Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 34 (2015), 199–219
DOI: 10.4171/ZAA/1536

Optimal Control in Matrix-Valued Coefficients for Nonlinear Monotone Problems: Optimality Conditions II

Peter I. Kogut, Ol'ga P. Kupenko, and Günter Leugering

Abstract. In this paper we study an optimal control problem for a nonlinear monotone Dirichlet problem where the controls are taken as the matrix-valued coefficients in $L^\infty(\Omega;\mathbb{R}^{N\times N})$. Given a suitable cost function, the objective is to provide a substantiation of the first order optimality conditions using the concept of convergence in variable spaces. While in the first part [Z. Anal. Anwend. 34 (2015), 85–108] optimality conditions have been derived and analysed in the general case under some assumptions on the quasi-adjoint states, in this second part, we consider diagonal matrices and analyse the corresponding optimality system without such assumptions.

Keywords. Nonlinear monotone Dirichlet problem, control in coefficients, adjoint equation, variable spaces.

Mathematics Subject Classification (2010). 35J70, 49J20, 49J45, 93C73.

1. Introduction

The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $y_d \in L^p(\Omega)$, where Ω is an open bounded Lipschitz domain in $\mathbb{R}^N$, and the solution of a nonlinear Dirichlet problem by choosing an appropriate matrix of coefficients $\mathcal{U} \in L^\infty(D;\mathbb{R}^{N\times N})$. Namely, we consider the following minimization problem:

$$\text{Minimize} \quad \left\{ I_\Omega(\mathcal{U},y) = \int_\Omega |y(x) - y_d(x)|^p \, dx \right\} \tag{1}$$

P. I. Kogut: Department of Differential Equations, Dnipropetrovsk National University, Gagarin av., 72, 49010 Dnipropetrovsk, Ukraine; p.kogut@i.ua
O. P. Kupenko: Institute for Applied System Analysis, National Academy of Sciences and Ministry of Education and Science of Ukraine, Peremogy av., 37/35, IPSA, 03056 Kyiv, Ukraine; kogut_olga@bk.ru
G. Leugering: Lehrstuhl für Angewandte Mathematik II, Universität Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany; Guenter.Leugering@fau.de

subject to the constraints

$$\mathcal{U} \in U_{ad} \subset L^\infty(\Omega; \mathbb{R}^{N\times N}), \quad y \in W_0^{1,p}(\Omega), \tag{2}$$

$$-\mathrm{div}\,\big(\mathcal{U}[(\nabla y)^{p-2}]\nabla y\big) + |y|^{p-2}y = f \quad \text{in } \Omega, \tag{3}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where U_{ad} is a class of admissible controls.

In [2] we have derived first-order optimality conditions for optimal control problem (1)–(4) and carried out their realization under some additional assumptions. We introduced the notion of a quasi-adjoint state ψ_ε to an optimal solution $y_0 \in W_0^{1,p}(\Omega)$ that was proposed for linear problems by Serovajskiy [5]) and showed that an optimality system can be recovered in an explicit form if the mapping $U_{ad} \ni \mathcal{U} \mapsto \psi_\varepsilon(\mathcal{U})$ possesses the so-called $\mathcal{H}$-property with respect to the pair of spaces $\big(L^\infty(\Omega; \mathbb{R}^{N\times N}), W_0^{1,p}(\Omega)\big)$. However, it should be stressed that the fulfilment of this property was not proved for the case $p > 2$ and, thus, had to be considered as some extra hypothesis. Moreover, the verification of the $\mathcal{H}$-property for quasi-adjoint states is not straightforward, in general. That is why, in order to derive optimality conditions in the framework of more appropriate assumptions, we propose in Section 3 of the current paper another approach which is based on the concept of convergence in variable spaces.

For simplicity, we restrict our consideration to the case when each admissible control $\mathcal{U} \in U_{ad} := U_b \cap U_{sol}$ has a diagonal form. Our main assumption in this section is as follows: For a given distribution $f \in W^{-1,q}(\Omega)$ with $q = \frac{p}{p-1}$ and $p \geq 2$ and a given class of admissible controls U_{ad}, there exists a non-negative function $\zeta \in L^1(\Omega)$ such that $\zeta^{-1} \in L^1(\Omega)$ and the corresponding weak solutions $y\,(\mathcal{U})$ of the nonlinear Dirichlet boundary value problem (3), (4) satisfy the relations

$$\big(\xi, \mathcal{U}[(\nabla y)^{p-2}]\xi\big)_{\mathbb{R}^N} \geq \zeta(x)\,\|\xi\|^2_{\mathbb{R}^N} \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{R}^N,\ \text{and } |y|^{2-p} \in L^1(\Omega)$$

for each admissible control $\mathcal{U} \in U_{ad}$.

Note that this assumption is fulfilled as far as the matrices $\mathcal{U}[(\nabla y)^{p-2}]$ have a non-degenerate spectrum for each $\mathcal{U} \in U_{ad}$ (for the details, we refer to [3]). The main argument given in Section 3 is to associate with each admissible pair $(\mathcal{U}, y(\mathcal{U}))$ an appropriate weighted Sobolev space $H^p_{\mathcal{U},y}(\Omega)$ with continuous embedding $H^p_{\mathcal{U},y}(\Omega) \subset W_0^{1,1}(\Omega)$. As a result, we show that each of the variational problems for the corresponding quasi-adjoint states has a unique solution, and these solutions form a weakly convergent sequence $\big\{\psi_{\varepsilon_\theta,\theta} \in H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)\big\}_{\theta\to 0}$ in the variable space. This property suffices in order to establish that the optimality system for the problem (1)–(4), that was derived in [2], remains valid even if $\mathcal{H}$-property does not hold for the quasi-adjoint states.

2. Problem setting

Monotone operators. Let α and β be constants such that $0 < \alpha \le \beta < +\infty$. We define $M_p^{\alpha,\beta}(\Omega)$ as a set of all square symmetric matrices $\mathcal{U}(x) = [a_{i\,j}(x)]_{1\le i,j\le N}$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$ such that the following conditions of growth, monotonicity, and strong coercivity are fulfilled:

$$|a_{ij}(x)| \le \beta \quad \text{a.e. in } \Omega\ \forall\, i,j \in \{1,\dots,N\}, \tag{5}$$

and

$$\big(\mathcal{U}(x)([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta), \zeta - \eta\big)_{\mathbb{R}^N} \ge 0 \quad \text{a.e. in } \Omega\ \forall\,\zeta,\eta \in \mathbb{R}^N, \tag{6}$$

$$\big(\mathcal{U}(x)[\zeta^{p-2}]\zeta,\zeta\big)_{\mathbb{R}^N} = \sum_{i,j=1}^{N} a_{i\,j}(x)|\zeta_j|^{p-2}\,\zeta_j\,\zeta_i \ge \alpha\,|\zeta|_p^p \quad \text{a.e. in } \Omega, \tag{7}$$

where $|\eta|_p = \Big(\sum_{k=1}^N |\eta_k|^p\Big)^{\frac{1}{p}}$ is the Hölder norm of $\eta \in \mathbb{R}^N$ and

$$[\eta^{p-2}] = \operatorname{diag}\{|\eta_1|^{p-2}, |\eta_2|^{p-2}, \dots, |\eta_N|^{p-2}\} \quad \forall\,\eta \in \mathbb{R}^N. \tag{8}$$

Let us consider the nonlinear operator $A : M_p^{\alpha,\beta}(\Omega) \times W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ defined as

$$A(\mathcal{U}, y) = -\operatorname{div}\big(\mathcal{U}(x)[(\nabla y)^{p-2}]\nabla y\big) + |y|^{p-2}y,$$

or via the paring

$$\langle A(\mathcal{U},y), v\rangle_{W_0^{1,p}(\Omega)} = \sum_{i,j=1}^{N} \int_\Omega \left(a_{ij}(x)\left|\frac{\partial y}{\partial x_j}\right|^{p-2}\frac{\partial y}{\partial x_j}\right)\frac{\partial v}{\partial x_i}\,dx + \int_\Omega |y|^{p-2}y\,v\,dx,$$

for all $v \in W_0^{1,p}(\Omega)$. In view of properties (5)–(7), for every $f \in W^{-1,q}(\Omega)$ the nonlinear Dirichlet boundary value problem

$$A(\mathcal{U}, y) = f \quad \text{in } \Omega, \quad y \in W_0^{1,p}(\Omega), \tag{9}$$

amits a unique weak solution in $W_0^{1,p}(\Omega)$ for every fixed matrix $\mathcal{U} \in M_p^{\alpha,\beta}(\Omega)$.

Optimal control problem. Let ξ_1, ξ_2 be given functions of $L^\infty(\Omega)$ such that $0 \le \xi_1(x) \le \xi_2(x)$ a.e. in Ω and $\{Q_1,\dots,Q_N\}$ be a collection of nonempty compact convex subsets of $W^{-1,q}(\Omega)$. To define the class of admissible controls, we introduce two sets

$$U_b = \big\{\mathcal{U} = [a_{i\,j}] \in M_p^{\alpha,\beta}(\Omega) \,\big|\, \xi_1(x) \le a_{i\,j}(x) \le \xi_2(x) \text{ a.e. in } \Omega,\ 1 \le i,j \le N\big\}$$

$$U_{sol} = \big\{\mathcal{U} = [u_1,\dots,u_N] \in M_p^{\alpha,\beta}(\Omega) \,\big|\, \operatorname{div} u_i \in Q_i,\ \forall\, i = 1,\dots,N\big\},$$

assuming that the intersection $U_b \cap U_{sol} \subset L^\infty(\Omega;\mathbb{R}^{N\times N})$ is nonempty.

Definition 2.1. We say that a matrix $\mathcal{U} = [a_{i\,j}]$ is an admissible control of solenoidal type to the nonlinear Dirichlet problem (9) if $\mathcal{U} \in U_{ad} := U_b \cap U_{sol}$.

Let us consider the optimal control problem

$$\text{Minimize } \left\{ I(\mathcal{U}, y) = \int_\Omega |y(x) - y_d(x)|^p \, dx \right\}, \tag{10}$$

subject to the constraints

$$\int_\Omega \big(\mathcal{U}(x)[(\nabla y)^{p-2}]\nabla y, \nabla v\big)_{\mathbb{R}^N} dx + \int_\Omega |y|^{p-2} y v dx = \langle f, v\rangle_{W_0^{1,p}(\Omega)} \quad \forall\, v \in W_0^{1,p}(\Omega), \tag{11}$$

$$\mathcal{U} \in U_{ad}, \quad y \in W_0^{1,p}(\Omega), \tag{12}$$

where $f \in W^{-1,q}(\Omega)$ and $y_d \in W_0^{1,p}(\Omega)$ are given distributions.

Hereinafter, $\Xi_{sol} \subset L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ denotes the set of all admissible pairs to optimal control problem (10)–(12), for which the following existence result takes place.

Theorem 2.2. *If $U_{ad} = U_b \cap U_{sol} \neq \emptyset$, then the optimal control problem* (10)–(12) *admits at least one solution*

$$(\mathcal{U}^{opt}, y^{opt}) \in \Xi_{sol} \subset L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega),$$
$$I(\mathcal{U}^{opt}, y^{opt}) = \inf_{(\mathcal{U},y)\in\Xi_{sol}} I(\mathcal{U}, y).$$

Optimality conditions. To derive the optimality conditions for given optimal control problem (10)–(12) in [2], we used the following concept of quasi-adjoint states, that was first introduced for linear problems by Serovajskiy [5].

Definition 2.3. We say that, for a given $\mathcal{U} \in U_{sol}$, a distribution ψ_ε is the quasi-adjoint state to $y_0 \in W_0^{1,p}(\Omega)$ if ψ_ε satisfies the following integral identity:

$$\left.\begin{aligned} &(p-1)\int_\Omega \big([(\nabla y_\varepsilon)^{p-2}]\mathcal{U}\nabla\psi_\varepsilon, \nabla\varphi\big)_{\mathbb{R}^N} dx \\ &+ (p-1)\int_\Omega |y_\varepsilon|^{p-2}\,\psi_\varepsilon\,\varphi\, dx + p\int_\Omega |y_\varepsilon - y_d|^{p-1}\varphi\, dx \\ &= 0 \end{aligned}\right\} \quad \forall\,\varphi \in W_0^{1,p}(\Omega).$$

Here, $y_\varepsilon = y_0 - \varepsilon(y - y_0)$, $y = y(\mathcal{U})$ is the solution of problem (11), (12), and $\varepsilon = \varepsilon(\mathcal{U}) \in [0, 1]$ (see [2, Lemma 4.8]).

As was shown in [2], in order to derive the optimality conditions to problem (10)–(12) in a correct way, the following property of quasi-adjoint states had to be fulfilled.

Definition 2.4. We say that mapping $U_{ad} \ni \mathcal{U} \mapsto \psi_\varepsilon(\mathcal{U})$ possesses the $\mathcal{H}$-property at the point $\widetilde{\mathcal{U}}$ with respect to the pair of spaces $\left(L^\infty(\Omega;\mathbb{R}^{N\times N}), W_0^{1,p}(\Omega)\right)$ if for each $\mathcal{U} \in U_{ad}$ we have: $\psi_{\varepsilon,\theta} := \psi_\varepsilon(\widetilde{\mathcal{U}} + \theta(\mathcal{U} - \widetilde{\mathcal{U}})) \in W_0^{1,p}(\Omega)$ for all $\theta \in [0,1]$ and the sequence $\{\psi_{\varepsilon,\theta}\}_\theta$ is uniformly bounded in $W_0^{1,p}(\Omega)$ with respect to $\theta \in [0,1]$.

Theorem 2.5. *Let us suppose that $f \in W^{-1,q}(\Omega)$, $y_d \in W_0^{1,p}(\Omega)$, and $U_{ad} \neq \emptyset$ are given with $p \geq 2$. Let $(\mathcal{U}_0, y_0) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ be an optimal pair to the problem* (10)–(12). *Assume that the quasi-adjoint state $\psi_\varepsilon(\mathcal{U})$ to $y_0 \in W_0^{1,p}(\Omega)$, defined by* (2.3), *possesses the $\mathcal{H}$-property at $\mathcal{U}_0$ in the sense of Definition* 2.4. *Then there exists an element $\overline{\psi} \in W_0^{1,p}(\Omega)$ such that*

$$\int_\Omega \left((\mathcal{U} - \mathcal{U}_0)[(\nabla y_0)^{p-2}]\, \nabla y_0, \nabla\overline{\psi}\right)_{\mathbb{R}^N} dx \geq 0, \quad \forall \mathcal{U} \in U_{ad}, \tag{13}$$

$$\begin{aligned} &\int_\Omega \left(\mathcal{U}_0[(\nabla y_0)^{p-2}]\nabla y_0, \nabla\varphi\right)_{\mathbb{R}^N} dx + \int_\Omega |y_0|^{p-2} y_0 \varphi\, dx \\ &= \langle f, \varphi\rangle_{W_0^{1,p}(\Omega)}, \quad \forall\, \varphi \in W_0^{1,p}(\Omega), \end{aligned} \tag{14}$$

$$\begin{aligned} &(p-1)\int_\Omega \left([(\nabla y_0)^{p-2}]\, \mathcal{U}_0 \nabla\overline{\psi}, \nabla\varphi\right)_{\mathbb{R}^N} dx + (p-1)\int_\Omega |y_0|^{p-2}\, \overline{\psi}\, \varphi\, dx \\ &= p\int_\Omega |y_0 - y_d| \varphi\, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega). \end{aligned} \tag{15}$$

3. On weighted Sobolev spaces and fine properties of quasi-adjoint states

The aim of this paper is to provide a substantiation of (13)–(15) regardless of the $\mathcal{H}$-property for quasi-adjoints states. For simplicity, we restrict our consideration to the case when each admissible control $\mathcal{U} \in U_{ad} := U_b \cap U_{sol}$ has a diagonal form

$$\begin{aligned} &\mathcal{U}(x) = \operatorname{diag}\{\delta_1(x), \delta_2(x), \dots, \delta_N(x)\}, \\ &\text{with} \;\; \alpha \leq \delta_i(x) \leq \beta \quad \text{a.e. in } \Omega,\ \forall\, i = 1, \dots, N. \end{aligned} \tag{16}$$

We define subset $\mathfrak{M}(\Omega) \subset W_0^{1,p}(\Omega)$ as follows: $y \in \mathfrak{M}(\Omega)$ if and only if

$$\exists\, \zeta \in L^1(\Omega) \text{ such that } \zeta > 0 \text{ a.e. in } \Omega,\ \zeta^{-1} \in L^1(\Omega), \tag{17}$$

$$\left(\xi, \mathcal{U}[(\nabla y)^{p-2}]\xi\right)_{\mathbb{R}^N} \geq \zeta(x) \|\xi\|^2_{\mathbb{R}^N} \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{R}^N, \text{ and } |y|^{2-p} \in L^1(\Omega). \tag{18}$$

We begin this section with the following hypothesis.

(H2) For a given distribution $f \in W^{-1,q}(\Omega)$ with $q = \frac{p}{p-1}$ and $p \geq 2$, the corresponding weak solutions $y(\mathcal{U})$ of the nonlinear Dirichlet boundary value problem (9) satisfy: $y(\mathcal{U}) \in \mathfrak{M}(\Omega)$ for each admissible control $\mathcal{U} \in U_{ad} := U_b \cap U_{sol}$.

Remark 3.1. Since (8) and (16) imply that $\mathcal{U}[(\nabla y)^{p-2}] = [(\nabla y)^{p-2}]\mathcal{U}$, it follows that the quadratic form $(\xi, \mathcal{U}[(\nabla y)^{p-2}]\xi)_{\mathbb{R}^N}, \forall\, \xi \in \mathbb{R}^N$ can be associated with the collection $\left\{\lambda_1^{\mathcal{U},y}, \dots, \lambda_N^{\mathcal{U},y}\right\}$ of eigenvalues of the symmetric matrix $\mathcal{U}[(\nabla y)^{p-2}]$, where each $\lambda_k^{\mathcal{U},y} = \lambda_k^{\mathcal{U},y}(x)$ is counted with its multiplicity. Hence, the assumptions (17), (18) can be fulfilled if the sets

$$S = \{x \in \Omega : \; y(x) = 0\} \quad \text{and} \quad \widehat{S} = \bigcup_{i=1}^{N} \left\{ x \in \Omega : \; \prod_{i=1}^{N} \lambda_i^{\mathcal{U},y}(x) = 0 \right\}$$

have zero Lebesgue measure. On the other hand, Hypothesis (H2) can be omitted if we consider conditions (17), (18) as extra state constraints to the original optimal control problem (10)–(12).

Let $(\mathcal{U}, y) \in L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ be an arbitrary pair such that the diagonal matrix $\mathcal{U}$ is an admissible control to problem (10)–(12). To each such pair $(\mathcal{U}, y)$ we associate two weighted Sobolev spaces:

$$H^p_{\mathcal{U},y}(\Omega) := H(\Omega; |y|^{p-2}dx \times [(\nabla y)^{p-2}]\mathcal{U}\, dx^N)$$

which we define as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\psi\|_{\mathcal{U},y} = \left(\int_\Omega \left(|y|^{p-2}\psi^2 + \left(\nabla\psi, \mathcal{U}[(\nabla y)^{p-2}]\nabla\psi\right)_{\mathbb{R}^N} \right) dx \right)^{\frac{1}{2}}, \tag{19}$$

and $W^p_{\mathcal{U},y}(\Omega)$ which is the set of all functions $y \in W_0^{1,1}(\Omega)$ with finite $\|\cdot\|_{\mathcal{U},y}$-norm.

In order to establish the main properties of the spaces $H^p_{\mathcal{U},y}(\Omega)$ and $W^p_{\mathcal{U},y}(\Omega)$, we make use of the following observation.

Lemma 3.2. *Let $(\mathcal{U}, y) \in U_{ad} \times W_0^{1,p}(\Omega)$ be an arbitrary pair. Then there exists a non-negative function $f \in L^1(\Omega)$ such that*

$$\begin{aligned} |y|^{p-2} &\leq f(x) && a.e. \text{ in } \Omega, \\ \left(\xi, \mathcal{U}[(\nabla y)^{p-2}]\xi\right)_{\mathbb{R}^N} &\leq f(x)\, \|\xi\|^2_{\mathbb{R}^N} && a.e. \text{ in } \Omega, \quad \forall\, \xi \in \mathbb{R}^N. \end{aligned}$$

Proof. We show that $|y|^{p-2} \in L^1(\Omega)$ and $[(\nabla y)^{p-2}]\mathcal{U} \in L^1(\Omega; \mathbb{R}^{N\times N})$. Indeed, using Hölder's inequality, we have

$$\||y|^{p-2}\|_{L^1(\Omega)} \leq \left(\int_\Omega |y|^p\, dx \right)^{\frac{p-2}{p}} |\Omega|^{\frac{2}{p}} \leq c\|y\|^{p-2}_{W_0^{1,p}(\Omega)} < +\infty,$$

$$
\begin{aligned}
\|[(\nabla y)^{p-2}]\mathcal{U}\|_{L^1(\Omega;\mathbb{R}^{N\times N})} &\le \|\mathcal{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial y}{\partial x_i}\right|^{p-2} dx \\
&\le \|\mathcal{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} |\Omega|^{\frac{2}{p}} \sum_{i=1}^{N} \left\|\frac{\partial y}{\partial x_i}\right\|_{L^p(\Omega)}^{p-2} \\
&\le c_1 \|y\|_{W_0^{1,p}(\Omega)}^{p-2} \\
&< +\infty.
\end{aligned}
$$

This immediately leads us to the required conclusion. □

Let $\zeta_{ad} : \Omega \to \mathbb{R}^1_+$ be a non-negative function satisfying the properties

$$
\begin{aligned}
&\zeta_{ad} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \in L^1(\Omega), \quad \zeta_{ad}^{-1} \not\in L^\infty(\Omega), \\
&\zeta_{ad} : \Omega \to \mathbb{R}^1_+ \ \text{ is smooth function along the boundary } \partial\Omega, \\
&\zeta_{ad} = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

In view of Lemma 3.2 and properties (17), (18), we pose the following hypothesis.

(H3) There exist elements f_* and ζ_* in $L^1(\Omega)$ such that $f_* > \zeta_* \ge \zeta_{ad}$ and, for each $\left(\widehat{\mathcal{U}}, \widehat{y}\right) \in U_{ad} \times \mathfrak{M}(\Omega)$, $\eta \in \mathbb{R}^1$, and $\xi \in \mathbb{R}^N$, the following conditions hold true almost everywhere in Ω:

$$
(\eta^2 + \|\xi\|^2_{\mathbb{R}^N})\zeta_* \le |\widehat{y}|^{p-2}\eta^2 + \left(\xi, \widehat{\mathcal{U}}[(\nabla\widehat{y})^{p-2}]\xi\right)_{\mathbb{R}^N} \le (\eta^2 + \|\xi\|^2_{\mathbb{R}^N}) f_*. \tag{20}
$$

We now concentrate on properties of the spaces $H^p_{\mathcal{U},y}(\Omega)$ and $W^p_{\mathcal{U},y}(\Omega)$. To this end, we note that due to Hypothesis (H2), inequality (20) and estimates

$$
\int_\Omega |\psi|\, dx \le \left(\int_\Omega |y|^{p-2}\psi^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |y|^{2-p}\, dx\right)^{\frac{1}{2}} \le C\|\psi\|_{\mathcal{U},y}, \tag{21}
$$

$$
\begin{aligned}
\int_\Omega \|\nabla\psi\|_{\mathbb{R}^N}\, dx &\le \left(\int_\Omega \|\nabla\psi\|^2_{\mathbb{R}^N}\zeta_*\, dx\right)^{\frac{1}{2}} \left(\int_\Omega \zeta_*^{-1}\, dx\right)^{\frac{1}{2}} \\
&\le C\left(\int_\Omega (\nabla\psi, \mathcal{U}[(\nabla y)^{p-2}]\nabla\psi)_{\mathbb{R}^N}\, dx\right)^{\frac{1}{2}} \\
&\le C\|\psi\|_{\mathcal{U},y},
\end{aligned} \tag{22}
$$

the spaces $H^p_{\mathcal{U},y}(\Omega)$ and $W^p_{\mathcal{U},y}(\Omega)$ are complete with respect to the norm $\|\cdot\|_{\mathcal{U},y}$ for every $(\mathcal{U}, y) \in U_{ad} \times \mathfrak{M}(\Omega)$. Moreover, following the initial definitions, we have $H^p_{\mathcal{U},y}(\Omega) \subseteq W^p_{\mathcal{U},y}(\Omega)$. However, for a 'typical' weight $|y|^{p-2}dx \times [(\nabla y)^{p-2}]\mathcal{U}\, dx^N$ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in $W^p_{\mathcal{U},y}(\Omega)$. Hence, the identity $W^p_{\mathcal{U},y}(\Omega) = H^p_{\mathcal{U},y}(\Omega)$ is not always valid (for the corresponding examples we refer to [7]). At the same time, it is clear that $H^p_{\mathcal{U},y}(\Omega) \subset W^{1,1}_0(\Omega)$ with continuous embedding (see (21), (22)), and due to the estimates

$$\int_\Omega |y|^{p-2}\psi^2\,dx \le \|y\|^{p-2}_{L^p(\Omega)}\|\psi\|^2_{L^p(\Omega)} \le C\|\psi\|^2_{W^{1,p}_0(\Omega)},$$

$$\int_\Omega (\nabla\psi, \mathcal{U}[(\nabla y)^{p-2}]\nabla\psi)_{\mathbb{R}^N}\,dx \le \|\mathcal{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})}\|\nabla y\|^{p-2}_{L^p(\Omega;\mathbb{R}^N)}\|\nabla\psi\|^2_{L^p(\Omega;\mathbb{R}^N)} \le C\|\psi\|^2_{W^{1,p}_0(\Omega)},$$

we have: $W^{1,p}_0(\Omega) \subset H^p_{\mathcal{U},y}(\Omega)$ with continuous embedding. Moreover, since $\mathcal{U}$ is a diagonal matrix, it follows that $\mathcal{U}[(\nabla y)^{p-2}]$ is symmetric and, hence, $H^p_{\mathcal{U},y}(\Omega)$ is a Hilbert space with the inner product

$$(z_1, z_2) = \int_\Omega \left(|y|^{p-2}z_1z_2 + \left(\nabla z_1, \mathcal{U}[(\nabla y)^{p-2}]\nabla z_2\right)_{\mathbb{R}^N}\right)dx. \tag{23}$$

Remark 3.3. Some spaces of more or less similar type have been studied by Casas and Fernández [1], Murthy and Stampacchia [4], Trudinger [6]. However, in contrast to the mentioned papers, we do not have a continuous embedding of $H^p_{\mathcal{U},y}(\Omega)$ into the reflexive Banach space $W^{1,2}(\Omega)$.

3.1. Weak convergence in variable $H^p_{\mathcal{U},y}$-spaces. Under the Hypotheses (H1)–(H3), we first provide some auxiliary results.

Lemma 3.4. *Assume that $(\mathcal{U}_k, y_k) \to (\mathcal{U}, y)$ strongly in $L^\infty(\Omega;\mathbb{R}^{N\times N})\times W^{1,p}_0(\Omega)$ as $k\to\infty$. Then*

$$\begin{aligned} |y_k|^{p-2} &\to |y|^{p-2} && \text{in } L^1(\Omega), \quad \text{and} \\ \mathcal{U}_k[(\nabla y_k)^{p-2}] &\to \mathcal{U}[(\nabla y)^{p-2}] && \text{in } L^1(\Omega;\mathbb{R}^{N\times N}) \quad \text{as } k\to\infty. \end{aligned} \tag{24}$$

Proof. In order to show that

$$\int_\Omega \left||y_k|^{p-2} - |y|^{p-2}\right|dx \to 0 \quad\text{and}\quad \int_\Omega \left\|\mathcal{U}_k[(\nabla y_k)^{p-2}] - \mathcal{U}[(\nabla y)^{p-2}]\right\|_{\mathbb{R}^{N\times N}}dx \to 0,$$

it is enough to consider two cases: $p > 3$ and $2 \le p \le 3$, and repeat the trick we made in the proof of [2, Theorem 5.4]. □

Lemma 3.5. *The set $U_{ad}\times\mathfrak{M}(\Omega)$ is sequentially closed with respect to the strong topology of $L^\infty(\Omega;\mathbb{R}^{N\times N})\times W^{1,p}_0(\Omega)$.*

Proof. Let $\{(\mathcal{U}_k, y_k)\}_{k\in\mathbb{N}}$ be an arbitrary sequence such that $(\mathcal{U}_k, y_k) \to (\mathcal{U}, y)$ strongly in $L^\infty(\Omega;\mathbb{R}^{N\times N})\times W^{1,p}_0(\Omega)$ as $k\to\infty$, and $(\mathcal{U}_k, y_k) \in U_{ad}\times\mathfrak{M}(\Omega)$ for all $k\in\mathbb{N}$. Since the inclusion $\mathcal{U}\in U_{ad}$ is guaranteed by Theorem [2, Theorem 3.3], it remains to show that $y\in\mathfrak{M}(\Omega)$. Due to Hypothesis (H3), there exists a couple of functions f_* and ζ_* in $L^1(\Omega)$ such that $f_* > \zeta_* \ge \zeta_{ad}$ and, for each $\xi \in \mathbb{R}^N$ and $k \in \mathbb{N}$,

$$\zeta_* \le |y_k|^{p-2} \le f_* \quad \text{and} \quad \zeta_* \|\xi\|^2_{\mathbb{R}^N} \le \big(\xi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\xi\big)_{\mathbb{R}^N} \le f_* \|\xi\|^2_{\mathbb{R}^N} \text{ a.e. in } \Omega. \quad (25)$$

Following the initial assumptions and Lemma 3.4, we can pass to the limit in (25) as $k \to \infty$. As a result, we get

$$\zeta_* \le |y|^{p-2} \le f_* \quad \text{and} \quad \zeta_* \|\xi\|^2_{\mathbb{R}^N} \le \big(\xi, \mathcal{U}[(\nabla y)^{p-2}]\xi\big)_{\mathbb{R}^N} \le f_* \|\xi\|^2_{\mathbb{R}^N} \text{ a.e. in } \Omega.$$

The proof is complete. □

As an obvious consequence of Lemma 3.5, we have the following result.

Corollary 3.6. *Let $\{(\mathcal{U}_k, y_k)\}_{k\in\mathbb{N}}$ be a sequence in $U_{ad} \times \mathfrak{M}(\Omega)$ such that $(\mathcal{U}_k, y_k) \to (\mathcal{U}, y)$ strongly in $L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ as $k \to \infty$. Then*

$$|y_k|^{2-p} \to |y|^{2-p} \text{ in } L^1(\Omega), \text{ and } \mathcal{U}_k^{-1}[(\nabla y_k)^{2-p}] \to \mathcal{U}^{-1}[(\nabla y)^{2-p}] \text{ in } L^1(\Omega; \mathbb{R}^{N\times N})$$

as $k \to \infty$.

Proof. Indeed, following the initial assumptions, we may assume that

$$|y_k|^{2-p} \to |y|^{2-p}, \quad \text{and} \quad \mathcal{U}_k^{-1}[(\nabla y_k)^{2-p}] \to \mathcal{U}^{-1}[(\nabla y)^{2-p}]$$

almost everywhere in Ω. Since, in view of (25), the sequences $\{|y_k|^{2-p}\}_{k\in\mathbb{N}}$ and $\{\mathcal{U}_k^{-1}[(\nabla y_k)^{2-p}]\}_{k\in\mathbb{N}}$ are equi-integrable, the required assertion immediately follows from Lebesgue's Theorem (see [2, Lemma 2.1]). □

Let $\{(\mathcal{U}_k, y_k)\}_{k\in\mathbb{N}}$ be an arbitrary sequence such that $(\mathcal{U}_k, y_k) \to (\mathcal{U}, y)$ strongly in $L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ as $k \to \infty$, and $(\mathcal{U}_k, y_k) \in U_{ad} \times \mathfrak{M}(\Omega)$ for all $k \in \mathbb{N}$. Hereinafter in this subsection we will associate this sequence with the collection of variable spaces $\{H^p_{\mathcal{U}_k, y_k}(\Omega)\}_{k\in\mathbb{N}}$.

Definition 3.7. We say that a sequence $\{\psi_k \in H^p_{\mathcal{U}_k, y_k}(\Omega)\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty} \|\psi_k\|^2_{\mathcal{U}_k, y_k} := \limsup_{k\to\infty} \int_\Omega \big(|y_k|^{p-2}\psi_k^2 + \big(\nabla\psi_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla\psi_k\big)_{\mathbb{R}^N}\big)\,dx < +\infty.$$

Definition 3.8. A bounded sequence $\{\psi_k \in H^p_{\mathcal{U}_k, y_k}(\Omega)\}_{k\in\mathbb{N}}$ is weakly convergent to a function $\psi \in H^p_{\mathcal{U}, y}(\Omega)$ in the variable space $H^p_{\mathcal{U}_k, y_k}(\Omega)$ if

$$\left.\begin{aligned} &\lim_{k\to\infty} \int_\Omega \big(|y_k|^{p-2}\psi_k\varphi + \big(\nabla\varphi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla\psi_k\big)_{\mathbb{R}^N}\big)\,dx \\ &= \int_\Omega \big(|y|^{p-2}\psi\varphi + \big(\nabla\varphi, \mathcal{U}[(\nabla y)^{p-2}]\nabla\psi\big)_{\mathbb{R}^N}\big)\,dx, \end{aligned}\right\} \quad \forall\,\varphi \in C_0^\infty(\Omega).$$

In order to motivate this definition, we provide an accurate analysis of this concept and introduce the following weighted spaces: $L^2(\Omega;\mathcal{U}[(\nabla y)^{p-2}]\,dx)$ as the set of measurable vector-valued functions $\mathbf{f}\in\mathbb{R}^N$ on Ω such that

$$\|\mathbf{f}\|_{L^2(\Omega,\mathcal{U}[(\nabla y)^{p-2}]\,dx)}=\left(\int_\Omega(\mathbf{f},\mathcal{U}[(\nabla y)^{p-2}]\mathbf{f})_{\mathbb{R}^N}\,dx\right)^{\frac{1}{2}}<+\infty,$$

and the space $L^2(\Omega;|y|^{p-2}\,dx)$ as the set of measurable functions $v\in\mathbb{R}$ on Ω such that

$$\|v\|_{L^2(\Omega,|y|^{p-2}\,dx)}=\left(\int_\Omega|y|^{p-2}v^2\,dx\right)^{\frac{1}{2}}<+\infty.$$

By analogy with Definition 3.8, the concept of weak convergence can be obviously reduced to the spaces $L^2(\Omega;\mathcal{U}[(\nabla y)^{p-2}]\,dx)$ and $L^2(\Omega;|y|^{p-2}\,dx)$. Now we come to the following results concerning certain convergence properties in variable Lebesgue's spaces.

Proposition 3.9. *If a sequence* $\left\{\psi_k\in H^p_{\mathcal{U}_k,y_k}(\Omega)\right\}_{k\in\mathbb{N}}$ *is bounded and* (16) *holds true for each* $\mathcal{U}_k\in U_{ad}$, *then the sequences* $\{\nabla\psi_k\in L^2(\Omega;\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)\}_{k\in\mathbb{N}}$ *and* $\{\psi_k\in L^2(\Omega;|y_k|^{p-2}\,dx)\}_{k\in\mathbb{N}}$ *are sequentially compact in the sense of weak convergence in variable spaces* $L^2(\Omega;\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)$ *and* $L^2(\Omega;|y|^{p-2}\,dx)$, *respectively.*

Proof. Having set

$$L_k(\phi)=\int_\Omega\left(\phi,\mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla\psi_k\right)_{\mathbb{R}^N}\,dx,\ \forall\,\phi\in C_0^\infty(\Omega)^N$$

and making use of the Hölder inequality, we get

$$\begin{aligned}
&|L_k(\phi)|\\
&=\left|\int_\Omega\left(\mathcal{U}_k^{\frac{1}{2}}[(\nabla y_k)^{p-2}]^{\frac{1}{2}}\phi,\mathcal{U}_k^{\frac{1}{2}}[(\nabla y_k)^{p-2}]^{\frac{1}{2}}\nabla\psi_k\right)_{\mathbb{R}^N}dx\right|\quad\{\text{by (16) and (8)}\}\\
&\le\left(\int_\Omega\|\mathcal{U}_k^{\frac{1}{2}}[(\nabla y_k)^{p-2}]^{\frac{1}{2}}\nabla\psi_k\|^2_{\mathbb{R}^N}dx\right)^{\frac{1}{2}}\left(\int_\Omega\|\mathcal{U}_k^{\frac{1}{2}}[(\nabla y_k)^{p-2}]^{\frac{1}{2}}\phi\|^2_{\mathbb{R}^N}dx\right)^{\frac{1}{2}}\\
&=\left(\int_\Omega(\nabla\psi_k,\mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla\psi_k)_{\mathbb{R}^N}\,dx\right)^{\frac{1}{2}}\left(\int_\Omega\left(\phi,\mathcal{U}_k[(\nabla y_k)^{p-2}]\phi\right)_{\mathbb{R}^N}\,dx\right)^{\frac{1}{2}}\\
&\le C\left(\int_\Omega\left(\phi,\mathcal{U}_k[(\nabla y_k)^{p-2}]\phi\right)_{\mathbb{R}^N}\,dx\right)^{\frac{1}{2}}\\
&\le C\left(\int_\Omega f(x)\,\|\phi(x)\|^2_{\mathbb{R}^N}\,dx\right)^{\frac{1}{2}}\quad\{\text{by Lemma 3.2}\}\\
&\le C\|\phi\|_{C(\Omega;\mathbb{R}^N)}\|f\|^{\frac{1}{2}}_{L^1(\Omega)}\quad\forall\,k\in\mathbb{N}.
\end{aligned}$$

Since the set $C_0^\infty(\Omega;\mathbb{R}^N)$ is separable with respect to the norm $\|\cdot\|_{C(\Omega;\mathbb{R}^N)}$ and $\{L_k(\varphi)\}_{k\in\mathbb{N}}$ is a uniformly bounded sequence of linear functionals, it follows that there exists a subsequence of positive numbers $\{k_j\}_{j=1}^\infty$ for which the limit (in the sense of pointwise convergence)

$$\lim_{j\to\infty} L_{k_j}(\phi) = L(\phi)$$

is well defined for every $\phi \in C_0^\infty(\Omega)^N$. As a result, by Lemma 3.4, we have

$$|L(\phi)| \leq C \lim_{k\to\infty} \left(\int_\Omega \big(\phi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\phi\big)_{\mathbb{R}^N} \, dx \right)^{\frac{1}{2}} = C \left(\int_\Omega \big(\phi, \mathcal{U}[(\nabla y)^{p-2}]\phi\big)_{\mathbb{R}^N} \, dx \right)^{\frac{1}{2}}.$$

Hence, $L(\phi)$ is a continuous functional on $L^2(\Omega;\mathcal{U}[(\nabla y)^{p-2}]\,dx)$ and, therefore, admits the following representation

$$L(\phi) = \int_\Omega \big(\mathbf{v}, \mathcal{U}[(\nabla y)^{p-2}]\phi\big)_{\mathbb{R}^N} \, dx,$$

where $\mathbf{v}$ is some element of Hilbert space $L^2(\Omega;\mathcal{U}[(\nabla y)^{p-2}]dx)$. Thus, $\mathbf{v}$ can be taken as the weak limit of $\{\nabla\psi_k\}_{k\in\mathbb{N}}$ in the variable space $L^2(\Omega;\mathcal{U}_k[(\nabla y_k)^{p-2}]dx)$.

Having set

$$L_k(\phi) = \int_\Omega |y_k|^{p-2}\psi_k\varphi \, dx = \int_\Omega \left(|y_k|^{\frac{p-2}{2}}\psi_k\right)\left(|y_k|^{\frac{p-2}{2}}\varphi\right) dx, \quad \forall\, \varphi \in C_0^\infty(\Omega)$$

and following the same reasoning, it can be easily proved that the compactness property of the sequence $\{\psi_k\}_{k\in\mathbb{N}}$ in the variable space $L^2(\Omega; |y_k|^{p-2}\,dx)$ holds true as well. □

Definition 3.10. We say that a sequence $\{\mathbf{v}_k \in L^2(\Omega,\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)\}_{k\in\mathbb{N}}$ strongly converges to a function $\mathbf{v} \in L^2(\Omega,\mathcal{U}[(\nabla y)^{p-2}]\,dx)$ if

$$\lim_{k\to\infty} \int_\Omega \big(\mathbf{b}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N} \, dx = \int_\Omega \big(\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N} \, dx \tag{26}$$

whenever $\mathbf{b}_k \rightharpoonup \mathbf{b}$ in $L^2(\Omega,\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)^N$ as $k\to\infty$.

The next property of weak convergence in $L^2(\Omega,\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)$ shows that the variable L^2-norm is lower semicontinuous with respect to weak convergence.

Proposition 3.11. *If a sequence $\{\mathbf{v}_k \in L^2(\Omega,\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)\}_{k\in\mathbb{N}}$ converges weakly to $\mathbf{v} \in L^2(\Omega,\mathcal{U}[(\nabla y)^{p-2}]\,dx)$, then*

$$\liminf_{k\to\infty} \int_\Omega \big(\mathbf{v}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N} \, dx \geq \int_\Omega \big(\mathbf{v}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N} \, dx. \tag{27}$$

Proof. Indeed, we have

$$\frac{1}{2}\int_\Omega \big(\mathbf{v}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N}\,dx =$$
$$= \frac{1}{2}\int_\Omega \| \left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{\frac{1}{2}}\mathbf{v}_k\|^2_{\mathbb{R}^N} dx$$
$$\geq \int_\Omega \big(\phi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N}\,dx - \frac{1}{2}\int_\Omega \big(\phi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\phi\big)_{\mathbb{R}^N}\,dx \quad \forall\,\phi \in C_0^\infty(\Omega)^N,$$
$$\frac{1}{2}\liminf_{k\to\infty}\int_\Omega \big(\mathbf{v}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N}\,dx$$
$$\geq \int_\Omega \big(\phi, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N}\,dx - \frac{1}{2}\int_\Omega \big(\phi, \mathcal{U}[(\nabla y)^{p-2}]\phi\big)_{\mathbb{R}^N}\,dx.$$

Since the last inequality is valid for all $\phi \in C_0^\infty(\Omega;\mathbb{R}^N)$ and $C_0^\infty(\Omega;\mathbb{R}^N)$ is a dense subset of $L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\,dx)$, it holds also true for $\phi \in L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\,dx)$. So, taking $\phi = \mathbf{v}$, we arrive at (27). □

For our further analysis we need the following property of strong convergence in variable $L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)$-spaces.

Proposition 3.12. *Weak convergence of* $\{\mathbf{v}_k \in L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}]dx)\}_{k\in\mathbb{N}}$ *to* $\mathbf{v} \in L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\,dx)$ *and*

$$\lim_{k\to\infty}\int_\Omega \big(\mathbf{v}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N}\,dx = \int_\Omega \big(\mathbf{v}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N}\,dx \tag{28}$$

are equivalent to strong convergence of $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ *to* $\mathbf{v} \in L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\,dx)$ *in the sense of Definition* 3.10.

Proof. It is easy to verify that strong convergence implies weak convergence and (28). Indeed, we use $\mathbf{b}_k = \phi \in C_0^\infty(\Omega)^N$ in (26) and then substitute $\mathbf{b}_k = \mathbf{v}_k$.

In view of Proposition 3.9, we may assume that there exist two values ν_1 and ν_2 such that (up to subsequences)

$$\lim_{k\to\infty}\int_\Omega \big(\mathbf{b}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N}\,dx = \nu_1, \quad \lim_{k\to\infty}\int_\Omega \big(\mathbf{b}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{b}_k\big)_{\mathbb{R}^N}\,dx = \nu_2.$$

Using lower semicontinuity (27), we obtain

$$\lim_{k\to\infty}\int_\Omega \big(\mathbf{v}_k + t\mathbf{b}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}](\mathbf{v}_k + t\mathbf{b}_k)\big)_{\mathbb{R}^N}\,dx$$
$$= \lim_{k\to\infty}\int_\Omega \big(\mathbf{v}_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\mathbf{v}_k\big)_{\mathbb{R}^N}\,dx + 2t\nu_1 + t^2\nu_2$$
$$\geq \int_\Omega \big(\mathbf{v} + t\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}](\mathbf{v} + t\mathbf{b})\big)_{\mathbb{R}^N}\,dx$$

$$= \int_\Omega \big(\mathbf{v}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N}\, dx + 2t \int_\Omega \big(\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N}\, dx +$$
$$+ t^2 \int_\Omega \big(\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{b}\big)_{\mathbb{R}^N}\, dx.$$

From this and (28), we conclude that

$$2t\nu_1 + t^2\nu_2 \geq 2t\int_\Omega \big(\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N}\, dx + t^2\int_\Omega \big(\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{b}\big)_{\mathbb{R}^N}\, dx \quad \forall\, t \in \mathbb{R}^1.$$

Hence, $\nu_1 = \int_\Omega \big(\mathbf{b}, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\big)_{\mathbb{R}^N}\, dx$. Thereby the strong convergence of the sequence $\{\mathbf{v}_k \in L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}]\, dx)\}_{k\in\mathbb{N}}$ is established. □

Remark 3.13. It is worth to notice that Propositions 3.11 and 3.12 remain valid if we reformulate them with respect to the variable spaces $L^2(\Omega; |y_k|^{p-2}\, dx)$.

For our further analysis, we make use of the following result.

Lemma 3.14. *Let $\{y_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}(\Omega)$ be a sequence such that $y_k \to y$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. Then*

$$\nabla y_k \to \nabla y \quad \textit{strongly in the variable space } L^2(\Omega, \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)$$

for any $\mathcal{U} \in U_{ad}$.

Proof. Let $\mathcal{U} \in U_{ad}$ be a given matrix. In view of Lemma 3.5, we have $y \in \mathfrak{M}(\Omega)$. Hence, the "limit" weighted space $L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\, dx)$ is correctly defined as a Hilbert space. Further, we note that, $\nabla y_k \in L^2(\Omega, \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)$ for all $k \in \mathbb{N}$. Indeed,

$$\begin{aligned}
\|\nabla y_k\|^2_{L^2(\Omega, \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)} &:= \int_\Omega \big(\nabla y_k, \mathcal{U}[(\nabla y_k)^{p-2}]\nabla y_k\big)_{\mathbb{R}^N}\, dx \\
&= \int_\Omega \operatorname{tr}\left(\mathcal{U}[(\nabla y_k)^p]\right) dx \\
&\leq \beta \|\nabla y_k\|^p_{L^p(\Omega)^N} \\
&\leq C \\
&< +\infty.
\end{aligned}$$

Hence, the sequence $\{\nabla y_k \in L^2(\Omega, \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)\}_{k\in\mathbb{N}}$ is bounded in variable spaces. Then, by Proposition 3.9, this sequence is sequentially compact with respect to the weak convergence in $L^2(\Omega; \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)$. Let us show that $\nabla y \in L^p(\Omega; \mathbb{R}^N)$ is its weak limit in variable space $L^2(\Omega; \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)$. Indeed, for any $\varphi \in C_0^\infty(\Omega)$, we have

$$I = \left| \int_\Omega \big(\nabla\varphi, \mathcal{U}[(\nabla y_k)^{p-2}]\nabla y_k\big)_{\mathbb{R}^N}\, dx - \int_\Omega \big(\nabla\varphi, \mathcal{U}[(\nabla y)^{p-2}]\nabla y\big)_{\mathbb{R}^N}\, dx \right|$$

and consequently

$$
\begin{aligned}
I &\le \|\varphi\|_{C^1(\Omega)} \|\mathcal{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} \int_\Omega \left\| [(\nabla y_k)^{p-2}]\nabla y_k - [(\nabla y)^{p-2}]\nabla y \right\|_{\mathbb{R}^N} dx \\
&\le \beta \|\varphi\|_{C^1(\Omega)} \int_\Omega \sum_{i=1}^N \left| \left(\frac{\partial y_k}{\partial x_i}\right)^{p-1} - \left(\frac{\partial y}{\partial x_i}\right)^{p-1} \right| dx \\
&\le (p-1)\beta \|\varphi\|_{C^1(\Omega)} \int_\Omega \sum_{i=1}^N \left| \frac{\partial y_k}{\partial x_i} - \frac{\partial y}{\partial x_i} \right| \left(\left| \frac{\partial y_k}{\partial x_i} \right| + \left| \frac{\partial y}{\partial x_i} \right| \right)^{p-2} dx.
\end{aligned}
$$

It remains to apply the trick we made in the proof of [2, Theorem 5.4] and use the strong convergence $\nabla y_k \to \nabla y$ in $L^p(\Omega)^N$. Thus, $I \to 0$ as $k \to \infty$ and, hence,

$$
\nabla y_k \rightharpoonup \nabla y \quad \text{in variable space } L^2(\Omega; \mathcal{U}[(\nabla y_k)^{p-2}]\, dx). \tag{29}
$$

In order to conclude the proof, we observe that

$$
\begin{aligned}
\|\nabla y_k\|^2_{L^2(\Omega, \mathcal{U}[(\nabla y_k)^{p-2}]\, dx)} &= \int_\Omega \operatorname{tr}\left(\mathcal{U}[(\nabla y_k)^p]\right) dx = \sum_{i=1}^N \int_\Omega \delta_i(x) \left| \frac{\partial y_k}{\partial x_i} \right|^p dx \\
\stackrel{k\to\infty}{\longrightarrow} \sum_{i=1}^N \int_\Omega \delta_i(x) \left| \frac{\partial y}{\partial x_i} \right|^p dx &= \int_\Omega \operatorname{tr}\left(\mathcal{U}[(\nabla y)^p]\right) dx = \|\nabla y\|^2_{L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\, dx)}
\end{aligned} \tag{30}
$$

by the strong convergence $\nabla y_k \to \nabla y$ in $L^p(\Omega)^N$. Hence, taking into account properties (29), (30), it remains to apply Proposition (3.12). The proof is complete. □

We are now in a position to give the main result of this subsection.

Proposition 3.15. *Let $\{(\mathcal{U}_k, y_k)\}_{k\in\mathbb{N}} \subset U_{ad} \times \mathfrak{M}(\Omega)$ be an arbitrary sequence such that $(\mathcal{U}_k, y_k) \to (\mathcal{U}, y)$ strongly in $L^\infty(\Omega; \mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ as $k \to \infty$. Let a sequence $\left\{\psi_k \in H^p_{\mathcal{U}_k, y_k}(\Omega)\right\}_{k\in\mathbb{N}}$ be bounded. Then there exists an element $\psi \in H^p_{\mathcal{U},y}(\Omega)$ such that within a subsequence $\psi_k \to \psi$ weakly in the variable space $H^p_{\mathcal{U}_k, y_k}(\Omega)$.*

Proof. Due to the compactness criterion for weak convergence in variable spaces (see Proposition 3.9), there exists a pair $(\psi, \mathbf{v}) \in L^2(\Omega; |y|^{p-2}\, dx) \times L^2(\Omega, \mathcal{U}[(\nabla y)^{p-2}]\, dx)$ such that, within a subsequence of $\{\psi_k\}_{k\in\mathbb{N}}$,

$$
\psi_k \rightharpoonup \psi \quad \text{in variable space } L^2(\Omega; |y_k|^{p-2}\, dx), \tag{31}
$$

$$
\nabla \psi_k \rightharpoonup \mathbf{v} \quad \text{in variable space } L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}]\, dx). \tag{32}
$$

Our aim is to show that $\mathbf{v} = \nabla \psi$, and $\psi \in H^p_{\mathcal{U},y}(\Omega)$.

To this end, we fix any test function $\phi \in C_0^\infty(\Omega)^N$ and make use of the following equality

$$\begin{aligned}
&\int_\Omega \left(\left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\zeta \right)_{\mathbb{R}^N} dx \\
&= \int_\Omega (\phi, \zeta)_{\mathbb{R}^N} \, dx \\
&= \int_\Omega \left(\left(\mathcal{U}[(\nabla y)^{p-2}]\right)^{-1} \phi, \mathcal{U}[(\nabla y)^{p-2}]\zeta \right)_{\mathbb{R}^N} dx,
\end{aligned} \tag{33}$$

which is obviously true for each $\zeta \in C_0^\infty(\Omega)^N$ and for all $k \in \mathbb{N}$. Since

$$\begin{aligned}
&\limsup_{k\to\infty} \int_\Omega \left(\left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi, \mathcal{U}_k[(\nabla y_k)^{p-2}] \left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi \right)_{\mathbb{R}^N} dx \\
&= \limsup_{k\to\infty} \int_\Omega \left(\phi, \left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi \right)_{\mathbb{R}^N} dx \quad \{\text{by (18)}\} \\
&\le \int_\Omega \zeta_*^{-1} \|\phi\|^2_{\mathbb{R}^N} \, dx \\
&\le \|\phi\|^2_{C(\Omega)^N} \|\zeta_*^{-1}\|_{L^1(\Omega)} \\
&< +\infty,
\end{aligned}$$

it follows that the sequence $\left\{ (\mathcal{U}_k[(\nabla y_k)^{p-2}])^{-1} \phi \in L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}] \, dx) \right\}_{k\in\mathbb{N}}$ is obviously bounded. Consequently, combining this fact with (33), we conclude that $(\mathcal{U}_k[(\nabla y_k)^{p-2}])^{-1} \phi \rightharpoonup (\mathcal{U}[(\nabla y)^{p-2}])^{-1} \phi$ in the variable space $L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}] \, dx)$. At the same time, strong convergence in (24) implies the relation (see Corollary 3.6)

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega \left(\left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi, \mathcal{U}_k[(\nabla y_k)^{p-2}] \left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi \right)_{\mathbb{R}^N} dx \\
&= \lim_{k\to\infty} \int_\Omega \left(\phi, \left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1} \phi \right)_{\mathbb{R}^N} dx \\
&= \int_\Omega \left(\phi, \left(\mathcal{U}[(\nabla y)^{p-2}]\right)^{-1} \phi \right)_{\mathbb{R}^N} dx \\
&= \int_\Omega \left(\left(\mathcal{U}[(\nabla y)^{p-2}]\right)^{-1} \phi, \mathcal{U}[(\nabla y)^{p-2}] \left(\mathcal{U}[(\nabla y)^{p-2}]\right)^{-1} \phi \right)_{\mathbb{R}^N} dx.
\end{aligned}$$

Hence (see Proposition 3.12),

$$\begin{aligned}
&(\mathcal{U}_k[(\nabla y_k)^{p-2}])^{-1}\phi \to \left(\mathcal{U}[(\nabla y)^{p-2}]\right)^{-1} \phi \\
&\text{strongly in } L^2(\Omega, \mathcal{U}_k[(\nabla y_k)^{p-2}] \, dx) \quad \forall \phi \in C_0^\infty(\Omega)^N.
\end{aligned} \tag{34}$$

Further, we note that for every measurable subset $K \subset \Omega$, the estimate

$$\begin{aligned}\int_K \|\nabla\psi_k\|_{\mathbb{R}^N} dx &\le \left(\int_K \|\nabla\psi_k\|^2_{\mathbb{R}^N}\zeta_*\, dx\right)^{\frac{1}{2}}\left(\int_K \zeta_*^{-1}\, dx\right)^{\frac{1}{2}}\\ &\le \left(\int_\Omega (\nabla\psi_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla\psi_k)_{\mathbb{R}^N} dx\right)^{\frac{1}{2}}\left(\int_K \zeta_*^{-1}\, dx\right)^{\frac{1}{2}}\\ &\le C\left(\int_K \zeta_*^{-1}\, dx\right)^{\frac{1}{2}}\end{aligned}$$

implies equi-integrability of the family $\{\|\nabla\psi_k\|_{\mathbb{R}^N}\}_{k\in\mathbb{N}}$. Therefore, the sequence $\{\|\nabla\psi_k\|_{\mathbb{R}^N}\}_{k\in\mathbb{N}}$ is weakly compact in $L^1(\Omega)$, which means the weak compactness of the vector-valued sequence $\{\nabla\psi_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^N)$. As a result, by the properties of the strong convergence in variable spaces, we obtain

$$\begin{aligned}&\int_\Omega (\phi, \nabla\psi_k)_{\mathbb{R}^N}\, dx = \int_\Omega \Big(\left(\mathcal{U}_k[(\nabla y_k)^{p-2}]\right)^{-1}\phi, \mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla\psi_k\Big)_{\mathbb{R}^N} dx\\ &\overset{\text{by (26), (32), (34)}}{\longrightarrow} \int_\Omega \Big(\left(\mathcal{U}[(\nabla y)^{p-2}]\right)^{-1}\phi, \mathcal{U}[(\nabla y)^{p-2}]\mathbf{v}\Big)_{\mathbb{R}^N} dx = \int_\Omega (\phi, \mathbf{v})_{\mathbb{R}^N}\, dx\end{aligned}$$

for all $\phi \in C_0^\infty(\Omega;\mathbb{R}^N)$. Thus, in view of the weak compactness property of $\{\nabla\psi_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega;\mathbb{R}^N)$, we conclude

$$\nabla\psi_k \rightharpoonup \mathbf{v} \text{ in } L^1(\Omega;\mathbb{R}^N) \quad \text{as } k\to\infty. \tag{35}$$

Following the same reasoning, it can be shown that

$$\psi_k \rightharpoonup \psi \text{ in } L^1(\Omega) \quad \text{as } k\to\infty. \tag{36}$$

Since $\psi_k \in W_0^{1,1}(\Omega)$ for all $k \in \mathbb{N}$ and the Sobolev space $W_0^{1,1}(\Omega)$ is complete, (35) and (36) imply $\nabla\psi = \mathbf{v}$, and consequently

$$\psi \in W_0^{1,1}(\Omega) \quad \text{and} \quad \psi_k \rightharpoonup \psi \text{ in } W_0^{1,1}(\Omega) \quad \text{as } k\to\infty.$$

To end the proof, it remains to show that $\psi \in H^p_{\mathcal{U},y}(\Omega)$. First we observe that the conditions (31), (32) guarantee the finiteness of the norm $\|\psi\|_{\mathcal{U},y}$ (see (19)), that is, $\psi \in W^p_{\mathcal{U},y}(\Omega)$. Therefore, if the space of smooth functions $C_0^\infty(\Omega)$ is dense in $W^p_{\mathcal{U},y}(\Omega)$, then we have the identity $W^p_{\mathcal{U},y}(\Omega) = H^p_{\mathcal{U},y}(\Omega)$ and it immediately leads us to the required conclusion: $\psi \in H^p_{\mathcal{U},y}(\Omega)$.

However, in view of (17), (18), it is possible that $H^p_{\mathcal{U},y}(\Omega) \subset W^p_{\mathcal{U},y}(\Omega)$, namely, the Banach space $H^p_{\mathcal{U},y}(\Omega)$ does not contain all functions $u \in W_0^{1,1}(\Omega)$ for which the $\|\cdot\|_{\mathcal{U},y}$-norm is finite. In this case the set

$$X^{\perp}_{H^p_{\mathcal{U},y}} = \Big\{(a,\nabla a): \ \|a\|_{\mathcal{U},y} < +\infty \text{ and } \int_\Omega \big[|y|^{p-2}ua + \big(\nabla a, \mathcal{U}[(\nabla y)^{p-2}]\nabla u\big)_{\mathbb{R}^N}\big]\,dx = 0 \ \forall\, u \in H^p_{\mathcal{U},y}(\Omega)\Big\}$$

which is obviously closed in $L^2(\Omega;|y|^{p-2}\,dx) \times L^2(\Omega,\mathcal{U}[(\nabla y)^{p-2}]\,dx)$, and it is not a singleton. Clearly, the same assertion holds true with respect to the sets $X^{\perp}_{H^p_{\mathcal{U}_k,y_k}}$.

Let $(a,\nabla a)$ be an arbitrary pair of $X^{\perp}_{H^p_{\mathcal{U},y}}$, and let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence such that $\sup_{k\in\mathbb{N}} \|a_k\|_{\mathcal{U}_k,y_k} < +\infty$, $a_k \to a$ in variable space $L^2(\Omega;|y_k|^{p-2}\,dx)$, $\nabla a_k \to \nabla a$ in variable space $L^2(\Omega,\mathcal{U}_k[(\nabla y_k)^{p-2}]\,dx)$, and $(a_k,\nabla a_k) \in X^{\perp}_{H^p_{\mathcal{U}_k,y_k}}$ for all $k \in N$ (existence of such sequences in weighted Sobolev spaces and methods of their construction are considered in [8, p. 110, Lemma 3.4]). Since $\psi_k \in H^p_{\mathcal{U}_k,y_k}(\Omega)$ for all $k \in N$, it follows that, for each $k \in \mathbb{N}$, the following equality holds true

$$\int_\Omega \big[|y_k|^{p-2}\psi_k a_k + \big(\nabla a_k, \mathcal{U}_k[(\nabla y_k)^{p-2}]\nabla \psi_k\big)_{\mathbb{R}^N}\big]\,dx = 0 \quad \forall\, k \in \mathbb{N}. \tag{37}$$

Then, taking into account the definition of strong convergence in variable spaces (see (26)), we can pass to the limit in (37) as k tends to ∞. As a result, we get

$$\int_\Omega \big[|y|^{p-2}\psi a + \big(\nabla a, \mathcal{U}[(\nabla y)^{p-2}]\nabla \psi\big)_{\mathbb{R}^N}\big]\,dx = 0.$$

Since the pair $(a,\nabla a)$ is arbitrary in $X^{\perp}_{H^p_{\mathcal{U},y}}$, it follows that $\psi \in H^p_{\mathcal{U},y}(\Omega)$ and this concludes the proof. □

3.2. Substantiation of the optimality conditions for the optimal control problem (10)–(12) in the framework of weighted Sobolev spaces. In this subsection we assume that $p > 2$ and Hypotheses (H2), (H3) are valid.

Let $(\mathcal{U}_0, y_0) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times W^{1,p}_0(\Omega)$ be an optimal pair to problem (10)–(12), and let $(\widehat{\mathcal{U}},\widehat{y}) \in \Xi_{sol}$ be any admissible pair. We set $\mathcal{U}_\theta = \mathcal{U}_0 + \theta(\widehat{\mathcal{U}} - \mathcal{U}_0)$, where $\theta \in [0,1]$. Then, by [2, Lemma 4.8], there exists a value $\varepsilon_\theta \in [0,1]$ such that the increment of Lagrangian

$$\begin{aligned}\Delta\Lambda &= \Lambda(\mathcal{U}_\theta, y_\theta, \lambda) - \Lambda(\mathcal{U}_0, y_0, \lambda)\\ &= \Lambda(\mathcal{U}_\theta, y_\theta, \lambda) - \Lambda(\mathcal{U}_0, y_\theta, \lambda) + \Lambda(\mathcal{U}_0, y_\theta, \lambda) - \Lambda(\mathcal{U}_0, y_0, \lambda)\\ &= \Lambda(\theta(\widehat{\mathcal{U}} - \mathcal{U}_0), y_\theta, \lambda) + \langle \mathcal{D}_y \Lambda(\mathcal{U}_0, y_0 + \varepsilon_\theta(y_\theta - y_0), \lambda), y_\theta - y_0\rangle_{W^{1,p}_0(\Omega)}\end{aligned}$$

can be represented in the form

$$\begin{aligned} \Delta\Lambda &= p\int_\Omega |\widetilde{y}_\theta - y_d|^{p-1}(y_\theta - y_0)\,dx + (p-1)\int_\Omega |\widetilde{y}_\theta|^{p-2}\,\lambda\,(y_\theta - y_0)\,dx \\ &\quad + (p-1)\int_\Omega \left([(\nabla\widetilde{y}_\theta)^{p-2}]\,\mathcal{U}_0\nabla\lambda, \nabla(y_\theta - y_0)\right)_{\mathbb{R}^N} dx \\ &\quad + \theta\int_\Omega \left((\widehat{\mathcal{U}} - \mathcal{U}_0)[(\nabla\widetilde{y}_\theta)^{p-2}]\,\nabla\widetilde{y}_\theta, \nabla\lambda\right)_{\mathbb{R}^N} dx \\ &\geq 0, \quad \forall \widehat{\mathcal{U}} \in U_{ad}, \end{aligned} \tag{38}$$

where $\widetilde{y}_\theta := y_0 + \varepsilon_\theta(y_\theta - y_0)$ (see, for comparison, [2, formula (5.12)]).

To begin with, we show that, for each $\theta \in [0,1]$, the multipliers λ and $(y_\theta - y_0)$ in [2, formula (5.12)] can be extended to elements of the weighted space $H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$.

Lemma 3.16. *Assume that one of the following conditions*

$$y_d \in L^\infty(\Omega) \quad \textit{or} \quad y_d \in W_0^{1,p}(\Omega) \textit{ and } \exists\, C > 0 \textit{ such that } y_d \leq C y_0 \textit{ a.e.in } \Omega. \tag{39}$$

holds true. Then

$$\left|\int_\Omega |\widetilde{y}_\theta - y_d|^{p-1}\varphi\,dx\right| \leq c(\widetilde{y}_\theta, y_d)\|\varphi\|_{\mathcal{U}_0,\widetilde{y}_\theta}, \tag{40}$$

$$\left|\int_\Omega |\widetilde{y}_\theta|^{p-2}\,\lambda\varphi\,dx\right| \leq \|\lambda\|_{\mathcal{U}_0,\widetilde{y}_\theta}\|\varphi\|_{\mathcal{U}_0,\widetilde{y}_\theta}, \tag{41}$$

$$\left|\int_\Omega \left([(\nabla\widetilde{y}_\theta)^{p-2}]\,\mathcal{U}_0\nabla\lambda, \nabla\varphi\right)_{\mathbb{R}^N} dx\right| \leq \|\lambda\|_{\mathcal{U}_0,\widetilde{y}_\theta}\|\varphi\|_{\mathcal{U}_0,\widetilde{y}_\theta}, \tag{42}$$

for each $\theta \in [0,1]$, $\lambda \in W_0^{1,p}(\Omega)$, *and* $\varphi \in W_0^{1,p}(\Omega)$.

Proof. Since estimates (41)–(42) are obvious consequences of the Cauchy-Bunyakovsky inequality and the fact that $W_0^{1,p}(\Omega) \subset H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$, we concentrate on the proof of (40). To this end, we note that

$$\left|\int_\Omega |\widetilde{y}_\theta - y_d|^{p-1}\varphi\,dx\right| \leq 2^{p-2}\int_\Omega \left(|\widetilde{y}_\theta|^{p-1} + |y_d|^{p-1}\right)|\varphi|\,dx.$$

Since

$$\begin{aligned} \int_\Omega |\widetilde{y}_\theta|^{p-1}\varphi\,dx &\leq \left(\int_\Omega |\widetilde{y}_\theta|^{p-2}|\widetilde{y}_\theta|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\widetilde{y}_\theta|^{p-2}\varphi^2\,dx\right)^{\frac{1}{2}} \\ &= \|\widetilde{y}_\theta\|^{\frac{p}{2}}_{L^p(\Omega)}\|\varphi\|_{L^2(\Omega,|\widetilde{y}_\theta|^{p-2}\,dx)} \\ &\leq \|\widetilde{y}_\theta\|^{\frac{p}{2}}_{W_0^{1,p}(\Omega)}\|\varphi\|_{\mathcal{U}_0,\widetilde{y}_\theta}, \end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |y_d|^{p-1}\varphi\,dx &\le \left(\int_\Omega \frac{|y_d|^{2p-2}}{|\widetilde{y}_\theta|^{p-2}}\right)^{\frac{1}{2}}\left(\int_\Omega |\widetilde{y}_\theta|^{p-2}\varphi^2\,dx\right)^{\frac{1}{2}} \quad \{\text{by } (39)_1\}\\
&\le \|y_d\|^{p-1}_{L^\infty(\Omega)}\||\widetilde{y}_\theta|^{2-p}\|_{L^1(\Omega)}\|\varphi\|_{L^2(\Omega,|\widetilde{y}_\theta|^{p-2}\,dx)} \quad \{\text{by (H3)}\}\\
&\le \|y_d\|^{p-1}_{L^\infty(\Omega)}\|\zeta_{ad}^{-1}\|_{L^1(\Omega)}\|\varphi\|_{\mathcal{U}_0,\widetilde{y}_\theta},\\
\int_\Omega |y_d|^{p-1}\varphi\,dx &\le \left(\int_\Omega \frac{|y_d|^{2p-2}}{|\widetilde{y}_\theta|^{p-2}}\right)^{\frac{1}{2}}\left(\int_\Omega |\widetilde{y}_\theta|^{p-2}\varphi^2\,dx\right)^{\frac{1}{2}} \quad \{\text{by } (39)_2\}\\
&\le C\|\widetilde{y}_\theta\|^{\frac{p}{2}}_{L^p(\Omega)}\|\varphi\|_{L^2(\Omega,|\widetilde{y}_\theta|^{p-2}\,dx)}\\
&\le C\|\widetilde{y}_\theta\|^{\frac{p}{2}}_{W_0^{1,p}(\Omega)}\|\varphi\|_{\mathcal{U}_0,\widetilde{y}_\theta},
\end{aligned}$$

it finally follows that estimate (40) holds true with

$$\begin{aligned}
c(\widetilde{y}_\theta, y_d) &= 2^{p-2}\left(\|\widetilde{y}_\theta\|^{\frac{p}{2}}_{W_0^{1,p}(\Omega)} + \|y_d\|^{p-1}_{L^\infty(\Omega)}\|\zeta_{ad}^{-1}\|_{L^1(\Omega)}\right) \quad \text{provided } (39)_1,\\
c(\widetilde{y}_\theta, y_d) &= 2^{p-2}(1+C)\|\widetilde{y}_\theta\|^{\frac{p}{2}}_{W_0^{1,p}(\Omega)} \quad \text{provided } (39)_2.
\end{aligned}$$

□

Using this lemma, we can define the element λ in [2, formula (5.12)] as $\lambda = \psi_{\varepsilon_\theta,\theta}$, where the quasi-adjoint state $\psi_{\varepsilon_\theta,\theta}$ satisfies the following integral identity:

$$\left.\begin{aligned}
&(p-1)\int_\Omega \left([(\nabla\widetilde{y}_\theta)^{p-2}]\,\mathcal{U}_0\nabla\psi_{\varepsilon_\theta,\theta}, \nabla\varphi\right)_{\mathbb{R}^N}\,dx\\
&+(p-1)\int_\Omega |\widetilde{y}_\theta|^{p-2}\,\psi_{\varepsilon_\theta,\theta}\,\varphi\,dx + p\int_\Omega |\widetilde{y}_\theta - y_d|^{p-1}\varphi\,dx\\
&=0,
\end{aligned}\right\}\quad \forall\,\varphi\in C_0^\infty(\Omega). \tag{43}$$

As a result, the increment of Lagrangian (38) we can be simplified to the form

$$\Delta\Lambda = \int_\Omega \left((\widehat{\mathcal{U}} - \mathcal{U}_0)[(\nabla\widetilde{y}_\theta)^{p-2}]\,\nabla\widetilde{y}_\theta, \nabla\psi_{\varepsilon_\theta,\theta}\right)_{\mathbb{R}^N}\,dx \ge 0, \quad \forall\widehat{\mathcal{U}}\in U_{ad}. \tag{44}$$

Remark 3.17. It is worth to notice that due to supposition (16), Hypotheses (H2), (H3), and the fact that $\mathcal{U}_\theta \to \mathcal{U}_0$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and $\widetilde{y}_\theta \to y_0$ in $W_0^{1,p}(\Omega)$ as $\theta\to 0$ (see the proof of Theorem 2.5), $H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$ is a Hilbert space for θ small enough.

Taking this remark into account, we can pass to the following variational formulation of the problem (43)

$$\left\{\begin{aligned}
&\text{Find }\ \psi_{\varepsilon_\theta,\theta}\in H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega) \text{ such that}\\
&(\psi_{\varepsilon_\theta,\theta},\varphi)_{H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)} = \frac{p}{1-p}\int_\Omega |\widetilde{y}_\theta - y_d|^{p-1}\varphi\,dx, \quad \forall\,\varphi\in H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega),
\end{aligned}\right. \tag{45}$$

where $(\cdot,\cdot)_{H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)}$ denotes the scalar product in $H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$ (see (23)).

Since the right-hand side of (45) is a linear bounded functional on $H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$ (see (40)), it follows that, for θ small enough, the variational problem (45) has a unique solution $\psi_{\varepsilon_\theta,\theta}\in H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$ by the Lax-Milgram lemma. Moreover, in this case we have the following a priori estimate

$$\|\psi_{\varepsilon_\theta,\theta}\|_{\mathcal{U}_0,\widetilde{y}_\theta}\le\frac{2^{p-2}p}{p-1}\left\{\begin{array}{ll}\|\widetilde{y}_\theta\|^{\frac{p}{2}}_{W_0^{1,p}(\Omega)}+\|y_d\|^{p-1}_{L^\infty(\Omega)}\|\zeta_{ad}^{-1}\|_{L^1(\Omega)} & \text{provided } (39)_1,\\ (1+C)\|\widetilde{y}_\theta\|^{\frac{p}{2}}_{W_0^{1,p}(\Omega)} & \text{provided } (39)_2.\end{array}\right\}$$

Using estimate given above and the fact that $(\mathcal{U}_0,\widetilde{y}_\theta)\to(\mathcal{U}_0,y_0)$ strongly in $L^\infty(\Omega;\mathbb{R}^{N\times N})\times W_0^{1,p}(\Omega)$ as $\theta\to 0$, we conclude: $\left\{\psi_{\varepsilon_\theta,\theta}\in H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)\right\}_{\theta\to 0}$ is a bounded sequence in the sense of Definition 3.7. Hence, there exists an element $\psi\in H^p_{\mathcal{U}_0,y_0}(\Omega)$ such that within a subsequence $\psi_{\varepsilon_\theta,\theta}\to\psi$ weakly in the variable space $H^p_{\mathcal{U}_0,\widetilde{y}_\theta}(\Omega)$. Since the strong convergence $\widetilde{y}_\theta\to y_0$ in $W_0^{1,p}(\Omega)$ implies $|\widetilde{y}_\theta-y_d|^{p-1}\to|y_0-y_d|^{p-1}$ strongly in $L^q(\Omega)$, we can pass to the limit in the integral identity (43) as $\theta\to 0$. As a result, we get

$$\left.\begin{array}{l}(p-1)\left[\int_\Omega\left([(\nabla y_0)^{p-2}]\,\mathcal{U}_0\nabla\psi,\nabla\varphi\right)_{\mathbb{R}^N}dx+\int_\Omega|y_0|^{p-2}\,\psi\varphi\,dx\right]\\ +\,p\int_\Omega|y_0-y_d|^{p-1}\varphi\,dx\\ =0,\end{array}\right\}\quad\forall\,\varphi\in C_0^\infty(\Omega).$$

It remains to study the asymptotic behavior of the inequality (44) as $\theta\to 0$. With that in mind, we make use of Lemma 3.14 and Definition 3.10. As a result, we get

$$\begin{aligned}&\lim_{\theta\to 0}\int_\Omega\left(\widehat{\mathcal{U}}[(\nabla\widetilde{y}_\theta)^{p-2}]\nabla\widetilde{y}_\theta,\nabla\psi_{\varepsilon_\theta,\theta}\right)_{\mathbb{R}^N}dx-\lim_{\theta\to 0}\int_\Omega\left(\mathcal{U}_0[(\nabla\widetilde{y}_\theta)^{p-2}]\nabla\widetilde{y}_\theta,\nabla\psi_{\varepsilon_\theta,\theta}\right)_{\mathbb{R}^N}dx\\ &=\int_\Omega\left(\widehat{\mathcal{U}}[(\nabla y_0)^{p-2}]\,\nabla y_0,\nabla\psi\right)_{\mathbb{R}^N}dx-\int_\Omega\left(\mathcal{U}_0[(\nabla y_0)^{p-2}]\nabla y_0,\nabla\psi\right)_{\mathbb{R}^N}dx\\ &=\int_\Omega\left((\widehat{\mathcal{U}}-\mathcal{U}_0)[(\nabla y_0)^{p-2}]\,\nabla y_0,\nabla\psi\right)_{\mathbb{R}^N}dx\\ &\ge 0\quad\forall\widehat{\mathcal{U}}\in U_{ad}.\end{aligned}$$

Thus, summing up the above obtained results, we arrive at the following final conclusion.

Theorem 3.18. *Assume that* $p>2$, $f\in W^{-1,q}(\Omega)$, $U_{ad}\neq\emptyset$ *is given by* (16), *and the element* y_d *possesses property* (39). *Let* $(\mathcal{U}_0,y_0)\in L^\infty(\Omega;\mathbb{R}^{N\times N})\times W_0^{1,p}(\Omega)$ *be an optimal pair to problem* (10)–(12). *Then the fulfilment of Hypotheses* (H2), (H3) *implies the existence of an element* $\psi\in H^p_{\mathcal{U}_0,y_0}(\Omega)$ *such that*

$$\int_\Omega\Big((\mathcal{U}-\mathcal{U}_0)[(\nabla y_0)^{p-2}]\,\nabla y_0,\nabla\psi\Big)_{\mathbb{R}^N}\,dx\geq 0,\qquad \forall\,\mathcal{U}\in U_{ad},$$

$$\int_\Omega\Big(\mathcal{U}_0[(\nabla y_0)^{p-2}]\nabla y_0,\nabla\varphi\Big)_{\mathbb{R}^N}\,dx+\int_\Omega|y_0|^{p-2}y_0\varphi\,dx=\langle f,\varphi\rangle_{W_0^{1,p}(\Omega)},\quad\forall\,\varphi\in W_0^{1,p}(\Omega),$$

and

$$\left.\begin{aligned}&(p-1)\int_\Omega\Big([(\nabla y_0)^{p-2}]\,\mathcal{U}_0\nabla\psi,\nabla\varphi\Big)_{\mathbb{R}^N}\,dx+(p-1)\int_\Omega|y_0|^{p-2}\,\psi\,\varphi\,dx\\&=p\int_\Omega|y_0-y_d|\varphi\,dx,\end{aligned}\right\}\quad\forall\,\varphi\in C_0^\infty(\Omega).$$

Acknowledgement. The last author gratefully acknowledges the support of the DFG-CRC 814 *"Additive Fertigung"* while the stay of the first author at the university of Erlangen-Nuremberg has been supported by the DFG Center of Excellence *Engineering of Advanced Materials.*

References

[1] Casas, E. and Fernández, A., Optimal control of quasilinear elliptic equations with non differentiable coefficients at the origin. *Rev. Mat. Complut.* 4 (1991)(2–3), 227 – 250.

[2] Kogut, P. I., Kupenko, O. P. and Leugering, G., Optimal control in matrix-valued coefficients for nonlinear monotone problems: Optimality conditions I. *Z. Anal. Anwend.* 34 (2015)(1), 85 – 108.

[3] Kogut, P. I. and Leugering, G., Optimal L^1-control in coefficients for Dirichlet elliptic problems: H-optimal solutions. *Z. Anal. Anwend.* 31 (2012)(1), 31 – 53.

[4] Murthy, M. K. V. and Stampacchia, G., Boundary value problems for some degenerate elliptic operators. *Ann. Mat. Pura Appl.* 80 (1968), 1 – 122.

[5] Serovajskij, S. Ya., Variational inequalities in non-linear optimal control problems. *Methods and Means of Mathematical Modelling.* 1 (1977), 156 – 169.

[6] Trudinger, N. S., Linear elliptic equations with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa* 27 (1973), 265 – 308.

[7] Zhikov, V. V., Weighted Sobolev spaces. *Sb. Math.* 189 (1998)(8), 27 – 58.

[8] Zhikov, V. V. and Pastukhova,S. E., Homogenization of degenerate elliptic equations. *Siberian Math. J.* 49 (2006)(1), 80 – 101.

Received August 14, 2013; revised July 30, 2014