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Behavior of Solutions of the Neumann Problem for the Poisson Equation near Straight Edges

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Abstract. The paper deals with the Neumann problem for the Poisson equation $\Delta u = f$ in the domain $\mathcal{D} = K \times \mathbb{R}^{n-m}$, where K is a cone in $\mathbb{R}^m$. The first part of the paper is concerned with the singularities of the Green function near the edge of the domain. Using the decomposition of the Green function given in the first part, the author obtains the asymptotics of the solution of the boundary value problem for a right-hand side f belonging to a weighted L_p Sobolev space. Precise formulas for all coefficients in the asymptotics are given.

Keywords. Poisson equation, Neumann problem, Green function, edge singularities
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1. Introduction

The paper is concerned with the Neumann problem for the Poisson equation

$$\Delta u = f \quad \text{in } \mathcal{D}, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\mathcal{D}\backslash M \tag{1}$$

in the domain

$$\mathcal{D} = \{x = (x', x'') : x' \in K,\ x'' \in \mathbb{R}^{n-m}\}.$$

Here, $K = \{x' \in \mathbb{R}^m : \frac{x'}{|x'|} \in \Omega\}$ is a cone in $\mathbb{R}^m$, $2 \le m < n$, and Ω denotes a subdomain of the unit sphere with smooth (of class C^∞) boundary $\partial\Omega$. The edge $\{x = (x', x'') \in \mathbb{R}^n : x' = 0,\ x'' \in \mathbb{R}^{n-m}\}$ of $\mathcal{D}$ is denoted by M. We define the space $\mathcal{H}_\mathcal{D}$ as the closure of the set $C_0^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{\mathcal{H}_\mathcal{D}} = \left(\int_\mathcal{D} \big|\nabla u(x)\big|^2\, dx\right)^{\frac{1}{2}}.$$

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Suppose that the mapping $\mathcal{H}_\mathcal{D} \ni v \to \int_\mathcal{D} f(x)\, v(x)\, dx$ defines a linear and continuous functional on $\mathcal{H}_\mathcal{D}$. Then there exists a uniquely determined solution $u \in \mathcal{H}_\mathcal{D}$ of the problem

$$-\int_\mathcal{D} \nabla u \cdot \nabla v\, dx = \int_\mathcal{D} f(x)\, v(x)\, dx \quad \text{for all } v \in \mathcal{H}_\mathcal{D}.$$

This function is called the variational solution of the problem (1). The goal of this paper is to describe the behavior of this solution near the edge M if $f \in V^{l-2}_{p,\beta}(\mathcal{D})$, where $l \ge 2$ and $0 < \beta + \frac{2}{p} < l$. Here, $V^l_{p,\beta}(\mathcal{D})$ is defined as the weighted Sobolev space (closure of the set $C_0^\infty(\overline{\mathcal{D}}\backslash M)$) with the norm

$$\|u\|_{V^l_{p,\beta}(\mathcal{D}} = \left(\int_\mathcal{D} \sum_{|\alpha|\le l} |x'|^{p(\beta-l+|\alpha|)} \left|\partial_x^\alpha u(x)\right|^p dx \right)^{\frac{1}{p}}.$$

The asymptotics of solutions of elliptic boundary value problems near edges was studied in many papers, see e.g. [2, 3, 7, 10, 12]. Mostly, the authors considered only the Dirichlet problem (for second and higher order elliptic equations) or assumed that the corresponding model problem arising after freezing of the coefficients is uniquely solvable in $V^l_{p,\beta}(\mathcal{D})$ for some β. This assumption is very restrictive and excludes e.g. the Neumann problem. Boundary value problems including the Neumann problem were handled e.g. in [1, 13]. But up to now, formulas for the coefficients in the asymptotics are published only under the above mentioned existence and uniqueness condition in the space $V^l_{p,\beta}(\mathcal{D})$. For example, one can find formulas for the coefficient of the leading term of the asymptotics in [7, 10, 12]. In [14], the author obtained precise formulas for the coefficients of all singular terms in the asymptotics of the solutions of the Dirichlet problem for the Poisson equation in the domain $\mathcal{D}$. The goal of the present paper is to obtain an analogous result for the Neumann problem.

The paper consists of two parts. In the first part (Section 2), we study the asymptotics of the Green function $G(x, y)$ for the problem (1) near the edge M. In the case of a cone ($m = n$), the asymptotics of Green's function is even known for general elliptic boundary value problems (cf. [9] and [12, Section 3.7]). The asymptotics of the Green function of the Dirichlet problem for the Poisson equation near the edge M of the domain $\mathcal{D}$ is described in [14]. In the last paper, the author employed well-known point-wise estimates of the Green function together with the asymptotics of solutions of the Dirichlet problem in the cone K. For the Neumann problem, such point-wise estimates were obtained by Solonnikov [15] (see also the monograph [11]). As in the case of the Dirichlet problem, the bounds for the Green function and its derivatives depend on the eigenvalues of the Beltrami operator $-\delta$ on Ω. Let $\{\Lambda_j\}_{j=0}^\infty$ be the nondecreasing sequence of eigenvalues of $-\delta$ on Ω with the Neumann boundary condition on $\partial\Omega$ counted with their multiplicities, and let $\{\phi_j\}_{j=0}^\infty$ be an orthonormal (in $L_2(\Omega)$) sequence of eigenfunctions corresponding to the eigenvalues Λ_j. In particular, $\phi_0 = (\text{mes}\,\Omega)^{-\frac{1}{2}}$. Furthermore, we define

$$\lambda_j^\pm = \frac{2-m}{2} \pm \sqrt{(1-\frac{m}{2})^2 + \Lambda_j}.$$

This means that $\lambda_j^\pm$ are the solutions of the quadratic equation $\lambda(m-2+\lambda) = \Lambda_j$,

$$\cdots \le \lambda_1^- < \lambda_0^- = 2-m \le 0 = \lambda_0^+ < \lambda_1^+ \le \cdots .$$

For an arbitrary point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $x' = (x_1, \ldots, x_m)$ and $x'' = (x_{m+1}, \ldots, x_n)$. Analogously, we set $\alpha' = (\alpha_1, \ldots, \alpha_m)$ and $\alpha'' = (\alpha_{m+1}, \ldots, \alpha_n)$ for an arbitrary multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the following estimate holds for all multi-indices α and γ (cf. [15] and [11, Theorems 6.6.2 and 6.6.5]):

$$\begin{aligned}\big|\partial_x^\alpha \partial_y^\gamma G(x,y)\big| \le c_{\alpha,\gamma}\, |x-y|^{2-n-|\alpha|-|\gamma|} \Big(\frac{|x'|}{|x'|+|x-y|}\Big)^{\min(0,\lambda_1^+-|\alpha'|-\varepsilon)} \\ \times \Big(\frac{|y'|}{|y'|+|x-y|}\Big)^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)}.\end{aligned} \tag{2}$$

Here, ε is an arbitrarily small positive number. For the description of the singularities of $G(x,y)$ near the edge M, we need the following functions

$$c_j(x'',y) = -\frac{\Gamma\big(\lambda_j^+ + \frac{n-2}{2}\big)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\lambda_j^+ + \frac{m}{2})}\; \frac{|y'|^{\lambda_j^+}\,\phi_j(\omega_y)}{(|y'|^2 + |x''-y''|^2)^{\lambda_j^+ + \frac{n-2}{2}}} \quad \text{for } j \ge 1 \tag{3}$$

and

$$c_0(x'',y) = -\frac{\Gamma\big(\frac{n-2}{2}\big)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\frac{m}{2})\,\text{mes}\,\Omega}\; \frac{1}{(|y'|^2 + |x''-y''|^2)^{\frac{n-2}{2}}}. \tag{4}$$

Furthermore, let σ be an arbitrary real number such that

$$\sigma > 0, \quad \sigma \ne \lambda_j^+ \quad \text{for all } j, \tag{5}$$

and let

$$m_{j,\sigma} = \Big[\frac{\sigma - \lambda_j^+}{2}\Big] \quad \text{for } j = 1,2,\ldots,$$

where $[s]$ denotes the integral part of s. It is proved in Section 2 (see Theorems 2.4 and 2.5) that $G(x,y)$ admits the decomposition

$$G(x,y) = \sum_{0<\lambda_j^+<\sigma} \sum_{k=0}^{m_{j,\sigma}} \frac{\Gamma(\lambda_j^+ + \frac{m}{2})\,(-\Delta_{x''})^k c_j(x'',y)}{4^k\, k!\; \Gamma(\lambda_j^+ + k + \frac{m}{2})}\, |x'|^{\lambda_j^+ + 2k}\, \phi_j(\omega_x) + P_\sigma + R_\sigma,$$

where

$$P_\sigma(x,y) = \sum_{k=0}^{[\frac{\sigma}{2}]} \frac{(-\Delta_{x''})^k c_0(x'',y)}{4^k\, k!\, (k-1+\frac{m}{2})_{(k)}}\, r^{2k} \tag{6}$$

is a quasipolynomial (polynomial in x' with coefficients depending on x'' and y), $s_{(k)} = \frac{\Gamma(s+1)}{\Gamma(s-k+1)}$ for $k < s+1$, and the remainder $R_\sigma(x,y)$ satisfies the estimate

$$\big|\partial_x^\alpha \partial_y^\gamma R_\sigma(x,y)\big| \le c_{\alpha\gamma} |x-y|^{2-n-|\alpha|-|\gamma|} \Big(\frac{|x'|}{|x-y|}\Big)^{\sigma-|\alpha'|} \Big(\frac{|y'|}{|x-y|}\Big)^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)} \tag{7}$$

for $2|x'| \le |x-y|$. The decomposition of the Green function of the Neumann problem is very similar to the case of the Dirichlet problem (see [14]). Of course, we have other eigenvalues Λ_j and eigenfunctions ϕ_j when considering the Neumann problem, but the principal new is the appearance of the quasipolynomial P_σ in the asymptotics. Although we have the same formulas for the coefficients c_j in the cases $m = 2$ and $m > 2$, the proof of the formula (4) in the case $m = 2$ differs from the proof for $m > 2$. The reason is that we have different formulas for the constant c_0 in the asymptotics of solutions of the Neumann problem in the cone K for the cases $m = 2$ and $m > 2$ (cf. formulas (20) and (21) in Lemma 2.2). For the Dirichlet problem, it was shortly discussed in [14] how the asymptotic of $G(x,y)$ can be obtained by integration with respect to the time t from the asymptotics of the Green function of the heat equation in a cone (cf. [4,5]). Since the condition II of [4, Section 1.2] is not satisfied for the Neumann problem if $m = 2$, this method is applicable for the problem (1) only if $m \ne 2$. In the present paper, we use only the estimate (2) for the Green function and the asymptotics of solutions of elliptic problems in a cone K.

In the second part of the paper (Section 3), we apply the result of Section 2 in order to describe the asymptotics of the solution $u \in \mathcal{H}_\mathcal{D}$ of the problem (1) with the right-hand side $f \in \mathcal{H}^*_\mathcal{D} \cap V^{l-2}_{p,\beta}(\mathcal{D})$. The main result (see Theorem 3.8) is the following. If the number $\sigma = l - \beta - \frac{m}{p}$ satisfies the inequalities (5), then the solution u admits the decomposition

$$u(x) = \Sigma(x) + Q(x) + v(x), \tag{8}$$

where

$$\Sigma(x) = \sum_{0<\lambda_j^+<\sigma} \sum_{k=0}^{m_{j,\sigma}} \frac{(-\Delta_{x''})^k (\mathcal{E}h_j)(x)}{4^k\, k!\, (\lambda_j^+ + k - 1 + \frac{m}{2})_{(k)}}\, r^{\lambda_j^+ + 2k} \phi_j(\omega_x), \tag{9}$$

$$Q(x) = \sum_{k=0}^{[\frac{\sigma}{2}]} \frac{(-\Delta_{x''})^k (\mathcal{E}h_0)(x)}{4^k\, k!\, (k-1+\frac{m}{2})_{(k)}}\, r^{2k}, \tag{10}$$

and the remainder v is a $V^l_{p,\beta}(\mathcal{D})$-function. The functions h_j are elements of the Besov space $B_p^{\sigma-\lambda_j^+}(\mathbb{R}^{n-m})$ and $\mathcal{E}$ is the extension operator (40). In the case

$m = 2$ and $p = 2$, such a decomposition of the solution into a "quasipolynomial" Q, a sum Σ of singular functions and a remainder $v \in V^l_{2,\beta}(\mathcal{D})$ was obtained by Costabel and Dauge in [1, Proposition 2.6] for a more general boundary value problem. The present paper contains not only a precise description of the singular terms but also precise formulas for the coefficients in (9) and (10). It is shown in Section 3 that

$$h_j(x'') = \int_{\mathcal{D}} c_j(x'', y)\, f(y)\, dy \tag{11}$$

for $j = 0, 1, \ldots$, where c_j is defined by (3) and (4).

Note that the "quasipolynomial" Q is a function of the space $W^l_{p,\beta}(\mathcal{D})$ if $\beta > -\frac{m}{p}$, where $W^l_{p,\beta}(\mathcal{D})$ is the closure of $C_0^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{W^l_{p,\beta}(\mathcal{D}} = \left(\int_{\mathcal{D}} \sum_{|\alpha| \le l} |x'|^{p\beta} \left|\partial_x^\alpha u(x)\right|^p dx \right)^{\frac{1}{p}}.$$

Thus, it makes sense to consider this term as a part of the regular remainder $Q + v$ in the decomposition (8). The singular part in this decomposition consists of the term Σ.

2. The Green function of the Neumann problem for the Poisson equation

Let $G(x, y)$ be the Green function of the problem (1). This means that

$$\Delta_x G(x, y) = \delta(x - y) \quad \text{for } x \in \mathcal{D},\ y \in \mathcal{D}, \tag{12}$$

$$\frac{\partial G(x, y)}{\partial n_x} = 0 \quad \text{for } x \in \partial\mathcal{D}\backslash M,\ y \in \mathcal{D}. \tag{13}$$

If ζ is an arbitrary function with bounded derivatives of order $\le l$ in $\mathcal{D}$, $\zeta = 1$ in a neighborhood of the point y, then the function $x \to \big(1 - \zeta(x)\big)\, G(x, y)$ belongs to the space $\mathcal{H}_{\mathcal{D}}$.

2.1. Equivalent norms in $\mathcal{H}_{\mathcal{D}}$. It follows from Hardy's inequality that

$$\int_{\mathcal{D}} |x'|^{-2}\, |u(x)|^2\, dx \le c\, \|u\|^2_{\mathcal{H}_{\mathcal{D}}}$$

for all $u \in C_0^\infty(\overline{\mathcal{D}})$ if $m \ge 3$. For the case $m \ge 3$, this means in particular that $\mathcal{H}_{\mathcal{D}} = V^1_{2,0}(\mathcal{D})$ and that the $\mathcal{H}_{\mathcal{D}}$- and $V^1_{2,0}(\mathcal{D})$-norms are equivalent. In the case $m = 2$, the norm

$$\|u\| = \left(\int_{\mathcal{D}} \left(|x|^{-2}\, |u(x)|^2 + |\nabla u(x)|^2\right) dx \right)^{\frac{1}{2}}$$

is equivalent to the $\mathcal{H}_\mathcal{D}$-norm. Obviously, the weight $|x|^{-2}$ can be replaced here by $|x-x_0|^{-2}$, where x_0 is an arbitrary point on the edge M.

2.2. Some properties of the Green function. First note that $G(x,y)$ is positively homogeneous of degree $2-n$, i.e.

$$G(ax,ay) = a^{2-n}\,G(x,y) \quad \text{for all } x,y\in\mathcal{D},\ a>0.$$

Obviously, it follows from (12), (13) that

$$\Delta_x G(x',x''-y'',y',0) = \delta(x'-y')\,\delta(x''-y'') \quad \text{for } x,y\in\mathcal{D},$$
$$\frac{\partial G(x',x''-y'',y',0)}{\partial n_x} = 0 \qquad \text{for } x\in\partial\mathcal{D}\backslash M,\ y\in\mathcal{D}.$$

This means that $G(x,y)$ depends only on x',y' and $x''-y''$. And what is more, the Green function depends only on x',y' and $|x''-y''|$, since the Laplace operator is invariant with respect to rotation and the outer normal vector n_x on $\partial\mathcal{D}\backslash M$ depends only on x'. Thus, the Green function has the representation

$$G(x',x'',y',y'') = \mathcal{G}(x',y',|x''-y''|).$$

If $\delta<1$ and one of the inequalities $|x'|\le\delta|x-y|$ or $|x'|^2\le\delta^2(|y'|^2+|x''-y''|^2)$ is satisfied, then

$$c_1\,|x-y|^2 \le |y'|^2+|x''-y''|^2 \le c_2\,|x-y|^2 \tag{14}$$

with certain positive constants c_1,c_2 depending on δ. Consequently, it follows from (2) that

$$\left|\partial_x^\alpha\partial_y^\gamma G(x,y)\right| \le \frac{c_{\alpha,\gamma}\,|x'|^{\min(0,\lambda_1^+-|\alpha'|-\varepsilon)}\,|y'|^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)}}{(|y'|^2+|x''-y''|^2)^{(n-2+|\alpha|+|\gamma|+\min(0,\lambda_1^+-|\alpha'|-\varepsilon)+\frac12\min(0,\lambda_1^+-|\gamma'|-\varepsilon))}}$$

for $|x'|\le\delta|x-y|$, $\delta<1$. An analogous estimate holds for $|y'|<\delta|x-y|$, while

$$\left|\partial_x^\alpha\partial_y^\gamma G(x,y)\right| \le c_{\alpha,\beta}\,|x-y|^{2-n-|\alpha|-|\beta|} \quad \text{for } \min(|x'|,|y'|)\ge\delta|x-y|.$$

In the following, we use the notation

$$r=|x'|,\quad \rho=|y'|,\quad \omega_x=\frac{x'}{|x'|},\quad \omega_y=\frac{y'}{|y'|}.$$

Lemma 2.1. *Let $G(x,y)$ be the Green function of the problem* (1). *Then*

$$\int_\Omega G(x',x'',y',y'')\,\phi_j(\omega_x)\,d\omega_x = G_j(r,\rho,|x''-y''|)\,\phi_j(\omega_y) \tag{15}$$

with a certain function G_j on $\mathbb{R}_+\times\mathbb{R}_+\times\overline{\mathbb{R}}_+$, where $\mathbb{R}_+$ is the interval $(0,\infty)$.

Proof. Suppose that $f \in \mathcal{H}_{\mathcal{D}}^*$ and $u \in \mathcal{H}_{\mathcal{D}}$ is a solution of the problem

$$\int_{\mathcal{D}} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \text{for all } v \in \mathcal{H}_{\mathcal{D}}. \tag{16}$$

Let $\mathcal{H}_j$ be the completion of $C_0^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^{n-m})$ with respect to the norm

$$\|g\|_{\mathcal{H}_j} = \left(\int_{\mathbb{R}^{n-m}} \int_0^\infty r^{m-1} \left(|\nabla_{x''} g|^2 + |\partial_r g|^2 + \Lambda_j \, r^{-2} \, |g|^2 \right) \, dr \, dx'' \right)^{\frac{1}{2}}.$$

Note that the norm in $\mathcal{H}_0$ is equivalent to

$$\|g\| = \left(\int_{\mathbb{R}^{n-m}} \int_0^\infty r^{m-1} \left(|\nabla_{x''} g|^2 + |\partial_r g|^2 + \left(r^2 + |x''|^2 \right)^{-1} |g|^2 \right) \, dr \, dx'' \right)^{\frac{1}{2}}.$$

We put $v(x) = g(r, x'') \, \phi_j(\omega_x)$, where $g \in \mathcal{H}_j$. Then $\int_{\mathcal{D}} |\nabla v|^2 \, dx = \|g\|^2_{\mathcal{H}_j}$. Inserting $v(x) = g(r, x'') \, \phi_j(\omega_x)$ into (16), we obtain

$$\int_{\mathcal{D}} \left(\nabla_{x''} u \cdot \nabla_{x''} g + (\partial_r u) \, \partial_r g + \Lambda_j \, r^{-2} u \, g \right) \phi_j(\omega_x) \, dx = \langle f, v \rangle.$$

We define $U_j(r, x'') = \int_\Omega u(x) \, \phi_j(\omega_x) \, d\omega_x$ and

$$\langle F_j, g \rangle = \langle f, v \rangle \quad \text{for all } g \in \mathcal{H}_j, \ \text{where } v(x) = g(r, x'') \, \phi_j(\omega_x).$$

Obviously, the mapping $g \to \langle F_j, g \rangle$ defines a linear and continuous functional on $\mathcal{H}_j$, and U_j is the uniquely determined solution of the problem

$$\int_0^\infty \int_{\mathbb{R}^{n-m}} r^{m-1} \left(\nabla_{x''} U_j \cdot \nabla_{x''} g + (\partial_r U_j) \, \partial_r g + \Lambda_j \, r^{-2} U_j \, g \right) dx'' \, dr = \langle F_j, g \rangle$$

for all $g \in \mathcal{H}_j$. Let $G_j(r, \rho, |x'' - y''|)$ be the Green function of this problem. Then

$$\begin{aligned} U_j(r, x'') &= \int_0^\infty \int_{\mathbb{R}^{n-m}} G_j(r, \rho, |x'' - y''|) \, F_j(\rho, y'') \, dy'' \, d\rho \\ &= \int_{\mathcal{D}} G_j(r, \rho, |x'' - y''|) \, \phi_j(\omega_y) \, f(y) \, dy. \end{aligned}$$

On the other hand, it follows from the definition of U_j that

$$U_j(r, x'') = \int_\Omega u(x) \, \phi_j(\omega_x) \, d\omega_x = \int_{\mathcal{D}} \int_\Omega G(x, y) \phi_j(\omega_x) \, f(y) \, d\omega_x \, dy.$$

Comparing the last two equalities, we obtain (15). □

2.3. The first terms in the asymptotics of Green's matrix. We assume that the integer number $l \geq 2$ and the real numbers $p \in (1,\infty)$ and β satisfy the inequalities

$$0 < \sigma \overset{def}{=} l - \beta - \frac{m}{p} < 2, \quad \sigma \neq \lambda_j^+ \quad \text{for all } j. \tag{17}$$

Using well-known results concerning the asymptotics of solutions of elliptic boundary value problems in a cone, we can prove the following lemma.

Lemma 2.2. *Let l, p, β satisfy the condition* (17). *Then*

$$G(x,y) = c_0(x'',y) + \sum_{0<\lambda_j^+<\sigma} c_j(x'',y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + R_\sigma(x',x'',y), \tag{18}$$

where

$$c_j(x'',y) = -\frac{1}{2\lambda_j^+ + m - 2} \int_K r^{\lambda_j^-}\, \phi_j(\omega_x)\, \Delta_{x'} G(x,y)\, dx' \quad \text{for } j \geq 1, \tag{19}$$

$$c_0(x'',y) = -\frac{1}{(m-2)\,\mathrm{mes}\,\Omega} \int_K r^{2-m}\, \Delta_{x'} G(x,y)\, dx' \quad \text{if } m > 2, \tag{20}$$

$$c_0(x'',y) = \frac{1}{\mathrm{mes}\,\Omega} \int_K \log r\, \Delta_{x'} G(x,y)\, dx' \quad \text{if } m = 2, \tag{21}$$

and $R_\sigma(\cdot,x'',y) \in V^l_{p,\beta}(K)$ for $x'' \neq y''$.

Proof. Suppose that $x'' \neq y''$. Then

$$\Delta_{x'} G(x,y) = -\Delta_{x''} G(x,y) \quad \text{for } x' \in K, \qquad \frac{\partial G(x,y)}{\partial n_x}\Big|_{x' \in \partial K \setminus \{0\}} = 0.$$

It follows from (2) that $\Delta_{x''} G(\cdot,x'',y) \in V^{l-2}_{p,\beta}(K)$. If $m > 2$, then $G(\cdot,x'',y) \in V^1_{2,0}(K)$. Consequently, we obtain the decomposition (18) with the coefficients (19) and (20) by means of [8, Theorems 3.2 and 3.4] (in the case $p = 2$ see also [6, Theorems 6.1.5 and 6.1.6]).

We consider the case $m = 2$. Let $\zeta_1 = \zeta_1(r)$ be a smooth cut-off function equal to 1 for $r < \frac{1}{2}$ and vanishing for $r > 1$. Furthermore, let $\zeta_2 = 1 - \zeta_1$. Then $\zeta_1 G(\cdot,x'',y) \in V^1_{2,\varepsilon}(K)$ and $\zeta_2 G(\cdot,x'',y) \in V^1_{2,-\varepsilon}(K)$ with arbitrary positive ε. Furthermore, it follows from the equality

$$\Delta_{x'}(\zeta_k G(\cdot,x'',y)) = -\zeta_k \Delta_{x''} G(\cdot,x'',y) + 2\nabla_{x'}\zeta_k \cdot \nabla_{x'} G(\cdot,x'',y) + G(\cdot,x'',y)\, \Delta_{x'}\zeta_k$$

that $\Delta_{x'}(\zeta_k G(\cdot,x'',y)) \in V^{l-2}_{p,\beta}(K)$ for $k = 1$ and $k = 2$. Consequently by [8, Theorem 3.2],

$$\zeta_1 G(x,y) = c_0(x'',y) + d_0(x'',y)\log r + \sum_{0<\lambda_j^+<\sigma} c_j^{(1)}(x'',y) r^{\lambda_j^+} \phi_j(\omega_x) + v_1(x,y) \tag{22}$$

and

$$\zeta_2 G(x,y) = \sum_{0<\lambda_j^+<\sigma} c_j^{(2)}(x'',y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + v_2(x,y), \tag{23}$$

where $v_k(\cdot,x'',y) \in V_{2,\beta}^l(K)$ for $k=1$ and $k=2$. Applying the coefficients formula in [8, Theorem 3.4], we get

$$c_j^{(k)}(x'',y) = -\frac{1}{2\lambda_j^+}\int_K r^{\lambda_j^-}\, \phi_j(\omega)\, \Delta_{x'}\zeta_k G(x,y)\, dx' \quad \text{for } k=1,2,\ j\ge 1,$$
$$c_0(x'',y) = \frac{1}{\operatorname{mes}\Omega}\int_K (1+\log r)\, \Delta_{x'}\zeta_1 G(x,y)\, dx',$$
$$d_0(x'',y) = -\frac{1}{\operatorname{mes}\Omega}\int_K \Delta_{x'}\zeta_1 G(x,y)\, dx'.$$

Since $\Delta_{x'}(1+\log r) = 0$ and $|\partial_{x'}^{\alpha'} G(x,y)| \le c\,|x'|^{2-n-|\alpha'|}$ for $|x'| > \frac{1}{2}$, $x'' \ne y''$, where the constant c depends only on x'' and y, integration by parts yields

$$\int_K (1+\log r)\, \Delta_{x'}\zeta_2 G(x,y)\, dx' = \int_K \Delta_{x'}\zeta_2 G(x,y)\, dx' = 0.$$

Consequently,

$$c_0 = \frac{1}{\operatorname{mes}\Omega}\int_K (1+\log r)\, \Delta_{x'} G(x,y)\, dx', \quad d_0 = -\frac{1}{\operatorname{mes}\Omega}\int_K \Delta_{x'} G(x,y)\, dx'.$$

Adding the equalities (22) and (23), we obtain

$$G(x,y) = c_0(x'',y) + d_0(x'',r)\, \log r + \sum_{0<\lambda_j^+<\sigma} c_j(x'',y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + v_1 + v_2.$$

Here, the coefficients $c_j = c_j^{(1)} + c_j^{(2)}$ are given by (19) for $j \ge 1$. Since $\nabla_{x'} G(\cdot,x'',y) \in \big(L_2(K)\big)^2$, we conclude that the coefficient

$$d_0(x'',y) = -\frac{1}{\operatorname{mes}\Omega}\int_K \Delta_{x'} G(x,y)\, dx'$$

in (22) is zero. This means in particular that the coefficient $c_0(x'',y)$ is given by the formula (21). The proof is complete. □

Next, we derive explicit formulas for the coefficients c_j in (18).

Lemma 2.3. *The coefficients c_j in* (18) *are given by the formulas* (3) *and* (4).

Proof. 1) We show that the functions c_j have the form

$$c_j(x'',y) = \rho^{2-n-\lambda_j^+}\, h_j\Big(\frac{|x''-y''|}{\rho}\Big)\, \phi_j(\omega_y) \tag{24}$$

for $j = 0, 1, \ldots$. For $j \ge 1$, the proof is the same as for the Dirichlet problem (see [14, Lemma 2.3]). We consider the case $j = 0$. By Lemma 2.2, we have

$$c_0(x'',y) = \int_K f(r)\, \Delta_{x'} G(x,y)\, dx' = \int_K f(r)\, \big(\delta(x-y) - \Delta_{x''} G(x,y)\big)\, dx',$$

where $f(r) = (\text{mes}\,\Omega)^{-1} \log r$ if $m = 2$ and $f(r) = \big((2-m)\,\text{mes}\,\Omega\big)^{-1} r^{2-m}$ if $m > 2$. Using the equality $\int_\Omega G(x,y)\, d\omega_x = G_0(r,\rho,|x''-y''|)$ (see Lemma 2.1), we obtain the representation $c_0(x'',y) = g(\rho,|x''-y''|)$. Furthermore, it follows from the obvious equality $c_0(x'',y) = G(0,x'',y)$ and from the homogeneity of the function $G(x,y)$ that the function g is positively homogeneous of degree $2-n$. Consequently,

$$c_0(x'',y) = g(\rho,|x''-y''|) = \rho^{2-n}\, g\Big(1, \frac{|x''-y''|}{\rho}\Big).$$

This proves (24) for $j = 0$.

2) Next, we show that

$$|c_j(x'',y)| \le c\,|x''-y''|^{2-n-\lambda_j^+} \quad \text{for } \rho = |y'| < |x''-y''|. \tag{25}$$

For $j = 0$, this follows directly from the equality $c_0(x'',y) = G(0,x'',y)$ and from the estimate (2). Suppose that $j \ge 1$. If $|y'| < |x''-y''|$, then

$$\frac{r^2}{2} + |x''-y''|^2 < 2|x-y|^2 < 4r^2 + 6|x''-y''|^2.$$

We define $K_1 = \{x' \in K : 2r < |x''-y''|\}$ and $K_2 = K \backslash K_1$. By (19),

$$c_j(x'',y) = -\int_{K_1} V_j(x')\, \Delta_{x''} G(x,y)\, dx' + \int_{K_2} V_j(x')\, \Delta_{x'} G(x,y)\, dx'.$$

where

$$V_j(x') = -\frac{1}{2\sigma_j}\, r^{\lambda_j^-}\, \phi_j(\omega_x), \quad \sigma_j = \lambda_j^+ + \frac{m-2}{2}.$$

By (2), we have

$$\big|\Delta_{x''} G(x,y)\big| \le c\,|x''-y''|^{-n} \quad \text{for } x' \in K_1,\ \rho < |x''-y''|$$

and

$$\big|\Delta_{x'} G(x,y)\big| \le c\, r^{-n} \qquad \text{for } x' \in K_2,\ \rho < |x''-y''|$$

Here we used the fact that $r^2 < 2|x-y|^2 < 28r^2$ if $\rho < |x''-y''| < 2r$. Thus,

$$|c_j(x'',y)| \le c\Big(|x''-y''|^{-n}\int_{K_1} r^{\lambda_j^-}\,dx' + \int_{K_2} r^{\lambda_j^- -n}\,dx'\Big).$$

Since $0 < \lambda_j^- + m = 2 - \lambda_j^+ < n$, we get (25).

3) Analogously to the Dirichlet problem (cf. [14, Lemma 2.3]), we obtain

$$\int_{\mathbb{R}^{n-m}} c_j(x'',y)\,dx'' = V_j(y') \quad \text{for } j \ge 1. \tag{26}$$

In the same way, the equality

$$\int_{\mathbb{R}^{n-m}} c_0(x'',y)\,dx'' = -\frac{\rho^{2-m}}{(m-2)\,\mathrm{mes}\,\Omega} \quad \text{for } m > 2 \tag{27}$$

holds. Suppose that $m = 2$ and $i = 1$ or $i = 2$. Using (21), we get

$$\begin{aligned}\int_{\mathbb{R}^{n-2}} \partial_{y_i} c_0(x'',y)\,dx'' &= \frac{1}{\mathrm{mes}\,\Omega}\int_{\mathbb{R}^{n-2}}\int_K \log r\,\partial_{y_i}\Delta_{x'}G(x,y)dx'\,dx'' \\ &= \frac{1}{\mathrm{mes}\,\Omega}\int_{\mathcal{D}} \log r\,\partial_{y_i}\big(\delta(x-y) - \Delta_{x''}G(x,y)\big)\,dx \\ &= -\frac{1}{\mathrm{mes}\,\Omega}\int_{\mathcal{D}} \log r\,\partial_{x_i}\,\delta(x-y)\,dx = \frac{1}{\mathrm{mes}\,\Omega}\,\partial_{y_i}\log\rho.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^{n-m}} \partial_{y_i} c_0(x'',y)\,dx'' = \frac{1}{\mathrm{mes}\,\Omega}\,\frac{y_i}{\rho^2} \quad \text{for } m = 2 \text{ and } i \in \{1,2\}. \tag{28}$$

4) Using (19)–(21) and the equality $\Delta_y G(x,y) = 0$ for $x'' \ne y''$, we obtain

$$\Delta_y c_j(x'',y) = 0 \quad \text{for } x'' \ne y''.$$

Hence, the function h_j in (24) is a solution of the ordinary differential equation

$$(1+t^2)h_j''(t) + \Big((a+2b+2)t + \frac{a}{t}\Big)\,h_j'(t) + 2(a+1)b\,h_j(t) = 0,$$

where $a = n-m-1$ and $b = \lambda_j^+ - 1 + \frac{n}{2}$. This equation has the solution

$$h_j(t) = (1+t^2)^{-b}\Big(C_j + D_j\int_1^t s^{-a}\,(1+s^2)^{b-1}\,ds\Big)$$

with arbitrary constants C_j and D_j. Thus,

$$c_j(x'',y) = \frac{\rho^{\lambda_j^+}\,\phi_j(\omega_y)}{(\rho^2+|x''-y''|^2)^b}\Big(C_j + D_j\int_1^{\frac{|x''-y''|}{\rho}} s^{-a}\,(s^2+1)^{b-1}\,ds\Big).$$

By (25), the constant D_j must be zero, and (26) implies

$$C_j = -\frac{\Gamma\big(\sigma_j + \frac{n-m}{2}\big)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\sigma_j+1)} \quad \text{for } j \ge 1.$$

Furthermore, the equalities (27) and (28) yield

$$C_0 = -\frac{\Gamma\big(\frac{n-2}{2}\big)}{2\pi^{\frac{n-m}{2}}\,\Gamma(\frac{m}{2})\,(\text{mes}\,\Omega)^{\frac12}}$$

both for $m > 2$ and $m = 2$. This proves (3) and (4). □

Next, we prove point estimates for the remainder $R_\sigma(x,y)$ in (18)

Theorem 2.4. *Let σ be an arbitrary real number satisfying the inequalities* (17)*. Then $G(x,y)$ admits the decomposition*

$$G(x,y) = c_0(x'',y) + \sum_{0<\lambda_j^+<\sigma} c_j(x'',y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + R_\sigma(x,y),$$

where the coefficients c_j are given by the formulas (3), (4) *and $R_\sigma(x,y)$ satisfies the estimate* (7) *for $2|x'|^2 < |y'|^2 + |x''-y''|^2$.*

Proof. Analogously to (18), we obtain

$$\partial_{x''}^{\alpha''}\partial_y^\gamma G(x,y) = c_0^{(\alpha'',\gamma)}(x'',y) + \sum_{0<\lambda_j^+<\sigma} c_j^{(\alpha'',\gamma)}(x'',y)\, r^{\lambda_j^+}\,\phi_j(\omega) + R_\sigma^{(\alpha'',\gamma)}(x,y),$$

where $R_\sigma^{(\alpha'',\gamma)}(\cdot,x'',y) \in V_{p,\beta}^l(K)$. For the coefficients, we have the formula

$$c_j^{(\alpha'',\gamma)}(x'',y) = -\frac{1}{2\lambda_j^+ + m - 2}\int_K r^{\lambda_j^-}\,\phi_j(\omega)\,\Delta_{x'}\partial_{x''}^{\alpha''}\partial_y^\gamma G(x',x'',y)\,dx'$$

if $j \ge 1$. This means in particular that $c_j^{(\alpha'',\gamma)}(x'',y) = \partial_{x''}^{\alpha''}\partial_y^\gamma c_j(x'',y)$ for $j \ge 1$, where c_j is given by (3). Analogously,

$$c_0^{(\alpha'',\gamma)}(x'',y) = \partial_{x''}^{\alpha''}\partial_y^\gamma c_0(x'',y) = \partial_{x''}^{\alpha''}\partial_y^\gamma G(0,x'',y).$$

Consequently, $\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma(\cdot,x'',y) = R_\sigma^{(\alpha'',\gamma)}(\cdot,x'',y) \in V_{p,\beta}^l(K)$. Let $\psi = \psi(r)$ be a smooth cut-off function which is equal to 1 for $r < \frac12$ and to zero for $r > \frac34$. Furthermore, let

$$\zeta(x,y) = \psi\left(\frac{|x'|^2}{|y'|^2 + |x''-y''|^2}\right). \tag{29}$$

Then the normal derivative of $(\zeta\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma)(\cdot,x'',y)$ is equal to zero on $\partial K\backslash\{0\}$ and

$$\Delta_{x'}\big(\zeta(x,y)\,\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma(x,y)\big) = f(x,y),$$

where

$$\begin{aligned} f(x,y) &= \zeta\Delta_{x'}\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma + 2\nabla_{x'}\zeta\cdot\nabla_{x'}\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma + \partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma\,\Delta_{x'}\zeta \\ &= -\zeta\,\Delta_{x''}\partial_{x''}^{\alpha''}\partial_y^\gamma G + 2\nabla_{x'}\zeta\cdot\nabla_{x'}\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma + \partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma\,\Delta_{x'}\zeta. \end{aligned}$$

By [8, Theorem 3.1], the function $\zeta\,\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma$ satisfies the estimate

$$\|\zeta\,\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma(\cdot,x'',y)\|_{V_{p,\beta}^l(K)} \le c\,\|f(\cdot,x'',y)\|_{V_{p,\beta}^{l-2}(K)}$$

with a constant c independent of x'' and y. We estimate the $V_{p,\beta}^{l-2}$-norm of f. The estimate (2) implies

$$\big|\partial_{x'}^{\alpha'}\big(\zeta\,\Delta_{x''}\partial_{x''}^{\alpha''}\partial_y^\gamma G\big)\big| \le \frac{c}{d^{n+|\alpha'|+|\alpha''|+|\gamma|}}\left(\frac{|x'|}{d}\right)^{\min(0,\lambda_1^+-|\alpha'|-\varepsilon)}\left(\frac{|y'|}{d}\right)^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)},$$

where $d=\sqrt{|y'|^2+|x''-y''|^2}$. Since $\sigma = l-\beta-\frac{m}{p}<2$, this implies

$$\big\|\partial_{x'}^{\alpha'}\big(\zeta\,\Delta_{x''}\partial_{x''}^{\alpha''}\partial_y^\gamma G(\cdot,x'',y)\big)\big\|_{V_{p,\beta}^{l-2}(K)} \le c\,d^{2-n-|\alpha''|-|\gamma|-\sigma}\left(\frac{|y'|}{d}\right)^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)}.$$

The same estimate holds for the $V_{p,\beta}^{l-2}$-norms of the terms $\nabla_{x'}\zeta\cdot\nabla_{x'}\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma$ and $\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma\,\Delta_{x'}\zeta$. Thus,

$$\|\zeta\,\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma(\cdot,x'',y)\|_{V_{p,\beta}^l(K)} \le c\,d^{2-n-|\alpha''|-|\gamma|-\sigma}\left(\frac{|y'|}{d}\right)^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)}.$$

If $|\alpha'|<l-\frac{m}{p}$, then

$$\|r^{|\alpha'|-\sigma}\partial_{x'}^{\alpha'}(\zeta\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma)(\cdot,x'',y)\|_{L_\infty(K)} \le c\,\|\zeta\partial_{x''}^{\alpha''}\partial_y^\gamma R_\sigma(\cdot,x'',y)\|_{V_{p,\beta}^l(K)}$$

with a constant c independent of x'' and y (see e.g. [11, Lemma 1.2.3]). This leads to the estimate

$$\big|\partial_x^\alpha\partial_y^\gamma R_\sigma(x,y)\big| \le c\,d^{2-n-|\alpha|-|\gamma|}\left(\frac{|x'|}{d}\right)^{\sigma-|\alpha'|}\left(\frac{|y'|}{d}\right)^{\min(0,\lambda_1^+-|\gamma'|-\varepsilon)}$$

for $2|x'|^2<|y'|^2+|x''-y''|^2$. Since d can be estimated by means of (14), we get (7). The theorem is proved. □

2.4. Terms of higher order in the asymptotics of Green's function. Now let σ be an arbitrary positive number such that $\sigma \neq \lambda_j^+$ for all j. We define

$$G_\sigma(x,y) = \sum_{0<\lambda_j^+<\sigma} \sum_{k=0}^{m_{j,\sigma}} \frac{(-\Delta_{x''})^k c_j(x'',y)}{4^k\, k!\, (\sigma_j+k)_{(k)}}\, r^{\lambda_j^+ + 2k}\, \phi_j(\omega_x) \tag{30}$$

($G_\sigma(x,y) = 0$ if $\sigma < \lambda_1^+$), where $c_j(x'',y)$ is defined by (3), $\sigma_j = \lambda_j^+ + \frac{m-2}{2}$, and

$$s_{(k)} = s(s-1)\cdots(s-k+1) \quad \text{for } k = 1,2,\dots, \quad s_{(0)} = 1.$$

Furthermore, let P_σ be the quasipolynomial (6).

Theorem 2.5. *Let σ satisfy the condition* (5)*. Then $G(x,y)$ admits the decomposition*

$$G(x,y) = G_\sigma(x,y) + P_\sigma(x,y) + R_\sigma(x,y), \tag{31}$$

where R_σ satisfies the estimate (7) *for* $2|x'|^2 < |y'|^2 + |x''-y''|^2$.

Proof. Obviously, $G_\sigma(x,y) = G_{\sigma+\varepsilon}(x,y)$ for sufficiently small positive ε. Consequently, we may assume without loss of generality that $\frac{\sigma-\lambda_j^+}{2}$ is not integer for $j = 0,1,\dots,\ \lambda_j^+ < \sigma$.

Let l be an arbitrary integer, $l \geq 2$, p an arbitrary real number, $p \in (1,\infty)$, and $\beta = l - \sigma - \frac{m}{p}$. We prove by induction in $m_0 = [\frac{\sigma}{2}]$ that $G(x,y)$ admits the decomposition (31), where $R_\sigma(x,y)$ satisfies (7) and

$$\partial_{x''}^\alpha \partial_y^\gamma R_\sigma(\cdot,x'',y) \in V_{p,\beta}^l(K) \quad \text{for all } \alpha,\gamma;\ x'' \in \mathbb{R}^{n-m},\ y \in \mathcal{D},\ x'' \neq y''.$$

For $m_0 = 0$, this was shown in the proof of Theorem 2.4. Suppose now that $m_0 = N \geq 1$, i.e. $2N < \sigma < 2N+2$, and the assertion is proved for $\sigma < 2N$. We set $\sigma' = \sigma - 2$. Then $m_{j,\sigma'} = m_{j,\sigma} - 1$ and, by the induction hypothesis, we have

$$G(x,y) = P_{\sigma'}(x,y) + G_{\sigma'}(x,y) + R_{\sigma'}(x,y), \tag{32}$$

where $R_{\sigma'}$ satisfies the estimate (7) with σ' instead of σ and

$$\partial_{x''}^\alpha \partial_y^\gamma R_{\sigma'}(\cdot,x'',y) \in V_{p,\beta+2}^l(K) \quad \text{for all } \alpha,\gamma;\ x'' \in \mathbb{R}^{n-m},\ y \in \mathcal{D},\ x'' \neq y''.$$

Since $G(x,y)$, $P_{\sigma'}(x,y)$ and $G_{\sigma'}(x,y)$ are positively homogenous of degree $2-n$ and depend only on $x', y'|x''-y''|$, the same is true for the remainder $R_{\sigma'}$. The equality $\Delta_x G(x,y) = 0$ for $x'' \neq y''$ implies

$$\Delta_{x'} R_{\sigma'}(x,y) = -\Delta_x\big(P_{\sigma'}(x,y) + G_{\sigma'}(x,y)\big) - \Delta_{x''} R_{\sigma'}(x,y)$$

for $x'' \neq y''$. Using formulas $\Delta_{x'} r^{2k} = 2k(2k+m-2) r^{2k-2}$ and $\Delta_{x'} r^{\lambda_j^+ + 2k}\phi_j(\omega_x) = 4k(\sigma_j+k)\, r^{\lambda_j^+ + 2k-2}\phi_j(\omega_x)$, we get

$$-\Delta_x\big(P_{\sigma'}(x,y) + G_{\sigma'}(x,y)\big) = \Delta_{x'}\Sigma'(x,y),$$

where

$$\Sigma' = \frac{(-\Delta_{x''})^N c_j(x'',y)\, r^{2N}}{4^N\, N!\, (N-1+\frac{m}{2})_{(N)}} + \sum_{0<\lambda_j^+<\sigma'} \frac{(-\Delta_{x''})^{m_{j,\sigma}} c_j(x'',y)\, r^{\lambda_j^+ + 2m_{j,\sigma}}\, \phi_j(\omega_x)}{4^{m_{j,\sigma}}\, m_{j,\sigma}!\, (\sigma_j + m_{j,\sigma})_{(m_{j,\sigma})}}.$$

Therefore,

$$\Delta_{x'} R_{\sigma'}(x,y) = \Delta_{x'} \Sigma'(x,y) - \Delta_{x''} R_{\sigma'}(x,y) \quad \text{for } x'' \neq y'', \tag{33}$$

Let ψ be a smooth function on $\mathbb{R}_+ = (0,\infty)$, $\psi(r) = 1$ for $r < \frac{1}{2}$, $\psi(r) = 0$ for $r > \frac{3}{4}$. Furthermore, let the function χ be defined as

$$\chi(x',x'',y'') = \psi\left(\frac{2r}{|x''-y''|}\right).$$

Then by (33), $\Delta_{x'}\big(R_{\sigma'} - \chi\Sigma'\big) = \Delta_{x'}\big(1-\chi\big)\Sigma' - \Delta_{x''} R_{\sigma'}$ for $x'' \neq y''$. Here, by the induction hypotheses and by the definition of Σ',

$$\partial_{x''}^\alpha \partial_y^\gamma (R_{\sigma'} - \chi\Sigma')(\cdot, x'', y) \in V_{p,\beta+2}^l(K), \quad \partial_{x''}^\alpha \partial_y^\gamma \big(\Delta_{x'}(1-\chi)\Sigma'\big)(\cdot,x'',y) \in V_{p,\beta}^{l-2}(K)$$

for arbitrary $x'' \in \mathbb{R}^{n-2}$, $y \in \mathcal{D}$, $x'' \neq y''$, and for all multi-indices α,γ. Furthermore, $\partial_{x''}^\alpha \partial_y^\gamma \Delta_{x''} R_{\sigma'}(\cdot,x'',y) \in V_{p,\beta}^{l-2}(K)$. Applying [8, Theorem 3.2], we obtain

$$\partial_{x''}^\alpha \partial_y^\gamma \left(R_{\sigma'}(x,y) - \chi\,\Sigma'(x,y)\right) = \sum_{\sigma'<\lambda_j^+<\sigma} d_{j,\alpha,\gamma}(x'',y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + v_{\alpha,\gamma}(x,y), \tag{34}$$

where $v_{\alpha,\gamma}(\cdot,x'',y) \in V_{p,\beta}^l(K)$. The coefficients $d_{j,\alpha,\gamma}$ are given by the formula (cf. [8, Theorem 3.4])

$$d_{j,\alpha,\gamma}(x'',y) = \int_K V_j(x')\, \partial_{x''}^\alpha \partial_y^\gamma \Delta_{x'} \Big(R_{\sigma'}(x,y) - \chi\Sigma'(x,y)\Big)\, dx',$$

where $V_j(x') = -\frac{1}{2\sigma_j} r^{\lambda_j^-}\, \phi_j(\omega_x)$. Obviously, $d_{j,\alpha,\gamma} = \partial_{x''}^\alpha \partial_y^\gamma d_{j,0,0}$. In the same way as in the proof of Lemma 2.3, one can show that the functions $d_j = d_{j,0,0}$ have the same properties as the functions (19), i.e. $\Delta_y d_j(x'',y) = 0$ for $x'' \neq y''$,

$$d_j(x'',y) = \rho^{2-n-\lambda_j^+}\, h_j\left(\frac{|x''-y''|}{\rho}\right) \phi_j(\omega_y), \qquad \int_{\mathbb{R}^{n-m}} d_j(x'',y)\, dx'' = V_j(y'),$$

and $|d_j(x'',y)| \leq c\,|x''-y''|^{2-n-\lambda_j^+}$ for $|y'| \leq |x''-y''|$. Thus, analogously to Lemma 2.3, the formula

$$d_j(x'',y) = -\frac{\Gamma\big(\lambda_j^+ + \frac{n-2}{2}\big)}{2\pi^{\frac{n-m}{2}}\, \Gamma(\lambda_j^+ + \frac{m}{2})}\; \frac{|y'|^{\lambda_j^+}\, \phi_j(\omega_y)}{(|y'|^2 + |x''-y''|^2)^{\lambda_j^+ + \frac{n-2}{2}}}$$

holds for $\sigma' < \lambda_j^+ < \sigma$. Furthermore, it follows from (34) that

$$R_{\sigma'}(x,y) = \Sigma'(x,y) + \sum_{\sigma'<\lambda_j^+<\sigma} d_j(x'',y)\, r^{\lambda_j^+}\, \phi_j(\omega_x) + R_\sigma(x,y),$$

where $R_\sigma(x,y) = v_{0,0}(x,y) - (1-\chi)\,\Sigma'(x,y)$ and $\partial_{x''}^\alpha \partial_y^\gamma v_{0,0}(\cdot,x'',y) = v_{\alpha,\gamma}(\cdot,x'',y) \in V_{p,\beta}^l(K)$. This together with (32) yields (31). The estimate (7) for R_σ holds in the same way as in the proof of Theorem 2.4 (see also the proof of [14, Theorem 2.2]). □

One can easily show by means of (2) and (14) that the estimate (7) for R_σ is valid if one of the inequalities $|x'| \le \delta |x-y|$ or $|x'|^2 \le \delta^2(|y'|^2 + |x''-y''|^2)$ with an arbitrary positive $\delta < 1$ is satisfied.

3. Asymptotics of solutions of the Neumann problem

Now, we consider the variational solution $u \in \mathcal{H}_\mathcal{D}$ of the problem (1) with the right-hand side $f \in \mathcal{H}_\mathcal{D}^* \cap V_{p,\beta}^{l-2}(\mathcal{D})$. Here and in the sequel, we assume that $1 < p < \infty$, l is an integer, $l \ge 2$, and that the number

$$\sigma = l - \beta - \frac{m}{p}$$

satisfies the condition (5). Using the asymptotics of the Green function $G(x,y)$, we are able to describe the singularities of the solution

$$u(x) = \int_\mathcal{D} G(x,y)\, f(y)\, dy$$

of the problem (1). Let ψ be a smooth function on $\mathbb{R}_+ = (0,\infty)$, $\psi(r) = 1$ for $r < \frac{1}{2}$, $\psi(r) = 0$ for $r > \frac{3}{4}$, and let the function ζ be defined by (29). Furthermore, let P_σ, G_σ be the same functions (6) and (30) as in Section 2. Then by Theorem 2.5, the function u admits the decomposition

$$u(x) = S(x) + q(x) + v(x),$$

where

$$S(x) = \int_\mathcal{D} \zeta(x,y)\, G_\sigma(x,y)\, f(y)\, dy, \quad q(x) = \int_\mathcal{D} \zeta(x,y) P_\sigma(x,y)\, f(y)\, dy$$

and

$$v(x) = \int_\mathcal{D} \zeta(x,y)\, R_\sigma(x,y)\, f(y)\, dy + \int_\mathcal{D} \big(1 - \zeta(x,y)\big)\, G(x,y)\, f(y)\, dy. \qquad (35)$$

Here, R_σ satisfies the estimate (7) on the support of the function ζ. Note that the functions $S(x)$, $q(x)$ and $v(x)$ satisfy the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\mathcal{D}\backslash M$ since $\zeta(x,y)$ depends only on $r = |x'|, x''$ and y. Obviously,

$$S(x) = \sum_{0<\lambda_j^+<\sigma} \sum_{k=0}^{m_{j,\sigma}} \frac{H_{j,k}(x)}{4^k\, k!\, (\lambda_j^+ + k - 1 + \frac{m}{2})_{(k)}}\, r^{\lambda_j^+ + 2k} \phi_j(\omega_x), \tag{36}$$

and

$$q(x) = \sum_{k=0}^{[\frac{\sigma}{2}]} \frac{H_{0,k}(x)}{4^k\, k!\, (k - 1 + \frac{m}{2})_{(k)}}\, r^{2k}, \tag{37}$$

where

$$H_{j,k}(x) = \int_{\mathcal{D}} \zeta(x,y)\, (-\Delta_{x''})^k\, c_j(x'',y)\, f(y)\, dy. \quad \text{for } j,k = 0,1,\dots.$$

3.1. Estimation of the remainder v. We will show that $v \in V^l_{p,\beta}(\mathcal{D})$ if $f \in V^{l-2}_{p,\beta}(\mathcal{D})$. Let $L_{p,\beta}(\mathcal{D}) = V^0_{p,\beta}(\mathcal{D})$. The proof of the following lemma can be found in [14].

Lemma 3.1. *Let v be defined as*

$$v(x) = \int_{\mathcal{D}} K(x,y)\, f(y)\, dy,$$

where $f \in L_{p,\alpha+\beta-\gamma+n}(\mathcal{D})$, $K(x,y) = 0$ for $|x-y| < \delta|x'|$ (δ is a given positive number) and

$$\big|K(x,y)\big| \le c\, \frac{|x'|^\alpha\, |y'|^\beta}{|x-y|^\gamma}.$$

If $\alpha > -\frac{m}{p}$ and $\alpha - \gamma < m - n - \frac{m}{p}$, then

$$\|v\|_{L_p(\mathcal{D})} \le c\, \|f\|_{L_{p,\alpha+\beta-\gamma+n}(\mathcal{D})}$$

with a constant c independent of f.

Using this lemma and the estimates for $G(x,y)$ and $R_\sigma(x,y)$ in the last section, one can easily prove the following estimates for v.

Lemma 3.2. *Let l, p, β satisfy the inequalities*

$$l \ge 2, \quad 1 < p < \infty, \quad l - \beta - \frac{m}{p} > 0, \quad l - \beta - \frac{m}{p} \ne \lambda_j^+ \quad \textit{for all } j. \tag{38}$$

Then the function v satisfies the estimate

$$\|v\|_{L_{p,\beta-l}(\mathcal{D})} \le c\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})} \tag{39}$$

with a constant c independent of f.

Proof. By (35), we have $v = v_1 + v_2$, where

$$v_1(x) = \int_{\mathcal{D}} \zeta(x,y)\, R_\sigma(x,y)\, f(y)\, dy, \quad v_2(x) = \int_{\mathcal{D}} \big(1-\zeta(x,y)\big)\, G(x,y)\, f(y)\, dy.$$

On the support of ζ, the inequality $4|x'|^2 < 3\big(|y'|^2 + |x''-y''|^2\big)$ and, therefore, $|x'| < C|x-y|$ with a certain positive constant C is satisfied. Since

$$|x'|^{\beta-l}\, \big|v_1(x)\big| \le c \int \frac{|x'|^{\beta-l+\sigma+\varepsilon}}{|x-y|^{n-2+\sigma+\varepsilon}}\, |f(y)|\, dy$$

(cf. (7)), where the integration is extended over the set of all $y \in \mathcal{D}$ such that $C|x-y| < |x'|$ and ε is a sufficiently small positive number, Lemma 3.1 implies $\|v_1\|_{L_{p,\beta-l}(\mathcal{D})} \le c\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$.

We consider the function v_2. The function $1-\zeta$ is zero for $|y'|^2+|x''-y''|^2 > 2|x'|^2$. Let $\mathcal{D}_x$ be the set of all $y\in\mathcal{D}$ such that $|y'|^2+|x''-y''|^2 < 2|x'|^2$. Then by (2),

$$|v_2(x)| \le A(x) + B(x),$$

where

$$A(x) = \int_{\substack{y\in\mathcal{D}_x \\ |x'|<2|y'|}} \frac{|f(y)|}{|x-y|^{n-2}}\, dy, \quad B(x) = \int_{\substack{y\in\mathcal{D}_x \\ |x'|>2|y'|}} \frac{|f(y)|}{|x-y|^{n-2}}\, dy.$$

If $y \in \mathcal{D}_x$, $|x'| < 2|y'|$, then $|x-y| < 2|x'| < 4|y'|$ and $|y'| < 2|x'| < 4|y'|$. Hence by means of Hölder's inequality, we get

$$\begin{aligned} |A(x)|^p &\le \Big(\int_{|x-y|<2|x'|} |x-y|^{2-n}\, dy\Big)^{p-1} \int_{\substack{y\in\mathcal{D}_x \\ |x'|<2|y'|}} |x-y|^{2-n}\, |f(y)|^p\, dy \\ &\le c\, |x'|^{2p-2} \int_{\substack{y\in\mathcal{D}_x \\ |x'|<2|y'|}} |x-y|^{2-n}\, |f(y)|^p\, dy. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathcal{D}} |x'|^{p(\beta-l)}\, |A|^p\, dx &\le c \int_{\mathcal{D}} |y'|^{p(\beta-l+2)-2}\, |f(y)|^p \Big(\int_{|x-y|<4|y'|} \frac{dx}{|x-y|^{n-2}}\Big)\, dy \\ &\le c \int_{\mathcal{D}} |y'|^{p(\beta-l+2)}\, |f(y)|^p\, dy. \end{aligned}$$

If $y\in\mathcal{D}_x$, $|x'|>2|y'|$, then $\frac{|x'|}{2} < |x-y| < 3|x'|$. Using Hölder's inequality, we get

$$\begin{aligned} B^p &\le \frac{c}{|x'|^{p(n-2)}} \int_{\substack{y\in\mathcal{D}_x \\ |x'|>2|y'|}} |y'|^{(p-1)(m-\varepsilon)}\, \big|f(y)\big|^p\, dy\, \Bigg(\int_{\substack{|y'|<|x'| \\ |x''-y''|<2|x'|}} \frac{dy}{|y'|^{m-\varepsilon}}\Bigg)^{p-1} \\ &\le c\, |x'|^{p(2-m+\varepsilon)-n+m-\varepsilon} \int_{\substack{y\in\mathcal{D}_x \\ |x'|>2|y'|}} |y'|^{(p-1)(m-\varepsilon)}\, \big|f(y)\big|^p\, dy. \end{aligned}$$

This leads to the estimate

$$\int_{\mathcal{D}} \frac{|B|^p}{|x'|^{p(l-\beta)}}\, dx \le c \int_{\mathcal{D}} |y'|^{(p-1)(m-\varepsilon)}\, |f(y)|^p \, \Big(\int \frac{dx}{|x'|^{p(l-\beta+m-2-\varepsilon)+n-m+\varepsilon}} \Big)\, dy,$$

where in the last integral the integration is extended over the set of all x such that $|x'| > 2|y'|$ and $|x''-y''| < |x'|\sqrt{2}$. Since $p(\beta - l + 2 - m) < -m$, it follows that

$$\begin{aligned} \int_{\mathcal{D}} \frac{|B|^p}{|x'|^{p(l-\beta)}}\, dx &\le c \int_{\mathcal{D}} |y'|^{(p-1)(m-\varepsilon)}\, |f(y)|^p \, \Big(\int_{|x'|>2|y'|} \frac{dx'}{|x'|^{p(l-\beta+m-2-\varepsilon)+\varepsilon}} \Big)\, dy \\ &\le c \int_{\mathcal{D}} |y'|^{p(\beta-l+2)}\, |f(y)|^p \, dy \end{aligned}$$

if ε is sufficiently small. This means that the $V^0_{p,\beta-l}(\mathcal{D})$-norm of v_2 can be estimated by the right-hand side of (39). The proof is complete. □

Next, we estimate the $V^{l-2}_{p,\beta}(\mathcal{D})$-norms of ΔS and Δq.

Lemma 3.3. *Let $f \in V^{l-2}_{p,\beta}(\mathcal{D})$, where l, p, β satisfy the condition* (38)*. Then the functions* (36) *and* (37) *satisfy the estimate*

$$\|\Delta S\|_{V^{l-2}_{p,\beta}(\mathcal{D})} + \|\Delta q\|_{V^{l-2}_{p,\beta}(\mathcal{D})} \le c\, \|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}$$

with a constant c independent of f.

Proof. The proof of the estimate for the $V^{l-2}_{p,\beta}$-norm of ΔS proceeds analogously to [14, Lemma 3.4] since the representation of S is the same as in [14]. We consider the term Δq. Obviously

$$\partial_x^\alpha \Delta_x q(x) = \int_{\mathcal{D}} K(x,y)\, f(y)\, dy,$$

where

$$K(x,y) = \partial_x^\alpha \Big(\zeta(x,y)\, \Delta_x P_\sigma(x,y) + 2\nabla_x \zeta \cdot \nabla_x P_\sigma(x,y) + P_\sigma(x,y)\, \Delta_x \zeta \Big).$$

Using the equality $\Delta_{x'} r^{2k} = 4k(k-1+\frac{m}{2})\, r^{2k-2}$, we get

$$\Delta_x P_\sigma(x,y) = -\frac{(-\Delta_{x''})^{k+1} c_0(x'',y)}{4^k\, k!\, (k-1+\frac{m}{2})_{(k)}}\, r^{2k},$$

where $k = [\frac{\sigma}{2}]$. From (4) it follows that $\big|\partial_{x''}^{\alpha''} c_0(x'',y)\big| \le c\big(|y'|^2 + |x''-y''|^2\big)^{\frac{2-n-|\alpha''|}{2}}$. This together with (14) implies

$$\big|\partial_x^\alpha \big(\zeta(x,y)\, \Delta_x P_\sigma(x,y)\big)\big| \le c\, |x-y|^{-n-2k}\, r^{2k-|\alpha|}.$$

The same estimate holds for the terms $\partial_x^\alpha(\nabla_x\zeta\cdot\nabla_x P_\sigma)$ and $\partial_x^\alpha(P_\sigma\,\Delta_x\zeta)$. Thus

$$|K(x,y)| \le c\,|x-y|^{-n-2k}\,r^{2k-|\alpha|}$$

with $k=[\frac{\sigma}{2}]$. Applying Lemma 3.1, we obtain the estimate

$$\|r^{\beta-l+2+|\alpha|}\,\partial_x^\alpha\Delta q\|_{L_p(\mathcal{D})} \le c\,\|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$$

since $2+2k+\beta-l = 2+2[\frac{\sigma}{2}]-\sigma-\frac{m}{p} > -\frac{m}{p}$. This proves the lemma. □

Now it is easy to prove the following theorem.

Theorem 3.4. *Suppose that $u\in\mathcal{H}_{\mathcal{D}}$ is a solution of the problem* (1) *with the right-hand side $f\in\mathcal{H}^*_{\mathcal{D}}\cap V^{l-2}_{p,\beta}(\mathcal{D})$, where l,p,β satisfy the condition* (38)*. Then*

$$u(x)=S(x)+q(x)+v(x),$$

where S, q are defined by (36) and (37)*, respectively, and $v\in V^l_{p,\beta}(\mathcal{D})$. Furthermore,*

$$\|v\|_{V^l_{p,\beta}(\mathcal{D})} \le c\,\|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}$$

with a constant c independent of f.

Proof. By Lemma 3.2, we have $u=S+q+v$, where $v\in V^0_{p,\beta-l}(\mathcal{D})$ and

$$\|v\|_{V^0_{p,\beta-l}(\mathcal{D})} \le c\,\|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}.$$

Using [11, Theorem 6.1.3], Lemma 3.3 and the equality $\Delta v = f-\Delta S-\Delta q$, we conclude that $v\in V^l_{p,\beta}(\mathcal{D})$ and

$$\|v\|_{V^l_{p,\beta}(\mathcal{D})} \le c\left(\|\Delta v\|_{V^{l-2}_{p,\beta}(\mathcal{D})}+\|v\|_{V^0_{p,\beta-l}(\mathcal{D})}\right) \le c'\,\|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}. \qquad \square$$

3.2. On the coefficients in the asymptotics. Now we consider the coefficients $H_{j,k}(x)$ of the terms $S(x)$ and $q(x)$ in Theorem 3.4. Let

$$H_j(x)=H_{j,0}(x)=\int_{\mathcal{D}}\zeta(x,y)\,c_j(x'',y)\,f(y)\,dy \quad \text{for } j=0,1,\ldots.$$

Analogously to [14, Lemma 3.6], we obtain the following lemma.

Lemma 3.5. *Suppose that the conditions of Theorem* 3.4 *on f are satisfied. Then the functions*

$$U_{j,k}(x)=\big(H_{j,k}(x)-(-\Delta_{x''})^k H_j(x)\big)\,r^{\lambda_j^+ +2k}\,\phi_j(\omega_x)$$

belong to the space $V^l_{p,\beta}(\mathcal{D})$ for $\lambda_j^+<\sigma$ and $k=0,1,\ldots,[\frac{\sigma-\lambda_j^+}{2}]$. Furthermore,

$$\|U_{j,k}\|_{V^l_{p,\beta}(\mathcal{D})} \le c\,\|f\|_{V^0_{p,\beta-l}(\mathcal{D})}$$

with a constant c independent of f.

Proof. By the definition of $H_{j,k}$ and H_j, we have

$$U_{j,k}(x) = (-1)^k \int_{\mathcal{D}} K_{j,k}(x,y)\, f(y)\, dy,$$

where

$$K_{j,k}(x,y) = \Big(\zeta(x,y)\Delta^k_{x''} c_j(y', x''-y'') - \Delta^k_{x''}\zeta(x,y)\, c_j(y', x''-y'')\Big)\, r^{\lambda_j^+ + 2k}\phi_j(\omega_x).$$

Using the estimates

$$\big|\partial_x^\alpha \zeta(x,y)\big| \le c\, r^{-|\alpha|}, \quad \big|\partial_{x''}^{\alpha''} c_j(x'',y)\big| \le c\left(|y'|^2 + |x''-y''|^2\right)^{\frac{2-n-2\lambda_j^+ - |\alpha''|}{2}} |y'|^{\lambda_j^+},$$

(14), and the fact that $c_1|x-y| \le |x'| \le c_2|x-y|$ on the support of the function $\nabla_{x''}\zeta$, we obtain $r^{\beta-l+|\alpha|}\,\big|\partial_x^\alpha K_{j,k}(x,y)\big| \le c\, \frac{|y'|^{\lambda_j^+}}{|x-y|^{l-\beta+n-2+\lambda_j^+}}$. Applying Lemma 3.1, we obtain the estimate

$$\|r^{\beta-l+|\alpha|}\, \partial_x^\alpha U_{j,k}\|_{L_p(\mathcal{D})} \le c\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}.$$

This proves the lemma. □

Due to the last lemma, it makes sense to replace the coefficients $H_{j,k}(x)$ in the definition of the functions (36) and (37) by $(-\Delta_{x''})^k H_j(x)$. Then we obtain the functions

$$\tilde{S}(x) = \sum_{0<\lambda_j^+<\sigma} \sum_{k=0}^{m_{j,\sigma}} \frac{(-\Delta_{x''})^k H_j(x)}{4^k\, k!\, (\lambda_j^+ + k - 1 + \frac{m}{2})_{(k)}}\, r^{\lambda_j^+ + 2k}\phi_j(\omega_x),$$

$$\tilde{q}(x) = \sum_{k=0}^{[\frac{\sigma}{2}]} \frac{(-\Delta_{x''})^k H_0(x)}{4^k\, k!\, (k-1+\frac{m}{2})_{(k)}}\, r^{2k},$$

By Lemma 3.5, the differences $S-\tilde{S}$ and $q-\tilde{q}$ are functions of the space $V^l_{p,\beta}(\mathcal{D})$. Consequently, we can deduce the following result directly from Theorem 3.4.

Theorem 3.6. *Suppose that $u \in \mathcal{H}_{\mathcal{D}}$ is a solution of the problem* (1) *with the right-hand side $f \in \mathcal{H}^*_{\mathcal{D}} \cap V^{l-2}_{p,\beta}(\mathcal{D})$, where l, p, β satisfy the condition* (38). *Then*

$$u(x) = \tilde{S}(x) + \tilde{q}(x) + v(x),$$

where $v \in V^l_{p,\beta}(\mathcal{D})$ and

$$\|v\|_{V^l_{p,\beta}(\mathcal{D})} \le c\, \|f\|_{V^{l-2}_{p,\beta}(\mathcal{D})}$$

with a constant c independent of f.

We consider the coefficients H_j of the functions $\tilde{S}$ and $\tilde{q}$. It is evident that $H_j(x)$ depends only on $r = |x'|$ and x'' but not on ω_x. Moreover, the following assertions hold.

Lemma 3.7. *Suppose that $f \in V^{l-2}_{p,\beta}(\mathcal{D})$ and l, p, β satisfy the condition* (38). *Then*

(i) $\partial_x^\alpha H_j = \partial_{x'}^{\alpha'} \partial_{x''}^{\alpha''} H_j \in L_{p,\beta-l+\lambda_j^+ +|\alpha|}(\mathcal{D})$ *if* $|\alpha'| \ge 1$.

(ii) $\partial_{x''}^{\alpha''} H_j \in L_{p,\beta-l+\lambda_j^+ +|\alpha''|}(\mathcal{D})$ *if* $|\alpha''| > \sigma - \lambda_j^+$.

(iii) *The trace of H_j on M coincides with the function* (11) *and belongs to the Besov space $B_p^{\sigma-\lambda_j^+}(\mathbb{R}^{n-m})$ for $j = 0, 1, \dots$.*

Proof. For $j > 0$, the proof is the same as for [14, Lemma 3.7], since we have the same formula (with other eigenvalues and eigenfunctions Λ_j and ϕ_j) for the functions $c_j(x'', y)$ as in the case of the Dirichlet problem. We consider the function

$$H_0(x) = \int_{\mathcal{D}} \zeta(x,y)\, c_0(x'',y)\, f(y)\, dy,$$

where $c_0(x'', y) = G(0, x'', y)$ is defined by (4). Obviously,

$$\left|\partial_x^\alpha \zeta(x,y)\, c_0(x'',y)\right| \le c_\alpha\, |x-y|^{2-n-|\alpha|}$$

for every multi-index α. If $\alpha = (\alpha', \alpha'')$ and $|\alpha'| \ge 1$, then the function $\partial_x^\alpha \zeta(x,y)\, c_0(x'',y)$ vanishes outside the region $2(|y'|^2 + |x''-y''|^2) < 4|x'|^2 < 3(|y'|^2 + |x''-y''|^2)$. Therefore, $r^{\beta-l+|\alpha|} \left|\partial_x^\alpha \zeta(x,y)\, c_0(x'',y)\right| \le c_\alpha\, |x-y|^{\beta-l+2-n}$. Consequently, it follows from Lemma 3.1 that

$$\|r^{\beta-l+|\alpha|}\, \partial_x^\alpha H_0\|_{L_p(\mathcal{D})} \le c_\alpha\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}$$

with a constant c_α independent of f. Now, let $\alpha' = 0$ and $|\alpha''| > \sigma = l - \beta - \frac{m}{p}$. Then

$$r^{\beta-l+|\alpha''|} \left|\partial_{x''}^{\alpha''} \zeta(x,y)\, c_0(x'',y)\right| \le c_{\alpha''}\, \frac{|x'|^{\beta-l+|\alpha''|}}{|x-y|^{n-2+|\alpha''|}}.$$

Applying again Lemma 3.1, we obtain

$$\|r^{\beta-l+|\alpha''|}\, \partial_{x''}^{\alpha''} H_0\|_{L_p(\mathcal{D})} \le c_{\alpha''}\, \|f\|_{L_{p,\beta-l+2}(\mathcal{D})}.$$

Thus, the assertions (i) and (ii) are true for $j = 0$. If we consider H_0 as a function on $\mathbb{R}_+ \times \mathbb{R}^{n-m}$ (with the variables r and x''), then it follows from (i) and (ii) in particular that

$$r^{\beta-l+k+|\alpha''|+\frac{m-1}{p}}\, \partial_r^k \partial_{x''}^{\alpha''} H_0 \in L_p(\mathbb{R}_+ \times \mathbb{R}^{n-m}) \quad \text{for } k + |\alpha''| > \sigma.$$

Hence by [16, Section 2.9.2, Theorem 1], the trace $h_0 = H_0|_{x'=0}$ of H_0 on the edge M exists and belongs to the Besov space $B_p^\sigma(M)$. This proves the lemma. □

Finally, we note that the coefficients H_j of Σ and Q can be replaced (as in the case of the Dirichlet problem, cf. [14]) by other extensions of h_j. Such a possible extension of h_j is

$$(\mathcal{E}h_j)(x', x'') = \psi(r) \int_{\mathbb{R}^{n-m}} T(y'')\, h_j(x'' - ry'')\, dy'', \tag{40}$$

where ψ is a smooth function with compact support on $[0, \infty)$ such that $\psi(r) = 1$ for $r < \frac{1}{2}$, T is a smooth function on $\mathbb{R}^{n-m}$ with support in the unit ball $|y''| \le 1$ satisfying the condition

$$\int_{\mathbb{R}^{n-m}} T(y'')\, dy'' = 1, \quad \int_{\mathbb{R}^{n-m}} (y'')^{\alpha''}\, T(y'')\, dy'' = 0 \quad \text{for } 1 \le |\alpha''| \le [\sigma].$$

As was shown in [14, Lemma 3.8], this extension has the same properties (i) and (ii) of Lemma 3.7 as H_j. From this it follows that $\Delta^k_{x''}(H_j - \mathcal{E}h_j)\, r^{\lambda_j^+ + 2k}\phi_j(\omega_x)$ is a function of the space $V^l_{p,\beta}(\mathcal{D}$ for $\lambda_j^+ < \sigma$, $k \le \left[\frac{\sigma - \lambda_j^+}{2}\right]$. Thus the following theorem holds.

Theorem 3.8. *Suppose that* $u \in \mathcal{H}_\mathcal{D}$ *is a solution of the problem* (1) *with the right-hand side* $f \in \mathcal{H}^*_\mathcal{D} \cap V^{l-2}_{p,\beta}(\mathcal{D})$, *where* l, p, β *satisfy the condition* (38). *Then*

$$u(x) = \Sigma(x) + Q(x) + v(x),$$

where $v \in V^l_{p,\beta}(\mathcal{D})$ *and* Σ, Q *are defined by* (9) *and* (10), *respectively.*

References

[1] Costabel, M. and Dauge, M., General edge asymptotics of solutions of second order elliptic boundary value problems II. *Proc. Roy. Soc. Edinburgh* 123A (1993), 157 – 184.

[2] Dauge, M., *Elliptic Boundary Value Problems in Corner Domains - Smoothness and Asymptotics of Solutions.* Lect. Notes Math. 1341. Berlin: Springer 1988.

[3] Kondrat'ev, V. A., Singularities of the solutions of the Dirichlet problem for elliptic equations of second order in a neighborhood of an edge (in Russian). *Diff. Uravn.* 13 (1977)(11), 2026 – 2032.

[4] Kozlov, V. A., On the asymptotics of Green's function and Poisson's kernels for a parabolic problem in a cone I (in Russian). *Z. Anal. Anwend.* 8 (1989)(2), 131 – 152.

[5] Kozlov, V. A., On the asymptotics of Green's function and Poisson's kernels for a parabolic problem in a cone II (in Russian). *Z. Anal. Anwend.* 10 (1991)(1), 27 – 42.

[6] Kozlov, V. A., Maz′ya, V. G. and Roßmann, J., *Elliptic Boundary Value Problems in Domains with Point Singularities.* Math. Surveys Monogr. 52. Providence (RI): Amer. Math. Soc. 1997.

[7] Maz'ya, V. G. and Plamenevskiĭ, B. A., On elliptic boundary value problems in domains with piecewise smooth boundary (in Russian). In: *Trudy Simp. Mekh. Sploshnoi Sredy* (Tbilissi 1971; ed.: I. N. Vekua). Tbilissi: Metsniereba 1973, pp. 171 – 182.

[8] Maz'ya, V. G. and Plamenevskiĭ, B. A., On the coefficients in the asymptotics of solutions of elliptic boundary value problems in domains with conical points (in Russian). *Math. Nachr.* 76 (1977), 29 – 60; Engl. transl.: *Amer. Math. Soc. Transl.* 123 (1984), 57 – 88.

[9] Maz'ya, V. G. and Plamenevskiĭ, B. A., On the asymptotics of the fundamental solutions of elliptic boundary value problems in regions with conical points (in Russian). *Probl. Mat. Anal.* 7 (1979), 100 – 145, Engl. transl.: *Sel. Math. Sov.* 4 (1985)(4), 363 – 397.

[10] Maz'ya, V. G. and Rossmann, J., Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten (in German). *Math. Nachr.* 138 (1988), 27 – 53.

[11] Maz'ya, V. and Rossmann, J., *Elliptic Equations in Polyhedral Domains.* Math. Surveys Monogr. 162. Providence (RI): Amer. Math. Soc. 2010.

[12] Nazarov, S. A. and Plamenevskiĭ, B. A., *Elliptic Problems in Domains with Piecewise Smooth Boundaries.* De Gruyter Exp. Math. 13. Berlin: De Gruyter 1994.

[13] Rossmann, J., The asymptotics of the solutions of linear elliptic variational problems in domains with edges. *Z. Anal. Anwend.* 9 (1990)(3), 565 – 575.

[14] Rossmann, J., Asymptotics of solutions of the Poisson equation near straight edges. *Z. Anal. Anwend.* 33 (2014)(2), 217 – 245.

[15] Solonnikov, V. A., Estimates of solutions of the Neumann problem for a second order elliptic equation in domains with edges on the boundary. Preprint P-4-83, Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 1983.

[16] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators.* Berlin: Dt. Verlag Wiss. 1978.

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