}-1} E_\nu(f)_p^{\widetilde{p}} \right)^{\frac{1}{\widetilde{p}}} \tag{47}$$

for $f \in L_p$ *and* $\delta > 0$.

(iv) *There exists a positve constant* c_3 *such that*

$$\omega_{\langle\beta\rangle}(f,t\delta)_p \;\le\; c_3\,(t+1)^{\beta+\frac{1}{\widetilde{p}}-1}\,\omega_{\langle\beta\rangle}(f,\delta)_p \tag{48}$$

for all $f \in L_p$ *and* $\delta,\ t \ge 0$.

(v) (Quality of approximation by Riesz means) *If, in addition,* $\alpha > 0$ *and* $p \ge 1$*, then it holds*

$$\left\| f - \mathcal{R}_n^{(\alpha,\beta)}(f) \right\|_p \;\asymp\; \omega_{\langle\beta\rangle}(f,(n+1)^{-1})_p \tag{49}$$

for all $f \in L_p$ *and* $n \in \mathbb{N}_0$.

(vi) (Quality of approximation by Riesz families) *If, in addition,* $\alpha > 0$ *and* $p > \frac{1}{\alpha+1}$*, then it holds*

$$\left\| f - \mathcal{R}_{n;\lambda}^{(\alpha,\beta)}(f) \right\|_{\overline{p}} \;\asymp\; \omega_{\langle\beta\rangle}(f,(n+1)^{-1})_p \tag{50}$$

for all $f \in L_p$ *and* $n \in \mathbb{N}_0$.

Proof. Combining Theorem 2.1 and Theorem 3.6 we immediately obtain part (i). Parts (ii)–(iv) follow from (i) and parts (i)–(iii) of Theorem 2.2, respectively. The combination of Theorem 2.3 and part (i) implies parts (v) and (vi). □

Remark 4.2. In the non-periodic case equivalences of type (45) and (49) have been established in [15] for moduli of smoothess of fractional order and $1 < p < \infty$. The corresponding extension to the multivariate case can be found in [17].

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References

[1] Artamonov, S., On some Properties of modulus of continuity related to Riesz derivative (in Russian). *Trudy TNU V. I. Vernadskogo* 24 (2011)(3), 10 – 22.

[2] Butzer, P. L. and Nessel, R., *Fourier Analysis and Approximation. Vol. 1.* New-York: Academic Press 1971.

[3] Grafakos L., *Classical Fourier Analysis.* New York: Springer 2008.

[4] Kuznetsova I. and Trigub, R. M., Two-sides estimates of the approximation of functions by Riesz and Marcinkiewicz means (in Russian). *Doklady Akad. Nauk SSSR.* 251 (1980), 34 – 36.

[5] Peetre, J., *New Thoughts on Besov Spaces.* Duke Univ. Math. Ser. 1. Durham: Duke Univ. 1976.

[6] Rukasov, V., Runovski, K. and Schmeisser, H.-J., On convergence of families of linear polynomial operators. *Funct. Approx. Comment. Math.* 41 (2009)(1), 41 – 54.

[7] Rukasov, V., Runovski, K. and Schmeisser, H.-J., Approximation by families of linear polynomial operators and smoothness properties of functions. *Math. Nachr.* 284 (2011)(11-12), 1523 – 1537.

[8] Runovski, K., *Methods of Trigonometric Approximation* (in Russian). Moscow: Lambert Acad. Publ. 2012.

[9] Runovski, K. and Schmeisser, H.-J., On some extensions of Bernstein inequalities for trigonometric polynomials. *Funct. Approx. Comment. Math.* 29 (2001), 125 – 142.

[10] Runovski, K. and Schmeisser, H.-J., On families of linear polynomial operators generated by Riesz kernels. *Eurasian Math. J.* 1 (2010)(4), 124 – 139.

[11] Runovski, K. and Schmeisser, H.-J., On modulus of continuity related to Riesz derivative. Jenaer Schriften zur Math. und Inf., Math/Inf/01/11 (2011).

[12] Runovski, K. and Schmeisser, H.-J., General moduli of smoothness and approximation by families of linear polynomial operators. In: *New Perspectives on Approximation and Sampling Theory* (Festschrift in Honor of Paul Butzer's 85th Birthday; eds.: G. Schmeisser et al.). Basel: Birkhäuser 2014, pp. 269 – 298.

[13] Schmeisser, H.-J. and Sickel, W., Characterization of periodic function spaces via means of Abel-Poisson and Bessel-potential type. *J. Approx. Theory* 61 (1990), 239 – 262.

[14] Trebels, W., *Multipliers for* (C, α)*-Bounded Fourier Expansions in Banach Spaces and Approximation Theory.* Lect. Notes Math. 329. Berlin: Springer 1973.

[15] Trebels, W., On the approximation behaviour of the Riesz means in $L^p(\mathbb{R}^n)$. In: *Approximation Theory* (Proceedings Bonn 1976; eds.: R. Schaback et al.). Lect. Notes Math. 556. Berlin: Springer 1976, pp. 428 – 438.

[16] Stein, E. M. and Weiss, G., *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton: Univ. Press 1971.

[17] Wilmes, G., On Riesz-type inequalities and K-functionals related to Riesz potentials in $\mathbb{R}^N$. *Numer. Funct. Anal. Optim.* 1 (1979)(1), 57 – 77.

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