

# A Smooth Solution of a Singular Fractional Differential Equation

*Kaido Lätt, Arvet Pedas and Gennadi Vainikko*

**Abstract.** In this article we examine the existence of a unique smooth solution to a singular fractional differential equation. We reformulate the singular equation with the help of cordial Volterra integral operators and then extend a result from cordial Volterra integral operator theory.

**Keywords.** Fractional derivatives, cordial Volterra integral operators, smooth solutions, singular fractional differential equations

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## 1. Introduction

Throughout the article we use the notations  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ ,  $\mathbb{R} = (-\infty, \infty)$ ,  $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ ,  $\lambda = \operatorname{Re}\lambda + i\operatorname{Im}\lambda$  for  $\lambda \in \mathbb{C}$ ,  $D^k = \left(\frac{d}{dt}\right)^k$ ,  $k \in \mathbb{N}$ ,  $D^0 = I$ , where  $I$  is the identity operator and  $i = \sqrt{-1}$  is the imaginary unit.

For the space of  $m$  times continuously differentiable functions  $u$  on  $[0, T]$  we use the abbreviated notation  $C^m = C^m[0, T]$ ,  $m \in \mathbb{N}_0$ ,  $C^0 = C$ ;

$$\|u\|_{C^m} = \max_{0 \leq k \leq m} \max_{0 \leq t \leq T} |u^{(k)}(t)|.$$

For Banach spaces  $X$  and  $Y$ , the notation  $\mathcal{L}(X, Y)$  stands for the space of linear bounded operators from  $X$  to  $Y$ , and  $\mathcal{L}(X) = \mathcal{L}(X, X)$ . By  $\rho_{\mathcal{L}(X)}(V)$  we denote the resolvent set of operator  $V \in \mathcal{L}(X)$ , and by  $\sigma_{\mathcal{L}(X)}(V) = \mathbb{C} \setminus \rho_{\mathcal{L}(X)}(V)$  its spectrum. In the case  $X = C^m$  we use the following abbreviated notations:

$$\sigma_m(V) = \sigma_{\mathcal{L}(C^m)}(V), \quad \rho_m(V) = \rho_{\mathcal{L}(C^m)}(V) \quad \text{for } V \in \mathcal{L}(C^m), \quad m \in \mathbb{N}_0.$$

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K. Lätt, A. Pedas, G. Vainikko: Institute of Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia;  
kaido.latt@ut.ee; arvet.pedas@ut.ee; gennadi.vainikko@ut.ee

The multiplication operator  $M_\alpha$  for  $\alpha \in \mathbb{R}$  is defined by

$$(M_\alpha u)(t) = t^\alpha u(t), \quad 0 < t \leq T, \quad u \in C.$$

In article [24], the singular system of ordinary differential equations

$$tu'(t) = A(t)u(t) + f(t), \quad 0 < t \leq T < \infty, \quad (1)$$

with given matrix function  $A = (a_{p,q})_{p,q=1}^n \in C_{n \times n}^m$ ,  $m \geq 0$ ,  $n \in \mathbb{N}$  and vector function  $f = (f_1, \dots, f_n)^T \in C_n^m$  was considered. It was shown how the unique solvability of this problem in  $C_n^m$  depends on the set of eigenvalues of the matrix  $A(0)$ . The central idea of [24] was the reduction of (1) to a system of cordial Volterra integral equations. Note that singular systems (1) can also be presented in the equivalent form

$$(tu(t))' = B(t)u(t) + f(t), \quad B(t) = A(t) + I, \quad 0 < t \leq T.$$

On the other hand, singular fractional differential equations of the form

$$(M_\alpha D_0^\alpha u)(t) = \sum_{k=1}^l a_k(t)(M_{\alpha_k} D_0^{\alpha_k} u)(t) + f(t), \quad 0 < t \leq T, \quad (2)$$

and

$$(D_0^\alpha M_\alpha u)(t) = \sum_{k=1}^l b_k(t)(D_0^{\alpha_k} M_{\alpha_k} u)(t) + f(t), \quad 0 < t \leq T, \quad (3)$$

are not equivalent and thus need independent treatments. In the present article we concentrate on (3) which in some aspects is simpler than (2).

In (3) the fractional differentiation operators  $D_0^\alpha$  ( $\alpha > 0$ ) and  $D_0^{\alpha_k}$  ( $\alpha_k \geq 0$ ) are defined as the inverses of the Riemann-Liouville integral operator  $J^\nu$  on  $J^\nu C$ , i.e.

$$D_0^\nu := (J^\nu)^{-1}, \quad \nu \geq 0.$$

The Riemann-Liouville fractional integral operator  $J^\nu$  is given by

$$(J^\nu u)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s) ds, \quad u \in C, \quad t > 0, \quad \nu > 0; \quad J^0 = I.$$

Here  $\Gamma$  is the Euler Gamma function. For  $\nu = m \in \mathbb{N}$ , the operator  $D_0^m$  is the restriction of  $D^m$  to the subspace

$$C_0^m := \{u \in C^m \mid u^{(k)}(0) = 0, \quad k = 0, \dots, m-1\}. \quad (4)$$

It is known (see e.g. [5]): if  $\alpha > 0$ ,  $\beta > 0$ , then

$$(J^\alpha J^\beta u)(t) = (J^{\alpha+\beta} u)(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} u(s) ds$$

with  $0 < t \leq T$ ,  $u \in C$ . Consequently, also  $D_0^\alpha D_0^\beta = D_0^{\alpha+\beta}$  for  $\alpha > 0$ ,  $\beta > 0$ . This property is important in theoretical considerations and it does not hold for Riemann-Liouville and Caputo fractional differentiation operators which are more popular in applications. Fortunately, as we soon explain, under natural conditions (see (5) below),  $u \in C^m$  remains to be a solution of (3) if we replace  $D_0^\alpha$ ,  $D_0^{\alpha_k}$  with either Riemann-Liouville or Caputo fractional differentiation operators.

The main goal of the present article is to extend the results of [24] to fractional differential equations (3). We study the unique solvability of (3) for  $\alpha, \alpha_k \in \mathbb{R}$ ,

$$m < \alpha \leq m+1, \quad \alpha > \alpha_k \geq 0, \quad b_k, f \in C^m, \quad k = 1, 2, \dots, l, \quad m \in \mathbb{N}_0. \quad (5)$$

Note that for a unique solution  $u \in C^m$  of (3), no initial or boundary conditions are permitted: imposing them one determines, as a rule, a solution of lesser regularity. Linear fractional differential equations without singularities, but with initial conditions, have been intensively discussed e.g. in [6,14], monographs [5,12,18], see also references therein.

The Riemann-Liouville fractional differentiation operator  $D_{R-L}^\alpha$  of order  $\alpha > 0$ ,  $m < \alpha \leq m+1$ ,  $m \in \mathbb{N}_0$ , is determined by the formula

$$D_{R-L}^\alpha u = D^{m+1} J^{m+1-\alpha} u \text{ provided that } J^{m+1-\alpha} u \in C^{m+1}.$$

The following claim is elementary (see [25]).

**Proposition 1.1.** *For  $m < \alpha \leq m+1$ ,  $m \in \mathbb{N}_0$ , a function  $u \in C^m$  is  $D_{R-L}^\alpha$ -differentiable if and only if  $u$  is  $D_0^\alpha$ -differentiable. Besides  $D_{R-L}^\alpha u = D_0^\alpha u$ .*

For  $u \in C^k$ ,  $k < m$ , the situation is more complicated [25].

Introduce the Taylor projection

$$(\Pi_m u)(t) = \sum_{k=0}^m \frac{u^{(k)}(0)}{k!} t^k, \quad 0 \leq t \leq T, \quad u \in C^m, \quad m \in \mathbb{N}_0.$$

The Caputo fractional differentiation operator  $D_{Cap}^\alpha$ , for  $m < \alpha \leq m+1$ ,  $m \in \mathbb{N}_0$ , is usually defined by

$$D_{Cap}^\alpha u = D^{m+1} J^{m+1-\alpha} (u - \Pi_m u)$$

where  $u \in C^m$  is such that  $J^{m+1-\alpha} (u - \Pi_m u) \in C^{m+1}$ . For  $u \in C^{m+1}$ , this is equivalent to  $D_{Cap}^\alpha u = J^{m+1-\alpha} D^{m+1} u$  (cf. [14]). The following claim is elementary (see [25]).

**Proposition 1.2.** *A function  $u \in C^m$  has the Caputo fractional derivative  $D_{Cap}^\alpha u \in C$ ,  $m < \alpha \leq m + 1$ ,  $m \in \mathbb{N}_0$ , if and only if  $u - \Pi_m u$  has the fractional derivative  $D_0^\alpha(u - \Pi_m u) \in C$ . Besides  $D_{Cap}^\alpha u = D_0^\alpha(u - \Pi_m u)$ .*

For  $u \in C^m$ ,  $\alpha > m$ ,  $m \in \mathbb{N}_0$ , it holds  $M_\alpha u \in C^m$ ,  $\Pi_m M_\alpha u = 0$ , and by Propositions 1.1 and 1.2

$$D_{R-L}^\alpha(M_\alpha u) = D_0^\alpha(M_\alpha u) = D_{Cap}^\alpha(M_\alpha u).$$

Similar relations hold for  $D_{R-L}^{\alpha_k}$ ,  $D_0^{\alpha_k}$  and  $D_{Cap}^{\alpha_k}$ ,  $k = 1, \dots, l$ . Thus for  $u \in C^m$  equation (3) with (5) is equivalent to the equation which we obtain from (3) replacing  $D_0^\alpha$ ,  $D_0^{\alpha_k}$  either by  $D_{R-L}^\alpha$ ,  $D_{R-L}^{\alpha_k}$  or by  $D_{Cap}^\alpha$ ,  $D_{Cap}^{\alpha_k}$ ,  $k = 1, \dots, l$ , consequently our results remain to be true also in the case with Riemann-Liouville or Caputo fractional derivatives.

Recently, the area of fractional derivatives and their applications, has seen a remarkable growth in popularity (for the main results in this field see, for example, [3, 5, 12, 18, 19] and the references cited in these monographs). Additionally, various existence and uniqueness results for fractional differential equations are given in [1, 2, 4, 21] and some recent results regarding the numerical solution of such equations can be found in [8–10, 15–17]. However, as far as we know, no contributions exists concerning the singular fractional differential equations of the form (3) with (5).

The main purpose of the present paper is to derive criteria for the existence of a smooth solution to (3). We exploit the concept of cordial Volterra integral operators [22, 23] recalled in Section 2.1. The results of the present article will play a fundamental role when constructing high order numerical methods for solving equations of the form  $\{(3), (5)\}$ . Having said that, the question regarding numerical methods, is beyond the scope of the present paper.

The article is broken up into four sections. In Section 2 we present some definitions and results required for our work. Section 3 is devoted to the main result of this article, Theorem 3.3. To prove Theorem 3.3 we also formulate an auxiliary result, Lemma 3.2. Section 4 contains the proof of Lemma 3.2.

## 2. Preliminaries

**2.1. Cordial Volterra integral operators.** The cordial Volterra integral operator  $V_\varphi$  with a core  $\varphi \in L^1(0, 1)$  is defined by

$$(V_\varphi u)(t) = \int_0^t \frac{1}{t} \varphi\left(\frac{s}{t}\right) u(s) ds = \int_0^1 \varphi(x) u(tx) dx, \quad 0 \leq t \leq T, \quad u \in C. \quad (6)$$

Denote

$$\widehat{\varphi}(\lambda) = \int_0^1 x^\lambda \varphi(x) dx \quad (7)$$

for such  $\lambda \in \mathbb{C}$  where the integral converges. From the second form of (6), we get that

$$V_\varphi w_\lambda = \widehat{\varphi}(\lambda)w_\lambda, \quad \text{where } w_\lambda(t) = t^\lambda, \quad 0 \leq t \leq T. \quad (8)$$

By differentiating the second form of (6) we have

$$(V_\varphi u)^{(m)}(t) = \int_0^1 \varphi(x)x^m u^{(m)}(tx)dx \quad \text{for } u \in C^m, \quad m \geq 0. \quad (9)$$

We also get

$$(aV_\varphi u)^{(m)}(t) = \sum_{i=0}^m \frac{m!}{i!(m-i)!} a^{(m-i)}(t) \int_0^1 \varphi(x)x^i u^{(i)}(tx)dx \quad (10)$$

for  $a, u \in C^m$ ,  $m \geq 0$ . The following theorem is a summary of results proved in [22] and [23].

**Theorem 2.1.** For  $\varphi \in L^1(0, 1)$ ,  $a \in C^m$ ,  $m \geq 0$ , it holds that  $V_\varphi, aV_\varphi \in \mathcal{L}(C^m)$  and

$$\sigma_0(V_\varphi) = \{\widehat{\varphi}(\lambda) \mid \operatorname{Re}\lambda \geq 0\} \cup \{0\}, \quad (11)$$

$$\sigma_m(V_\varphi) = \{\widehat{\varphi}(\lambda) \mid \operatorname{Re}\lambda \geq m\} \cup \{0\} \cup \{\widehat{\varphi}(j) \mid j=0, 1, \dots, m-1\} \quad \text{for } m \geq 1, \quad (12)$$

$$\sigma_m(aV_\varphi) = a(0)\sigma_m(V_\varphi) \quad \text{for } m \geq 0.$$

Moreover,  $\|V_\varphi\|_{\mathcal{L}(C)} = \|\varphi\|_1$  and  $\|aV_\varphi\|_{\mathcal{L}(C)} \leq \|a\|_\infty \|\varphi\|_1$  with

$$\|\varphi\|_1 = \int_0^1 |\varphi(x)|dx, \quad \|a\|_\infty = \max_{0 \leq t \leq T} |a(t)|.$$

If  $a(0) = 0$ , then operator  $aV_\varphi \in \mathcal{L}(C^m)$  is compact and  $\sigma_m(aV_\varphi) = \{0\}$ .

**Proposition 2.2** ([22, Remark 4.9]). For  $\varphi \in L^1(0, 1)$ ,  $\mu \notin \sigma_0(V_\varphi)$  it holds

$$(\mu I - V_\varphi)^{-1} = \mu^{-1}I + V_\psi$$

where  $\psi \in L^1(0, 1)$  is uniquely determined by  $\mu$  and  $\varphi$ .

**Proposition 2.3** ([22, Remark 4.8]). For  $\varphi \in L^1(0, 1)$ ,  $\mu \in \sigma_0(V_\varphi)$ ,  $\mu \neq \widehat{\varphi}(0)$ , the set  $(\mu I - V_\varphi)C$  is dense in  $C$ . For  $\mu = \widehat{\varphi}(0)$ , the functions  $f \in (\mu I - V_\varphi)C$  satisfy  $f(0) = 0$ , hence the set  $(\widehat{\varphi}(0)I - V_\varphi)C$  is not dense in  $C$ .

**Proposition 2.4** ([22, Theorem 4.10]). For  $\varphi \in L^1(0, 1)$ ,  $\mu \neq 0$ , the operator  $\mu I - V_\varphi : C \rightarrow C$  has the right hand inverse if and only if  $\mu - \widehat{\varphi}(i\xi) \neq 0$  for any  $\xi \in \mathbb{R}$ ; further,  $\mu I - V_\varphi : C \rightarrow C$  has the (two side) inverse if and only if, in addition,  $\arg[\mu - \widehat{\varphi}(i\xi)]_{\xi=-\infty}^\infty = 0$ .

For  $\alpha > 0$  and  $u \in C$  it holds that

$$(J^\alpha u)(t) = \frac{t^\alpha}{\Gamma(\alpha)} \int_0^t \frac{1}{t} \left(1 - \frac{s}{t}\right)^{\alpha-1} u(s)ds, \quad 0 < t \leq T,$$

hence  $M_{-\alpha}J^\alpha$  is for any  $\alpha > 0$  a cordial Volterra integral operator with the core  $\varphi(x) = \frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1}$  in  $L^1(0, 1)$ .

**2.2. Fredholm type operators.** Let us recall the definition and some results from the theory of Fredholm operators of index 0 in a Banach space  $X$ .

**Definition 2.5.** For a Banach space  $X$ , an operator  $A \in \mathcal{L}(X)$  is called Fredholm (or, Noether) if its null-space  $\mathcal{N}(A) := \{u \in X \mid Au = 0\}$  is finite dimensional, and its range  $\mathcal{R}(A) = AX$  is closed and of a finite codimension in  $X$ ; the integer  $\dim \mathcal{N}(A) - \text{codim } \mathcal{R}(A)$  is called index of  $A$ . By  $\Phi_\kappa(X)$  we denote the class of Fredholm operators of index  $\kappa \in \mathbb{Z}$ .

Here,  $\text{codim } \mathcal{R}(A) = \dim(X/\mathcal{R}(A))$  and  $X/\mathcal{R}(A)$  is the factor space of  $X$  over  $\mathcal{R}(A)$ .

**Proposition 2.6** (See e.g. [20, Theorem 1.3.2]). *For  $A \in \mathcal{L}(X)$  the following conditions are equivalent:*

1.  $A \in \Phi_0(X)$ ;
2.  $A$  admits a representation  $A = B + K$  where  $B \in \mathcal{L}(X)$  possesses the inverse  $B^{-1} \in \mathcal{L}(X)$  and  $K \in \mathcal{L}(X)$  is compact.

**Corollary 2.7.** *Let  $\mu I - A \in \Phi_0(X)$  for a  $\mu \in \mathbb{C}$ . If  $\mathcal{N}(\mu I - A) = \{0\}$  then  $\mu \in \rho_X(A)$ .*

**Corollary 2.8.** *Suppose that  $\mu I - A \notin \Phi_0(X)$  for a  $\mu \in \mathbb{C}$ . Then  $\mu \in \sigma_X(A)$ .*

**Corollary 2.9.** *The set  $\Phi_0(X)$  is open in  $\mathcal{L}(X)$ .*

The following proposition is a simple consequence of Definition 2.5.

**Proposition 2.10.** *Let  $X$  be representable in a direct sum  $X = X_0 \oplus X_1$ , where  $X_0$  and  $X_1$  are subspaces of  $X$  and  $X_0$  is finite dimensional. Let  $A \in \mathcal{L}(X)$  be such that  $AX_0 \subset X_0$ ,  $AX_1 \subset X_1$ . Then  $A \in \Phi_\kappa(X)$  if and only if  $A_1 \in \Phi_\kappa(X_1)$  where  $A_1 = A|_{X_1} \in \mathcal{L}(X_1)$  is the restriction of  $A$  onto  $X_1$ .*

### 3. Formulation of the main result

As stated before, the main goal of this article is to study the unique solvability of singular fractional differential equations (3) with conditions (5).

We start off by considering the simplified version of equation (3) with constant coefficients:

$$(D_0^\alpha M_\alpha u)(t) = \sum_{k=1}^l b_k(0)(D_0^{\alpha_k} M_{\alpha_k} u)(t) + f(t), \quad 0 < t \leq T. \quad (13)$$

Here  $\alpha, \alpha_k \in \mathbb{R}$  and

$$m < \alpha \leq m + 1, \quad \alpha > \alpha_k \geq 0, \quad k = 1, 2, \dots, l, \quad f \in C^m, \quad m \in \mathbb{N}_0. \quad (14)$$

By the change of variables  $v = D_0^\alpha M_\alpha u$  in (13), we get due to  $u = (D_0^\alpha M_\alpha)^{-1}v = M_{-\alpha} J^\alpha v$  that equation (13) is equivalent to

$$v = \sum_{k=1}^l b_k(0) [D_0^{\alpha_k} M_{\alpha_k}] [M_{-\alpha} J^\alpha] v + f. \quad (15)$$

Note that for any  $v \in C$  and  $k = 1, 2, \dots, l$ , function  $M_{-\alpha} J^\alpha v$  belongs to the domain of operator  $D_0^{\alpha_k} M_{\alpha_k}$ , or to the range of  $(D_0^{\alpha_k} M_{\alpha_k})^{-1} = M_{-\alpha_k} J^{\alpha_k}$ , i.e. there exists a  $w \in C$  such that

$$M_{-\alpha} J^\alpha v = M_{-\alpha_k} J^{\alpha_k} w. \quad (16)$$

Namely, we claim that the last equality holds for  $w = V_{\varphi_{\alpha, \alpha_k}} v$ , where  $V_{\varphi_{\alpha, \alpha_k}}$  is a cordial Volterra integral operator with the core

$$\varphi_{\alpha, \alpha_k}(x) = \frac{1}{\Gamma(\alpha - \alpha_k)} (1-x)^{\alpha - \alpha_k - 1} x^{\alpha_k}, \quad \varphi_{\alpha, \alpha_k} \in L^1(0, 1). \quad (17)$$

In other words, we claim that there holds the equality of cordial Volterra integral operators:

$$M_{-\alpha} J^\alpha = [M_{-\alpha_k} J^{\alpha_k}] V_{\varphi_{\alpha, \alpha_k}}. \quad (18)$$

The three cordial Volterra integral operators  $M_{-\alpha} J^\alpha$ ,  $M_{-\alpha_k} J^{\alpha_k}$  and  $V_{\varphi_{\alpha, \alpha_k}}$  in (18) are well-defined and bounded in  $C$ , hence (18) holds if

$$M_{-\alpha} J^\alpha w_n = [M_{-\alpha_k} J^{\alpha_k}] V_{\varphi_{\alpha, \alpha_k}} w_n, \quad \forall n \in \mathbb{N}_0, \quad (19)$$

where  $w_n(t) = t^n$ . For  $n \in \mathbb{N}_0$ , it holds

$$\begin{aligned} M_{-\alpha} J^\alpha w_n &= \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} w_n, & M_{-\alpha_k} J^{\alpha_k} w_n &= \frac{\Gamma(n+1)}{\Gamma(\alpha_k+n+1)} w_n, \\ V_{\varphi_{\alpha, \alpha_k}} w_n &= \frac{\Gamma(\alpha_k+n+1)}{\Gamma(\alpha+n+1)} w_n \end{aligned}$$

and we see that (16), (18) and (19) hold.

Equation (15) can be rewritten as

$$v = \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} v + f.$$

By (7), we get  $\widehat{\varphi}_{\alpha, \alpha_k}(\lambda) = \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$ ,  $\operatorname{Re} \lambda \geq 0$ ,  $k = 1, 2, \dots, l$ . According to (11) and (12), remembering (8), we obtain that the spectrum of operator  $\sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}}$  has the form

$$\sigma_0 \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right) = \left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re} \lambda \geq 0 \right\} \cup \{0\} \quad (20)$$

and

$$\begin{aligned} \sigma_m \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right) &= \left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \mid q = 0, 1, \dots, m - 1 \right\} \\ &\cup \left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re} \lambda \geq m \right\} \cup \{0\}, \quad (21) \\ &m \geq 1. \end{aligned}$$

The following result is a consequence of (20) and (21).

**Lemma 3.1.** *Let  $\alpha, \alpha_k \in \mathbb{R}$ , and conditions (14) hold. Equation (15) has a unique solution  $v \in C$  for any  $f \in C$ , i.e.  $1 \notin \sigma_0 \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ , if and only if*

$$\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \neq 1, \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq 0.$$

*Equation (15) has a unique solution  $v \in C^m$  for any  $f \in C^m$ ,  $m \geq 1$ , i.e.  $1 \notin \sigma_m \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ , if and only if*

$$\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \neq 1, \quad q = 0, 1, \dots, m - 1,$$

and

$$\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \neq 1, \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq m.$$

Having found the solution  $v \in C$  ( $v \in C^m$ ) of equation (15), the solution of equation (13) has the form  $u = M_{-\alpha} J^\alpha v$ .

To study the unique solvability of singular fractional differential equations (3) with (5), we prove (see Section 4) the following result.

**Lemma 3.2.** *Under conditions  $b_k \in C^m$ ,  $m \geq 0$ ,  $k = 1, 2, \dots, l$ , it holds that*

$$\sigma_m \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right) = \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right),$$

with  $\varphi_{\alpha, \alpha_k}$  from (17).



Similarly as for equation (13), equation (3) can be rewritten as

$$v = \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} v + f, \quad (22)$$

where  $v = D_0^\alpha M_\alpha u$  is the unknown and  $V_{\varphi_{\alpha, \alpha_k}}$  is a cordial integral operator with core  $\varphi_{\alpha, \alpha_k}$  from (17). We can now formulate the main result of the present article.

**Theorem 3.3.** *Let  $\alpha, \alpha_k \in \mathbb{R}$ , and (5) hold. For any  $f \in C$ , equation (22) has a unique solution  $v \in C$  and equation (3) has a unique solution  $u = M_{-\alpha} J^\alpha v \in C$  if and only if  $1 \notin \sigma_0 \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ , i.e.*

$$\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \neq 1, \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq 0.$$

For any  $f \in C^m$ ,  $m \geq 1$ , equation (22) has a unique solution  $v \in C^m$  and equation (3) has a unique solution  $u = M_{-\alpha} J^\alpha v \in C^m$  if and only if  $1 \notin \sigma_m \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ , i.e.

$$\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \neq 1, \quad q = 0, 1, \dots, m-1,$$

and

$$\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \neq 1, \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq m.$$

*Proof.* The claims of Theorem 3.3 regarding the solution  $v$  of (22) are direct consequences of Lemma 3.2 and (20), (21). Furthermore,  $M_{-\alpha} J^\alpha$  is a cordial Volterra integral operator and thus according to Theorem 2.1,  $v \in C^m$ ,  $m \in \mathbb{N}_0$ , implies  $u = M_{-\alpha} J^\alpha v \in C^m$ ; recall that  $M_{-\alpha} J^\alpha v$  belongs to the domain of  $D_0^{\alpha_k} M_{\alpha_k}$  and  $u = M_{-\alpha} J^\alpha v$  really satisfies (3). Theorem 3.3 is proved.  $\square$

**Example 3.4.** Consider the equation

$$D_0^\alpha M_\alpha u = bu + f, \quad m < \alpha \leq m+1, \quad b, f \in C^m, \quad m \in \mathbb{N}_0. \quad (23)$$

By Theorem 3.3, equation (23) has a unique solution  $u \in C$  for any  $f \in C$  if and only if  $1 \notin \sigma_0 (b(0) V_{\varphi_{\alpha, 0}})$ , i.e.

$$b(0) \neq \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1)}, \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq 0;$$

for real  $b(0)$  and  $\alpha \in (0, 1]$  ( $m = 0$ ), this condition, or the condition  $1 \notin \sigma_0(b(0)V_{\varphi_{\alpha,0}})$ , takes the form  $b(0) < \Gamma(\alpha + 1)$ , since  $\sigma_0(V_{\varphi_{\alpha,0}}) \cap \mathbb{R} = \left[0, \frac{1}{\Gamma(\alpha+1)}\right]$ . For  $b(0) = \Gamma(\alpha + 1)$ , according to Proposition 2.4, operator  $I - bV_{\varphi_{\alpha,0}}$  is non-Fredholm; for  $b(0) > \Gamma(\alpha + 1)$  it holds  $I - bV_{\varphi_{\alpha,0}} \in \Phi_1(C)$  (is Fredholm operator of index 1).

Equation (23) has a unique solution  $u \in C^m$  for any  $f \in C^m$ ,  $m \geq 1$ , if and only if  $1 \notin \sigma_m(b(0)V_{\varphi_{\alpha,0}})$ , i.e.

$$b(0) \neq \frac{\Gamma(q + \alpha + 1)}{q!}, \quad q = 0, 1, \dots, m - 1,$$

and

$$b(0) \neq \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1)}, \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq m.$$

**Example 3.5.** As stated before, equations  $tu'(t) = a(t)u(t) + f(t)$ ,  $0 < t \leq T$  and  $(tu(t))' = b(t)u(t) + f(t)$ , where  $b(t) = a(t) + 1$  and  $0 < t \leq T$ , are equivalent. Thus, according to [24], if  $\operatorname{Re} b(0) < 1$ , then equation  $(tu(t))' = b(t)u(t) + f(t)$  has for any  $f \in C$  a unique solution in  $C$  (if  $b \in C$  is such that a finite limit  $\lim_{t \rightarrow 0} \frac{b(t) - b(0)}{t^\beta}$  exists for a  $\beta > 0$ , then condition  $\operatorname{Re} b(0) < 1$  is also necessary for the unique solution of  $(tu(t))' = b(t)u(t) + f(t)$  in  $C$  for all  $f \in C$ ). Equation  $D_0^\alpha(t^\alpha u(t)) = b(t)u(t) + f(t)$ ,  $0 < t \leq T$ ,  $0 < \alpha < 1$  (equation (23) for  $0 < \alpha < 1$ ) has for any  $f \in C$  a unique solution in  $C$  if and only if  $b(0) \neq \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\lambda + 1)}$  for any  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ . As  $\alpha \rightarrow 1$ , the last condition takes the form  $\operatorname{Re} b(0) < 1$ .

### 4. Proof of Lemma 3.2

To prove Lemma 3.2, we show that for  $b_k \in C^m$ ,  $k = 1, 2, \dots, l$  and  $m \geq 0$  the following relations hold:

$$\rho_m \left( \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}} \right) \subset \rho_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right), \tag{24}$$

$$\sigma_m \left( \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}} \right) \subset \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right). \tag{25}$$

The proof of (24) and (25) is presented in six parts. In part a) we show that (24) holds for  $m = 0$  under stricter conditions  $\varphi_{\alpha,\alpha_k} \in C^1[0, 1]$ , and  $b_k \in C^1$  for  $k = 1, 2, \dots, l$ . Part b) is dedicated to extending the results from part a) to  $m = 0$ ,  $\varphi_{\alpha,\alpha_k} \in L^1(0, 1)$  and  $b_k \in C$ ,  $k = 1, 2, \dots, l$ . With part c) the inclusion (24) is proved for  $m \geq 1$ . Part d) shows that (25) holds for  $m = 0$  and prepares the proof for  $m \geq 1$ , the last two parts e) and f) complete the proof of (25) for  $m \geq 1$ .

**4.1. Part a).** Let  $m = 0$ ,  $\varphi_{\alpha, \alpha_k} \in C^1[0, 1]$ , and  $b_k \in C^1$  for  $k = 1, 2, \dots, l$ . We show that  $\mu \in \rho_0 \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ ,  $\mu \in \sigma_0 \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$  leads to a contradiction, i.e. relation (24) holds for  $m = 0$ .

Note that,  $0 \in \sigma_0 \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$  (see (20)), thus  $\mu \neq 0$ . According to Theorem 2.1,  $\sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} : C \rightarrow C$  is compact, thus (see Proposition 2.6)  $\mu I - \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \in \Phi_0(C)$ . Now, based upon Corollary 2.7,  $\mu \in \sigma_0 \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$  is the eigenvalue of operator  $\sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}}$ . Let  $u_0 \in C$ ,  $\|u_0\|_\infty = 1$ , be the corresponding eigenfunction:  $\left( \mu I - \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right) u_0 = 0$  or

$$u_0 = \left( \mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)^{-1} \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u_0.$$

Since  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})^{-1} = \mu^{-1} I + V_\psi$  with a  $\psi \in L^1(0, 1)$  (see Proposition 2.2), it holds

$$u_0 = (\mu^{-1} I + V_\psi) \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u_0. \quad (26)$$

We assumed that  $b_k \in C^1$  for  $k = 1, 2, \dots, l$ , consequently

$$|b_k(t) - b_k(0)| \leq c_k t, \quad k = 1, 2, \dots, l, \quad 0 \leq t \leq T, \quad (27)$$

with some constants  $c_k > 0$ ,  $k = 1, 2, \dots, l$ . Using (26) and (27), we now evaluate  $|u_0(t)|$  step-by-step. The first step is

$$\begin{aligned} \left| \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u_0 \right| (t) &\leq \sum_{k=1}^l |b_k(t) - b_k(0)| \int_0^t t^{-1} |\varphi_{\alpha, \alpha_k}(t^{-1}s)| |u_0(s)| ds \\ &\leq t \sum_{k=1}^l c_k \|\varphi_{\alpha, \alpha_k}\|_1 \\ &\leq ct \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}\|_1, \end{aligned}$$

$$\left| \left( V_\psi \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u_0 \right) \right| (t) \leq ct \|\psi\|_1 \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}\|_1,$$

$$|u_0(t)| \leq ct (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}\|_1,$$

with a constant  $c > 0$  independent of  $t \in [0, T]$ . Let us assume that after the  $n$ -th step we have the estimate  $|u_0(t)| \leq \tilde{c}_n t^n$  with  $0 \leq t \leq T$  and constant  $\tilde{c}_n$ . Then

$$\begin{aligned} \left| \left( \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u_0 \right) (t) \right| &\leq \sum_{k=1}^l |b_k(t) - b_k(0)| \int_0^t t^{-1} |\varphi_{\alpha, \alpha_k}(t^{-1}s)| \tilde{c}_n s^n ds \\ &\leq c \tilde{c}_n t^{n+1} \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[n]}\|_1, \end{aligned}$$

$$\begin{aligned} \left| \left( V_\psi \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u_0 \right) (t) \right| &\leq c \tilde{c}_n t^{n+1} \|\psi\|_1 \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[n]}\|_1, \\ |u_0(t)| &\leq c \tilde{c}_n t^{n+1} (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[n]}\|_1, \end{aligned}$$

$0 \leq t \leq T$ , where

$$\varphi_{\alpha, \alpha_k}^{[n]}(x) = \varphi_{\alpha, \alpha_k}(x) x^n. \tag{28}$$

Therefore,  $|u_0(t)| \leq \tilde{c}_{n+1} t^{n+1}$  for  $0 \leq t \leq T$  with constant  $\tilde{c}_{n+1} = c \tilde{c}_n (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[n]}\|_1$ . As easily verified  $\|\varphi_{\alpha, \alpha_k}^{[n]}\|_1 \leq \frac{1}{n} \|\varphi_{\alpha, \alpha_k}^{[1]}\|_\infty$ , thus

$$\begin{aligned} \tilde{c}_{n+1} &\leq \frac{c (|\mu^{-1}| + \|\psi\|_1) \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[1]}\|_\infty}{n} \tilde{c}_n \\ &\leq \frac{c^2 (|\mu^{-1}| + \|\psi\|_1)^2 \left( \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[1]}\|_\infty \right)^2}{n(n-1)} \tilde{c}_{n-1} \\ &\vdots \\ &\leq \frac{c^n (|\mu^{-1}| + \|\psi\|_1)^n \left( \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[1]}\|_\infty \right)^n}{n!} \tilde{c}_1. \end{aligned}$$

Hence, for  $n \in \mathbb{N}_0$ ,  $0 \leq t \leq T$  we get

$$|u_0(t)| \leq \frac{c^n (|\mu^{-1}| + \|\psi\|_1)^n \left( \sum_{k=1}^l \|\varphi_{\alpha, \alpha_k}^{[1]}\|_\infty \right)^n T^{n+1}}{n!} \tilde{c}_1,$$

from which it follows that  $u_0(t) \equiv 0$ . This contradicts the fact that  $u_0 \in C$  is an eigenfunction with  $\|u_0\|_\infty = 1$ .

**4.2. Part b).** Let  $m = 0$ ,  $\varphi_{\alpha, \alpha_k} \in L^1(0, 1)$  and  $b_k \in C$ ,  $k = 1, 2, \dots, l$ . We begin the discussion as in part a) and then interpret relation (26) as follows: the eigenvalue problem

$$\lambda u = (\mu^{-1}I + V_\psi) \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) u$$

with compact operator  $(\mu^{-1}I + V_\psi) \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right)$  has an eigensolution  $(\lambda_0, u_0)$ ,  $\lambda_0 = 1$ ,  $u_0 \in C$ ,  $\|u_0\|_\infty = 1$ .

We approximate functions  $\varphi_{\alpha, \alpha_k}$  and  $b_k$  by  $\varphi_{\alpha, \alpha_k}^\varepsilon \in C^1[0, 1]$  and  $b_k^\varepsilon \in C^1$  so that  $\|\varphi_{\alpha, \alpha_k} - \varphi_{\alpha, \alpha_k}^\varepsilon\|_1 \leq \varepsilon$ ,  $b_k^\varepsilon(0) = b_k(0)$  and  $\|b_k - b_k^\varepsilon\|_\infty \leq \varepsilon$  for  $k = 1, 2, \dots, l$ , where  $\varepsilon > 0$  is a given small number. The operator  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}^\varepsilon}$  from  $C$  into  $C$  is still invertible and can be expressed as

$$\left( \mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}^\varepsilon} \right)^{-1} = \mu^{-1}I + V_{\psi_\varepsilon} \quad \text{with a } \psi_\varepsilon \in L^1(0, 1)$$

(see Proposition 2.2),  $\|\psi - \psi_\varepsilon\|_1 \leq c'\varepsilon$ ,  $c' > 0$ . We get

$$\left\| (\mu^{-1}I + V_\psi) \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha, \alpha_k}} \right) - (\mu^{-1}I + V_{\psi_\varepsilon}) \left( \sum_{k=1}^l [b_k^\varepsilon - b_k^\varepsilon(0)] V_{\varphi_{\alpha, \alpha_k}^\varepsilon} \right) \right\|_{\mathcal{L}(C)} \leq c''\varepsilon.$$

For a sufficiently small  $\varepsilon > 0$ , the perturbed eigenvalue problem

$$\lambda u = (\mu^{-1}I + V_{\psi_\varepsilon}) \left( \sum_{k=1}^l [b_k^\varepsilon - b_k^\varepsilon(0)] V_{\varphi_{\alpha, \alpha_k}^\varepsilon} \right) u$$

has a solution  $(\lambda_\varepsilon, u_\varepsilon)$ ,  $\|u_\varepsilon\|_\infty = 1$  such that,  $\lambda_\varepsilon \rightarrow 1$  as  $\varepsilon \rightarrow 0$ . Using a similar discussion as in part a) we get that  $u_\varepsilon \equiv 0$ . This is a contradiction since  $\|u_\varepsilon\|_\infty = 1$ . Consequently, for  $m = 0$ ,  $\varphi_{\alpha, \alpha_k} \in L^1(0, 1)$  and  $b_k \in C$ ,  $k = 1, 2, \dots, l$ , relation (24) holds.

**4.3. Part c).** In this part of the proof, we will show that inclusion (24) holds for  $m \geq 1$ . Let  $\mu \in \rho_m \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ . We may assume that,  $\mu \neq 0$  since according to (20) we have  $0 \in \sigma_0 \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$ . Proposition 2.6 yields  $\mu I - \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \in \Phi_0(C^m)$ . To prove that  $\mu \in \rho_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$ , it is sufficient to show that the homogeneous equation  $\mu u = \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} u$  has in  $C^m$  only the trivial solution (see Corollary 2.7). Let  $u_0 \in C^m$  be a solution:

$$\mu u_0 = \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right) u_0. \quad (29)$$

We first show by induction that

$$u_0^{(q)}(0) = 0, \quad q = 0, \dots, m - 1. \tag{30}$$

Note that, for  $u_\lambda(t) = t^\lambda$ ,  $\text{Re}\lambda > 0$ ,

$$\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}} u_\lambda = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} u_\lambda. \tag{31}$$

For  $t = 0$ , (29) takes the form

$$\mu u_0(0) = u_0(0) \sum_{k=1}^l b_k(0) \int_0^1 \varphi_{\alpha,\alpha_k}(x) dx. \tag{32}$$

If  $u_0(0) \neq 0$ , then (32) can be interpreted as follows:  $\mu$  is an eigenvalue of  $\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}$  corresponding to eigenfunction 1 (see (31)). This contradicts the assumption  $\mu \in \rho_m \left( \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}} \right)$ . Hence  $u_0(0) = 0$ . We show that the induction hypothesis  $u_0^{(j)}(0) = 0$  for  $j = 0, \dots, n - 1$ , where  $n \leq m - 1$ , leads to  $u_0^{(n)}(0) = 0$ . Indeed, from (10) it follows that

$$\left( \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)^{(n)} = \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}}^{[n]} u_0^{(n)} + \sum_{q=0}^{n-1} \frac{n!}{q!(n-q)!} \sum_{k=1}^l b_k^{(n-q)} V_{\varphi_{\alpha,\alpha_k}}^{[q]} u_0^{(q)}, \tag{33}$$

with  $\varphi_{\alpha,\alpha_k}^{[q]}$ ,  $q = 0, 1, \dots, n$ , given by (28). Since  $u_0^{(j)}(0) = 0$  for  $j = 0, \dots, n - 1$ , we get (see (29))

$$\mu u_0^{(n)}(0) = u_0^{(n)}(0) \sum_{k=1}^l b_k(0) \int_0^1 x^n \varphi_{\alpha,\alpha_k}(x) dx.$$

Now,  $u_0^{(n)}(0) = 0$ , since otherwise  $\mu$  would be an eigenvalue of  $\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}$  corresponding to eigenfunction  $t^n$  (see (31)). This completes the proof of (30).

Next, to obtain  $u_0 \equiv 0$ , it is sufficient to show that  $u_0^{(m)} \equiv 0$  (see (30)). We know that  $\mu u_0^{(m)} = \left( \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right) u_0 \right)^{(m)}$ , thus (see (33))

$$\begin{aligned} \left( \mu I - \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}} \right) u_0^{(m)} &= \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}}^{[m]} u_0^{(m)} \\ &\quad + \sum_{q=0}^{m-1} \frac{m!}{q!(m-q)!} \sum_{k=1}^l b_k^{(m-q)} V_{\varphi_{\alpha,\alpha_k}}^{[q]} u_0^{(q)}. \end{aligned}$$

Since  $\mu \in \rho_m \left( \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}} \right)$  for  $m \geq 1$ , it holds  $\mu \in \rho_0 \left( \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}^{[m]}} \right)$ . Hence, operator  $\mu I - \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}^{[m]}} : C \rightarrow C$  is invertible, and according to Proposition 2.2 the inverse has the form

$$\left( \mu I - \sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}^{[m]}} \right)^{-1} = \mu^{-1}I + V_{\psi_m}, \quad \text{with a } \psi_m \in L^1(0, 1).$$

Relations (30) imply that  $u_0^{(q)}(t) = \frac{1}{(m-q-1)!} \int_0^t (t-s)^{m-q-1} u_0^{(m)}(s) ds$ , where  $0 \leq t \leq T$  and  $q = 0, \dots, m-1$ . In conclusion,

$$u_0^{(m)} = (\mu^{-1}I + V_{\psi_m}) \times \left\{ \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}^{[m]}} u_0^{(m)} + \sum_{q=0}^{m-1} \frac{m!}{q!(m-q)!} \sum_{k=1}^l b_k^{(m-q)} V_{\varphi_{\alpha,\alpha_k}^{[q]}} G_q u_0^{(m)} \right\}, \quad (34)$$

where  $(G_q v)(t) = \frac{1}{(m-q-1)!} \int_0^t (t-s)^{m-q-1} v(s) ds$ ,  $q = 0, \dots, m-1$ . Note that,

$$|v(t)| \leq ct^p, \quad p \geq 0 \Rightarrow |(G_q v)(t)| \leq c_1 t^{p+m-q}, \quad \left| \left( \sum_{k=1}^l [b_k - b_k(0)] V_{\varphi_{\alpha,\alpha_k}^{[m]}} v \right)(t) \right| \leq c_2 t^{p+1}.$$

Also, operators  $\sum_{k=1}^l b_k^{(m-q)} V_{\varphi_{\alpha,\alpha_k}^{[q]}}$  and  $V_{\psi_m}$  preserve the convergence order of  $v(t)$  for  $t \rightarrow 0$ . Approximating  $u_0^{(m)}(t)$ , with the help of (34), step-by-step as in part a), we obtain that  $u_0^{(m)} \equiv 0$ . Therefore (24) holds for  $m \geq 1$ .

**4.4. Part d).** We now turn to the proof of (25). By (20) and (21), the inclusion (25) is equivalent to the following inclusions: for  $m = 0$ ,

$$\left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re} \lambda \geq 0 \right\} \cup \{0\} \subset \sigma_0 \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right); \quad (35)$$

for  $m \geq 1$ ,

$$\left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \mid q = 0, 1, \dots, m-1 \right\} \cup \left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re} \lambda \geq m \right\} \cup \{0\} \subset \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right).$$

First off all, the fact  $0 \in \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right)$  for  $m \geq 0$  follows directly from the closedness of the spectrum  $\sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}} \right)$ : in accordance with (24)

the spectrum  $\sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$  contains points  $\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$  (see (20) and (21)) for arbitrary large  $\lambda \in \mathbb{R}$  and  $\sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \rightarrow 0$  as  $\lambda \rightarrow \infty$ , since  $\alpha_k < \alpha$ .

Next, to establish the inclusions

$$\left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re} \lambda \geq m \right\} \subset \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right), \quad m \geq 0,$$

it is sufficient to show that

$$\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \notin \Phi_0(C^m), \quad \forall \mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \text{ with } \operatorname{Re} \lambda \geq m. \quad (36)$$

Indeed, relation (36) implies that also  $\mu I - \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \notin \Phi_0(C^m)$  and thus, by Corollary 2.8 we get that  $\mu \in \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$  for  $\operatorname{Re} \lambda \geq m$ .

Let us establish (36) for  $m = 0$ . For  $\mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$ ,  $\operatorname{Re} \lambda > 0$ , according to Propositions 2.3 and 2.4,  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})C = C$  holds and  $\mu$  is an eigenvalue of operator  $\sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}}$  with eigenfunction  $t^\lambda$  in  $C$  (see (31)), so  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \notin \Phi_0(C^m)$ . For  $\mu_0 = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda_0 + 1)}{\Gamma(\alpha + \lambda_0 + 1)}$ ,  $\operatorname{Re} \lambda_0 = 0$ , relation  $\mu_0 I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \in \Phi_0(C)$  cannot hold since otherwise it would also be true for a  $\lambda$  with  $\operatorname{Re} \lambda > 0$  that is close to  $\lambda_0$  (see Corollary 2.9), but this is not the case. Thus (36), and as a consequence (35) and (25) hold for  $m = 0$ .

**4.5. Part e).** In this part we prove (36) for  $m \geq 1$ . Let  $\mathcal{P}_{m-1}$  be the space of all polynomials with degree equal to or less than  $m - 1$ . It holds  $C^m = C_0^m \oplus \mathcal{P}_{m-1}$ , where  $C_0^m$  is defined in (4). Our aim is to use Proposition 2.10: we show that  $C_0^m$  and  $\mathcal{P}_{m-1}$  fulfill the presumptions about  $X_1$  and  $X_0$  respectively and  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \notin \Phi_0(C_0^m)$  for  $\mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$  with  $\operatorname{Re} \lambda > m$ . In such case, by Proposition 2.10  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}}$  belongs to  $\Phi_\kappa(C^m)$ ,  $\kappa > 0$  for  $\mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$  with  $\operatorname{Re} \lambda > m$ .

We start our discussion by noting that according to (9), for any  $\mu \in \mathbb{C}$  and  $u \in C_0^m$  it holds that  $((\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})u)^{(n)} = (\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}}^{[n]})u^{(n)}$  for  $n = 1, 2, \dots, m$ . Thus  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})C_0^m \subset C_0^m$ . In accordance with (31), the inclusion  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})\mathcal{P}_{m-1} \subset \mathcal{P}_{m-1}$  also holds. Fur-



thermore, we claim that for  $\operatorname{Re}\lambda > m$  we have

$$\left( \mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right) C_0^m = C_0^m \quad \text{for every } \mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)},$$

i.e. that equation  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})u = v$  has for every  $v \in C_0^m$  and every  $\mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$  with  $\operatorname{Re}\lambda > m$  a solution  $u \in C_0^m$ . Since operator  $D^m : C_0^m \rightarrow C$  is an isomorphism, the last statement is equivalent to the following: equation  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}^m})\bar{u} = \bar{v}$  has for every  $\bar{v} \in C$  and every  $\mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda' + 1)}{\Gamma(\alpha + \lambda' + 1)}$  with  $\operatorname{Re}\lambda' > 0$  ( $\lambda' = \lambda - m$ ) a solution  $\bar{u} \in C$  ( $\bar{u} = u^{(m)}$ ,  $\bar{v} = v^{(m)}$ ). Now, according to Propositions 2.3 and 2.4, we get  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}^{[m]}})C = C$  and  $(\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}})C_0^m = C_0^m$ . Also, due to (31), for  $u_\lambda(t) = t^\lambda$  which belongs for  $\operatorname{Re}\lambda > m$  (also for  $\lambda = m$ ) to  $C_0^m$ , we have

$$\left( \mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right) u_\lambda = 0 \quad \text{if and only if } \mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)},$$

thus  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \in \Phi_\kappa(C_0^m)$  with  $\kappa > 0$  for  $\operatorname{Re}\mu > m$ . For us it is important that  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \notin \Phi_0(C_0^m)$  for  $\operatorname{Re}\lambda > m$ .

In conclusion, by Proposition 2.10 we get that  $\mu I - \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}}$  belongs to  $\Phi_\kappa(C^m)$ ,  $\kappa > 0$  for  $\mu = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)}$ , with  $\operatorname{Re}\lambda > m$ . Therefore, for  $\operatorname{Re}\lambda > m$ , relation (36) is established. If  $\operatorname{Re}\lambda = m$ , then a similar approximation argument as at the end of d) can be applied.

**4.6. Part f).** To conclude the proof of Lemma 3.2, it remains to show that for  $m \geq 1$ ,

$$\left\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)} \mid q = 0, 1, \dots, m-1 \right\} \subset \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right).$$

Denote  $\mu_q = \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + q + 1)}{\Gamma(\alpha + q + 1)}$  for  $q = 0, 1, \dots, m-1$ . We may assume, that  $\mu_q \notin \{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re}\lambda \geq m \}$ , since otherwise it follows from previous discussions that  $\mu_q \in \sigma_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$ .

Let us fix a sufficiently small  $\delta > 0$  such that: 1) the ball  $|\mu - \mu_q| \leq \delta$  does not contain  $\mu_j$  different from  $\mu_q$ ,  $j = 0, 1, \dots, m-1$  and 2) the intersection of  $\{ \sum_{k=1}^l b_k(0) \frac{\Gamma(\alpha_k + \lambda + 1)}{\Gamma(\alpha + \lambda + 1)} \mid \operatorname{Re}\lambda \geq m \}$  and the ball  $|\mu - \mu_q| \leq \delta$  is empty.

Note that, under our assumptions, the sphere  $|\mu - \mu_q| = \delta$  is contained in  $\rho_m \left( \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \right)$  and as a consequence of (24) also in  $\rho_m \left( \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} \right)$ .

Let  $V_\psi^\theta$  be an operator depending on parameter  $\theta \in [0, 1]$  defined by

$$V_\psi^\theta := \theta \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}} + (1 - \theta) \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}} \in \mathcal{L}(C^m), \quad 0 \leq \theta \leq 1.$$

Obviously,  $V_\psi^0 = \sum_{k=1}^l b_k(0) V_{\varphi_{\alpha, \alpha_k}}$  and  $V_\psi^1 = \sum_{k=1}^l b_k V_{\varphi_{\alpha, \alpha_k}}$ . The inclusion (24) for operator  $V_\psi^\theta$  implies that the sphere  $|\mu - \mu_q| = \delta$  is in  $\rho_m(V_\psi^\theta)$  for  $0 \leq \theta \leq 1$ .

Furthermore, the Riesz projector defined for  $V_\psi^\theta$  by (see [7])

$$P_q^\theta := \frac{1}{2\pi i} \int_{|\mu - \mu_q| = \delta} (\mu I - V_\psi^\theta)^{-1} d\mu \in \mathcal{L}(C^m), \quad 0 \leq \theta \leq 1,$$

projects the space  $C^m$  onto an invariant subspace of operator  $V_\psi^\theta$  corresponding to its spectrum part in the ball  $|\mu - \mu_q| \leq \delta$ . Since, by assumptions 1) and 2) above, the only possible point from this spectrum part in ball  $|\mu - \mu_q| \leq \delta$  is  $\mu_q$ , we have that  $\mu_q \in \sigma_m(V_\psi^\theta)$  if and only if  $P_q^\theta \neq 0$ .

The operator  $V_\psi^\theta$  is continuously dependent on parameter  $\theta$  on the sphere  $|\mu - \mu_q| = \delta$ , therefore there exists a constant  $c_\delta$ , such that

$$\|(\mu I - V_\psi^\theta)^{-1}\|_{\mathcal{L}(C^m)} \leq c_\delta, \quad \text{for } |\mu - \mu_q| = \delta, \quad 0 \leq \theta \leq 1.$$

Since  $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ , there exists a constant  $c'_\delta$  such that

$$\|P_q^\theta - P_q^{\theta'}\|_{\mathcal{L}(C^m)} \leq c_\delta^2 \delta \|V_\psi^\theta - V_\psi^{\theta'}\|_{\mathcal{L}(C^m)} \leq c'_\delta |\theta - \theta'|$$

for  $0 \leq \theta \leq \theta' \leq 1$ .

Consider the gap [11, 13] between subspaces  $X^\theta = P_q^\theta X$  and  $X^{\theta'} = P_q^{\theta'} X$  of  $X = C^m$ :

$$\begin{aligned} \text{gap}(X^\theta, X^{\theta'}) &:= \max \left\{ \sup_{u \in X^\theta, \|u\|_X=1} \inf_{v \in X^{\theta'}} \|u - v\|_X, \sup_{u \in X^{\theta'}, \|u\|_X=1} \inf_{v \in X^\theta} \|u - v\|_X \right\} \\ &\leq \max \left\{ \sup_{u \in X, \|u\|_X=1} \|P_q^\theta u - P_q^{\theta'} u\|_X, \sup_{u \in X, \|u\|_X=1} \|P_q^{\theta'} u - P_q^\theta u\|_X \right\} \\ &\leq \|P_q^\theta - P_q^{\theta'}\|_{\mathcal{L}(X)} \\ &\leq c'_\delta |\theta - \theta'| \rightarrow 0 \quad \text{as } |\theta - \theta'| \rightarrow 0, \quad 0 \leq \theta \leq \theta' \leq 1. \end{aligned}$$

It is known [11] that for a Banach space  $X$  and its closed subspaces  $X_1, X_2$  the inequality  $\text{gap}(X_1, X_2) < 1$  implies that  $\dim X_1 = \dim X_2$ . Thus  $\dim X^\theta = \dim X^{\theta'}$  for  $\theta, \theta' \in [0, 1]$  such that  $c'_\delta |\theta - \theta'| < 1$ , hence also for all  $\theta, \theta' \in [0, 1]$ .

As  $\mu_q$  is an eigenvalue of operator  $\sum_{k=1}^l b_k(0)V_{\varphi_{\alpha,\alpha_k}}$ , then  $\dim(P_q^\theta C^m) \geq 1$ . Consequently,  $P_q^\theta \neq 0$  for every  $\theta \in [0, 1]$ , in particular for  $\theta = 1$ , i.e.  $\mu_q \in \sigma_m\left(\sum_{k=1}^l b_k V_{\varphi_{\alpha,\alpha_k}}\right)$ .

With parts d), e) and f) we have shown that (25) holds for  $m \geq 1$ . This concludes the proof of Lemma 3.2.

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