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Regularity of Solutions to the G-Laplace Equation Involving Measures

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Abstract. We establish regularity of solutions to the equation $-\Delta_G u = \mu$, provided that $\mu(B_r(x_0)) \leq Cr^\beta$ for any ball $B_r(x_0) \subset \Omega$ with $r \leq 1$, where $\beta > 0$ and G satisfies certain structural conditions.

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1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n (n \geq 2)$, and μ a nonnegative Radon measure in Ω. We consider the equation

$$-\Delta_G u := -\text{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in } \mathcal{D}'(\Omega), \tag{1}$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$, $g(t)$ is a nonnegative C^1 function in $[0, +\infty)$, satisfying $g(0) = 0$ and the following structural condition

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall\ t > 0,\ \delta, g_0 \text{ are positive constants.} \tag{2}$$

The operator Δ_G includes not only the case of the p-Laplacian $\triangle_p$ ($\delta = g_0 = p - 1 > 0$), but also the interesting case of a variable exponent $p = p(t) > 0$:

$$-\Delta_G u = -\text{div}\,(|\nabla u|^{p(|\nabla u|)-2}\nabla u),$$

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corresponding to set $g(t)=t^{p(t)-1}$, for which (2) holds if $\delta \le t(\ln t)p'(t)+p(t)-1 \le g_0$ for all $t>0$.

Other examples of functions satisfying (2) are given by $g(t)=t^\alpha \ln(\beta t+\theta)$, with $\alpha,\beta,\theta>0$, or by discontinuous power transitions as $g(t)=C_1 t^\alpha$, if $0\le t<t_0$, and $g(t)=C_2 t^\beta + C_3$, if $t\ge t_0$, where α,β,t_0 are positive numbers, $C_{1,2,3}$ are real numbers such that $g(t)$ is a C^1 function.

In this paper we consider the regularity of solutions for the G-Laplace operator involving measures, which continues the work by Challal and Lyaghfouri [2]. In [2], the authors proved that any local bounded solution of (1) is locally Hölder continuous for $\beta\in[n-1,n)$. We would like to extend the regularity result to the general case $\beta>0$. More precisely, we prove that any bounded solution of (1) is locally $C^{1,\alpha}$-continuous for $\beta>n$, Log-Lipschitz-continuous for $\beta=n$, and Hölder continuous for $\beta<n$ and $g_0<\frac{\beta+1}{n-1}$, respectively. We should also note a result in [7] (see also [8]) for the p-Laplace type operator which states that solutions are in $C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha\in(0,1)$ when $\beta=n-p+\alpha(p-1)$ and $1<p\le n$. The proofs of the main results are based on estimates for G-harmonic functions.

Throughout this paper we always assume (1.2) holds. The main result is

Theorem 1.1. *Suppose* $\frac{g(t)}{t}$ *is non-decreasing in* $t>0$. *Let* u *satisfy* (1) *with* $\beta>0$. *The following regularity results hold:*

1. *If* $\beta>n$, *then* $u\in C^{1,\alpha}_{loc}(\Omega)$ *for some* $\alpha\in(0,1)$.
2. *If* $\beta=n$, *then* u *is Log-Lipshitz continuous, thus* $u\in C^{0,\alpha}_{loc}(\Omega)$ *for any* $\alpha\in(0,1)$.
3. *If* $\beta\in[n-1,n)$, *then* $u\in C^{0,\alpha}_{loc}(\Omega)$ *for some* $\alpha\in(0,1)$.
4. *If* $\beta\in(0,n-1)$ *and* $g_0<\frac{\beta+1}{n-1}$, *then* $u\in C^{0,\alpha}_{loc}(\Omega)$ *for some* $\alpha\in(0,1)$.

2. Some auxiliary results

In this section we state some properties of function G and its derivative g that are used throughout the paper. We also state some real analytic properties for functions with finite $\int_\Omega G(|\nabla u|)\mathrm{d}x$, and some properties for G-harmonic functions which will be applied to establish $C^{1,\alpha}$-estimates for solutions.

Lemma 2.1 ([11, Lemma 2.1])**.** *Function* g *satisfies the following properties:*

(g_1) $\min\{s^\delta, s^{g_0}\}g(t)\le g(st)\le \max\{s^\delta, s^{g_0}\}g(t)$.

(g_2) *G is convex and C^2.*

(g_3) $\frac{tg(t)}{1+g_0}\le G(t)\le tg(t)$, *for all* $t\ge 0$.

Lemma 2.2 ([11, Remark 2.1])**.** *Function* G *satisfies the following properties:*

(G_1) $\min\{s^{\delta+1}, s^{g_0+1}\}\frac{G(t)}{1+g_0}\le G(st)\le (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t)$.

(G_2) $G(a+b)\le 2^{g_0}(1+g_0)(G(a)+G(b))$, *for all* $a,b>0$.

As g is increasing we can define its inverse function g^{-1}. Then g^{-1} satisfies a similar condition to (2).

Proposition 2.3 ([11, Lemma 2.2]). *Function g^{-1} satisfies the following property:*

$$\frac{1}{g_0} \leq \frac{t(g^{-1})'(t)}{g^{-1}(t)} \leq \frac{1}{\delta}, \quad \textit{for all } t > 0.$$

Set $\widetilde{G}$ such that $\widetilde{G}'(t) = g^{-1}(t)$. For more properties of $\widetilde{G}$, we refer the reader to [1,3,4,9,11,13], etc.

Recall that

$$\|u\|_{L^G(\Omega)} := \inf\left\{k > 0;\ \int_\Omega G\left(\frac{|u(x)|}{k}\right) \mathrm{d}x \leq 1\right\}$$

is a norm on the Orlicz space $L^G(\Omega)$ which is the linear hull of the Orlicz class

$$\mathcal{K}_G(\Omega) := \left\{u \text{ measurable};\ \int_\Omega G(|u|)\mathrm{d}x < \infty\right\}.$$

Notice that this set is convex, since G is also convex (g_2). The Orlicz-Sobolev space $W^{1,G}(\Omega)$ is defined as

$$W^{1,G}(\Omega) := \{u \in L^G(\Omega); \nabla u \in L^G(\Omega)\},$$

which is the usual subspace of $W^{1,1}(\Omega)$ associated with the norm $\|u\|_{W^{1,G}(\Omega)} := \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}$. $L^G(\Omega)$ and $W^{1,G}(\Omega)$ are reflexive, Moreover, $L^{\widetilde{G}}(\Omega)$ is the dual space of $L^G(\Omega)$ (see [11]).

Lemma 2.4 ([11, Lemma 2.3]). *There exists a constant $C = C(\delta, g_0)$ such that*

$$\|u\|_{L^G(\Omega)} \leq C \max\left\{\left(\int_\Omega G(|u|)\,dx\right)^{\frac{1}{1+\delta}}, \left(\int_\Omega G(|u|)\,dx\right)^{\frac{1}{1+g_0}}\right\}.$$

The following result is Hölder's inequality.

Lemma 2.5 ([1, 8.11]). *For any $u \in L^G(\Omega)$ and any $v \in L^{\widetilde{G}}(\Omega)$, there holds*

$$\left|\int_\Omega uv\,dx\right| \leq 2\|u\|_{L^G(\Omega)}\|v\|_{L^{\widetilde{G}}(\Omega)}.$$

Let $(h)_r := \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} h\mathrm{d}x$ denote the average value of function h on $B_r(x_0)$. Throughout this paper, without special states, by B_R and B_r we denote the balls contained in $\mathbb{R}^n$ or Ω with the same center.

The following two lemmas will pave the way to the $C^{1,\alpha}$-estimates.

Lemma 2.6 ([9, Lemma 5.1]). *Let h be a G-harmonic function in B_R, i.e., $\Delta_G h = 0$ in B_R. Then for some positive constant $0 < \sigma < 1$, there holds*

$$\int_{B_r} G(|\nabla h - (\nabla h)_r|)\,dx \le C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla h - (\nabla h)_R|)\,dx,$$

where $C = C(n, \delta, g_0) > 0$ is a positive constant.

Lemma 2.7 ([10, Lemma 2.7]). *Let $\overline{\phi}(s)$ be a non-negative and non-decreasing function. Suppose that*

$$\overline{\phi}(r) \le C_1\left[\left(\frac{r}{R}\right)^{\alpha} + \vartheta\right]\overline{\phi}(R) + C_2 R^{\beta},$$

for all $r \le R \le R_0$, with C_1, α, β positive constants and C_2, ϑ non-negative constants. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $\vartheta_0 = \vartheta_0(C_1, \alpha, \beta, \tau)$ such that if $\vartheta < C_1, \vartheta_0$, then for all $r \le R \le R_0$ we have

$$\overline{\phi}(r) \le C_3\left(\frac{r}{R}\right)^{\tau}[\overline{\phi}(R) + C_2 R^{\tau}],$$

where $C_3 = C_3(C_1, \tau - \min\{\alpha, \beta\})$ is a positive constant. In turn,

$$\overline{\phi}(r) \le C_4 r^{\tau},$$

where $C_4 = C_4(C_2, C_3, R_0, \overline{\phi}, \tau)$ is a positive constant.

3. Proofs of the main results

We first prove the following comparison theorem.

Lemma 3.1 (Comparison with G-harmonic functions). *Let $u \in W^{1,G}(B_R)$ and $h \in W^{1,G}(B_R)$ satisfying $\Delta_G h = 0$ in B_R in the distributional sense. Then there exists a positive constant $C = C(n, \delta, g_0)$ such that for each $0 < r \le R$, there holds*

$$\int_{B_r} G(|\nabla u - (\nabla h)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla h)_R|)\mathrm{d}x + C\int_{B_R} G(|\nabla u - \nabla h|)\,dx,$$

where $0 < \sigma < 1$ is the exponent in Lemma 2.6.

Proof. For each $r \in (0, R]$, by (G_2) and the nondecreasing monotonicity of G, we get

$$\begin{aligned}\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\int_{B_r} G(|\nabla u - (\nabla h)_r|)\mathrm{d}x \\ + C\int_{B_r} G(|(\nabla u)_r - (\nabla h)_r|)\mathrm{d}x,\end{aligned} \tag{3}$$

where C is a positive constant depending on g_0. Analogously, there holds

$$\begin{aligned}\int_{B_r} G(|\nabla u-(\nabla h)_r|)\mathrm{d}x \le C\int_{B_r} G(|\nabla u-\nabla h|)\mathrm{d}x \\ + C\int_{B_r} G(|\nabla h-(\nabla h)_r|)\mathrm{d}x,\end{aligned} \tag{4}$$

where C is a positive constant depending on g_0.

Convexity of G implies that

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r} G(|(\nabla u)_r-(\nabla h)_r|)\mathrm{d}x &= G(|(\nabla u)_r-(\nabla h)_r|)\\ &\le G\left(\frac{1}{|B_r|}\int_{B_r}|\nabla u-\nabla h|\mathrm{d}x\right)\\ &\le \frac{1}{|B_r|}\int_{B_r} G(|\nabla u-\nabla h|)\mathrm{d}x.\end{aligned}$$

Therefore

$$\int_{B_r} G(|(\nabla u)_r-(\nabla h)_r|)\mathrm{d}x \le \int_{B_r} G(|\nabla u-\nabla h|)\mathrm{d}x. \tag{5}$$

From (3) to (5), it follows

$$\begin{aligned}\int_{B_r} G(|\nabla u-(\nabla u)_r|)\mathrm{d}x \le C\int_{B_r} G(|\nabla u-\nabla h|)\mathrm{d}x \\ + C\int_{B_r} G(|\nabla h-(\nabla h)_r|)\mathrm{d}x.\end{aligned} \tag{6}$$

Since in the above sequel only properties of G have been used, interplaying the roles of u and h in (6) and arguing in the larger ball B_R, then we arrive at

$$\begin{aligned}\int_{B_R} G(|\nabla h-(\nabla h)_R|)\mathrm{d}x \le C\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x \\ + C\int_{B_R} G(|\nabla u-(\nabla u)_R|)\mathrm{d}x.\end{aligned} \tag{7}$$

Now, in view of Lemma 2.6 and (6), we have further estimate

$$\begin{aligned}\int_{B_r} G(|\nabla u-(\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla h-(\nabla h)_R|)\mathrm{d}x \\ + C\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x.\end{aligned} \tag{8}$$

Combining (7) and (8), we conclude

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\left(1+\left(\frac{r}{R}\right)^{n+\sigma}\right)\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x,$$

which finally implies

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x. \qquad \square$$

Lemma 3.2. *Suppose $\frac{g(t)}{t}$ is non-decreasing in $t > 0$, then there exists a positive constant C depending only on g_0 such that*

$$\left(\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right)(\xi - \eta) \ge CG(|\xi - \eta|) \quad \textit{for all } \xi, \eta \in \mathbb{R}^n.$$

Proof. Since $\frac{g(t)}{t}$ is non-decreasing in $t > 0$, we deduce that (see proof of [6, Lemma 3.1])

$$\left(\frac{g(|\xi|)}{|\xi|}\xi - \frac{g(|\eta|)}{|\eta|}\eta\right)(\xi-\eta) \ge \frac{1}{3}\left(\frac{g(|\xi|)}{|\xi|} + \frac{g(|\eta|)}{|\eta|}\right)|\xi-\eta|^2, \quad \text{for all } \xi, \eta \in \mathbb{R}^n\backslash\{0\}.$$

It suffices to show that $\left(\frac{g(|\xi|)}{|\xi|} + \frac{g(|\eta|)}{|\eta|}\right)|\xi-\eta|^2 \ge CG(|\xi-\eta|)$ for all $\xi, \eta \in \mathbb{R}^n\backslash\{0\}$. Without loss of generality, assume $|\xi| > |\eta|$. We get by the non-decreasing monotonicity of $\frac{g(t)}{t}$ that

$$\frac{g(|\xi-\eta|)}{|\xi-\eta|} \le \frac{g(|\xi|+|\eta|)}{|\xi|+|\eta|} \le \frac{g(2|\xi|)}{|\xi|} \le \frac{2^{g_0} g(|\xi|)}{|\xi|} \le 2^{g_0}\left(\frac{g(|\xi|)}{|\xi|} + \frac{g(|\eta|)}{|\eta|}\right).$$

It follows

$$G(|\xi-\eta|) \le |\xi-\eta| g(|\xi-\eta|) \le 2^{g_0}\left(\frac{g(|\xi|)}{|\xi|} + \frac{g(|\eta|)}{|\eta|}\right)|\xi-\eta|^2. \qquad \square$$

We have now gathered all the tools and ingredients we need to establish local Hölder continuity of the gradient of solutions of (1).

Proof of Theorem 1.1. Since only the local property will be considered, we may assume $\|u\|_{L^\infty(B_R(x_0))} \le M$ on the ball $B_R(x_0)(\subset \Omega)$, which we will work on. We prove the three results in Theorem 1.1 respectively.

1. $C^{1,\alpha}_{loc}$*-regularity for* $\beta > n$. Let h be the G-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\operatorname{div}\left(\frac{g(|\nabla h|)}{|\nabla h|}\nabla h\right) = 0 \text{ in } B_R \quad \text{and} \quad h - u \in W_0^{1,G}(B_R).$$

By Lemma 3.1, it follows

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x. \tag{9}$$

On the other hand, since u is a solution of (1), then we have

$$\int_{B_R}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u - \frac{g(|\nabla h|)}{|\nabla h|}\nabla h\right)\nabla(u-h)\mathrm{d}x = \int_{B_R}(u-h)\mathrm{d}\mu.$$

Using Lemma 3.2, we deduce that $\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x \le C\int_{B_R}(u-h)\mathrm{d}\mu \le C\mu(B_R) \le CR^\beta$. Therefore (9) becomes

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^\beta.$$

In view of Lemma 2.7 we conclude that there is a constant $\tau = \min\{\sigma, \beta - n\}$ such that

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le Cr^{n+\tau}. \tag{10}$$

Now we are expected to show that there is a constant $\kappa \in (0,1)$ such that

$$\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \le Cr^{n+\kappa}, \tag{11}$$

which and Campanato's embedding theorem (see [12] for instance) will give the desired Hölder continuity of the gradient of u.

Indeed, convexity of G and (10) implies that

$$G\left(\frac{1}{|B_r|}\int_{B_r}|\nabla u - (\nabla u)_r|\mathrm{d}x\right) \le \frac{1}{|B_r|}\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le Cr^\tau. \tag{12}$$

Let κ be a positive constant lying in $(0, \frac{\tau}{g_0})$. If $r^{-n-\kappa}\int_{B_r}|\nabla u - (\nabla u)_r|\mathrm{d}x \le 1$, then $\int_{B_r}|\nabla u - (\nabla u)_r|\mathrm{d}x \le r^{n+\kappa}$. Therefore, (11) holds.

If $r^{-n-\kappa}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x>1$, we define $\mathscr{G}(t)=G(t)-G(1)t, t\geq 1$. Since $\mathscr{G}(t)$ is increasing in t when $t\geq 1$, then we find

$$G\left(r^{-n-\kappa}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\right)\geq G(1)\left(r^{-n-\kappa}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\right),$$

which and (12) imply that

$$\begin{aligned}r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x &\leq Cr^{\kappa}G\left(r^{-n-\kappa}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\right)\\ &\leq Cr^{\kappa}(r^{-\kappa})^{1+g_0}G\left(r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\right)\\ &\leq Cr^{\tau-\kappa g_0}.\end{aligned}$$

This reveals that (11) holds and the proof of the first result in Theorem 1.1 is completed.

2. *Log-Lipschitz regularity for $\beta=n$.* We follow the initial steps of the proof of Theorem 1.1. Let h be the G-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\operatorname{div}\left(\frac{g(|\nabla h|)}{|\nabla h|}\nabla h\right)=0 \text{ in } B_R \quad\text{and}\quad h-u\in W_0^{1,G}(B_R).$$

Arguing as before, we have

$$\int_{B_r}G(|\nabla u-(\nabla u)_r|)\mathrm{d}x\leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R}G(|\nabla u-(\nabla u)_R|)\mathrm{d}x+CR^n.$$

In view of Lemma 2.7, there holds $\int_{B_r}G(|\nabla u-(\nabla u)_r|)\mathrm{d}x\leq Cr^n$. As before, we deduce that

$$\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\leq Cr^n,$$

which shows that the gradient of u lies in BMO space and for any fixed subdomain $\Omega'\subset\subset\Omega$, there holds

$$\|u\|_{BMO(\Omega')}\leq C(\Omega',n,\delta,g_0,G(1),M).$$

Then proceeding as in [10, pp 19–20], we obtain Log-Lipschitz regularity for solutions.

3. *$C^{0,\alpha}_{loc}$-regularity for $\beta<n$.* In this part, we address $C^{0,\alpha}_{loc}$-regularity for the solution of (1) when $\beta<n$. Here we deal with only the case $\beta<n-1$ and refer the reader to [2] for a proof when $\beta\in[n-1,n)$.

Let h be the G-harmonic function in B_R that agrees with u on the boundary. Proceeding as above, and by the locally Lipschitz estimate of solutions for G-harmonic functions (see [5]), we get

$$\begin{aligned}\int_{B_{\frac{R}{2}}} G(|\nabla u|)\mathrm{d}x &\leq C\left(\int_{B_{\frac{R}{2}}} G(|\nabla u-\nabla h|)\mathrm{d}x+\int_{B_{\frac{R}{2}}} G(|\nabla h|)\mathrm{d}x\right)\\ &\leq CR^{\beta}+C\int_{B_{\frac{R}{2}}} G(\|\nabla h\|_{L^{\infty}(B_{\frac{R}{2}})})\mathrm{d}x\\ &\leq CR^{\beta}+CR^{n}\\ &\leq CR^{\beta},\end{aligned}$$

where the last constant C depends only on $\|\nabla h\|_{L^{\infty}(B_{\frac{R}{2}})}$. On the other hand, by the local boundedness of u, which is the boundary value of h, we know that $\|h\|_{L^{\infty}(B_{\frac{R}{2}})}$ depends only on $n, \delta, g_0, \widetilde{G}(1), M$. Therefore, local Lipschitz estimate for solutions of G-Laplace equations (see [5]) give

$$\begin{aligned}\|\nabla h\|_{L^{\infty}(B_{\frac{R}{2}})} &\leq C(n,\delta,g_0,g(1),\|h\|_{L^{\infty}(B_{\frac{R}{2}})},\operatorname{dist}(\partial B_{\frac{R}{2}},B_R))\\ &= C(n,\delta,g_0,g(1),\widetilde{G}(1),M,\operatorname{dist}(\partial B_{\frac{R}{2}},B_R)).\end{aligned}$$

By Hölder's inequality (Lemma 2.5) and Lemma 2.4, we deduce, for $R<1$, $\int_{B_{\frac{R}{2}}}|\nabla u|\mathrm{d}x \leq 2\|\nabla u\|_{L^G}\|1\|_{L^{\widetilde{G}}} \leq C\Big(\int_{B_{\frac{R}{2}}}\widetilde{G}(1)\mathrm{d}x\Big)^{\frac{1}{1+g_0}}\Big(\int_{B_{\frac{R}{2}}}G(|\nabla u|)\mathrm{d}x\Big)^{\frac{1}{1+g_0}} \leq CR^{\frac{n+\beta}{1+g_0}} = CR^{n-1+\alpha}$, where $\alpha = \frac{n+\beta}{1+g_0}-(n-1)>0$ due to the structural condition $g_0<\frac{\beta+1}{n-1}$. By [12, Theorem(Morrey) 1.53 p. 30] we conclude that $u\in C^{0,\alpha}_{loc}(B_R)$. □

Remark 3.3. Furthermore, if $\beta = n$, we may deduce that any solution of (1) is locally Lipschitz continuous. Indeed, let h be the G-harmonic function in B_R that agrees with u on the boundary. Arguing as proof of Theorem 1.1, and by the locally Lipschitz estimate of solutions for G-harmonic functions (see [5]), we get

$$\begin{aligned}\int_{B_{\frac{R}{2}}} G(|\nabla u|)\mathrm{d}x &\leq C\left(\int_{B_{\frac{R}{2}}} G(|\nabla u-\nabla h|)\mathrm{d}x+\int_{B_{\frac{R}{2}}} G(|\nabla h|)\mathrm{d}x\right)\\ &\leq CR^{n}+C\int_{B_{\frac{R}{2}}} G(\|\nabla h\|_{L^{\infty}(B_{\frac{R}{2}})})\mathrm{d}x\\ &\leq CR^{n}.\end{aligned}$$

Now applying Lebesgue theorem, (G_1), and covering theorem, we conclude that for any $\Omega'\subset\subset\Omega$

$$\sup_{x\in\Omega'}|\nabla u(x)|\leq C,$$

where the constant C depends only on $n,\delta,g_0,g(1),\widetilde{G}(1),M$ and Ω'.

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