Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 34 (2015), 277–284
DOI: 10.4171/ZAA/1539

A Regularity Criterion for the Density-Dependent Hall-Magnetohydrodynamics

Jishan Fan, Ahmed Alsaedi, Yasuhide Fukumoto, Tasawar Hayat and Yong Zhou

Abstract. This paper proves a regularity criterion for the density-dependent Hall-magnetohydrodynamics with positive density.

Keywords. Hall-MHD, regularity criterion, positive density
Mathematics Subject Classification (2010). Primary 35L60, 35K55, secondary 35Q80, 70S15

1. Introduction

In this paper, we consider the density-dependent incompressible Hall-magnetohydrodynamics system:

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b \cdot \nabla b, \tag{1.1}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u + \operatorname{curl}\left(\frac{\operatorname{curl} b \times b}{\rho}\right) = \Delta b, \tag{1.2}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.3}$$

J. Fan: Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China; fanjishan@njfu.edu.cn
A. Alsaedi: Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; aalsaedi@kau.edu.sa
Y. Fukumoto: Institute of Mathematics for Industry, Kyushu University, Fukuoka 819-0395, Japan; yasuhide@imi.kyushu-u.ac.jp
T. Hayat: Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan; NAAM Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; tahaksag@yahoo.com
Y. Zhou: School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China; NAAM Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; yzhou@sufe.edu.cn

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.4}$$

$$(\rho, u, b) \mid_{t=0} = (\rho_0, u_0, b_0), \tag{1.5}$$

$$\lim_{|x|\to\infty} \rho = \tilde{\rho}, \ \lim_{|x|\to\infty} (u, b) = (0, 0). \tag{1.6}$$

Here ρ is the density of the fluid, u is the fluid velocity field, π is the pressure and b is the magnetic field. $\tilde{\rho}$ is a positive constant.

The applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo. The baroclinic creation of vorticity by density stratification and interaction of the vorticity and magnetic field play an important role, for example, in a series of processes of a supernova explosion followed by scattering supernova remnants. For the compressible model corresponding to (1.1)–(1.6), we refer to [2].

When $\rho = 1$, the density-dependent Hall-MHD system reduces to the standard Hall-MHD system, which has received many studies [2–8, 10, 12]. [2] gave a derivation of Hall-MHD system from a two-fluid Euler-Maxwell system. In [3], Chae-Degond-Liu proved the local existence of smooth solutions. Chae-Lee [4] and Fan-Ozawa [12] proved some regularity criteria.

When the Hall effect term $\operatorname{curl}\left(\frac{\operatorname{curl} b \times b}{\rho}\right)$ is neglected, the density-dependent Hall-MHD system reduces to the well-known density-dependent MHD system. Abidi-Hmidi [1] and Wu [13] proved the local existence of strong solutions. Fan-Li-Nakamura-Tan [11] proved some regularity criteria.

The aim of this paper is to prove a regularity criterion of the system (1.1)–(1.6). We will prove

Theorem 1.1. *Let $0 < \frac{1}{C} \le \tilde{\rho}$, $\rho_0 \le C$, $\nabla\rho_0 \in H^1$ and $u_0, b_0 \in H^2$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ in $\mathbb{R}^3$. Let (ρ, u, b) be a local strong solution to the problem (1.1)–(1.6). If*

$$u \in L^2(0, T; L^\infty), \ \nabla u \in L^1(0, T; L^\infty) \quad \text{and} \quad \nabla b \in L^{\frac{2q}{q-3}}(0, T; L^q) \tag{1.7}$$

with $3 < q \le \infty$, holds true, then the solution (ρ, u, b) can be extended beyond $T > 0$.

Remark 1.2. When $b = 0$, we are unable to prove the following regularity criterion

$$\nabla u \in L^1(0, T; L^\infty), \tag{1.8}$$

while it is easy to prove the following regularity criterion

$$u \in L^2(0, T; L^\infty). \tag{1.9}$$

However, (1.9) is not enough for the Hall-MHD system, because we must bound the gradient of the density due to the Hall effect. Thus we need assume both (1.8) and (1.9).

2. Proof of Theorem 1.1

By similar calculations as that in [9] (also see [13]), we can prove the local existence of strong solutions to the problem (1.1)–(1.6) and thus we omit the details here. We only need to establish a priori estimates.

First, thanks to the maximum principle, we have

$$0 < \frac{1}{C} \le \rho \le C < \infty. \tag{2.1}$$

Multiplying (1.1) by u and integrating over the whole space, thanks to (1.3) and (1.4), after integration by parts, we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \rho|u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx = \int_{\mathbb{R}^3} (b\cdot\nabla)b\cdot u dx. \tag{2.2}$$

Similarly, by using energy method, we infer that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |b|^2 dx + \int_{\mathbb{R}^3} |\nabla b|^2 dx = \int_{\mathbb{R}^3} (b\cdot\nabla)u\cdot b dx. \tag{2.3}$$

Summing up (2.2) and (2.3) and using $\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot u dx + \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot b dx = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} (\rho|u|^2 + |b|^2) dx + \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla b|^2) dx = 0.$$

This proves

$$\|(u,b)\|_{L^\infty(0,T;L^2)} + \|(u,b)\|_{L^2(0,T;H^1)} \le C.$$

Multiplying (1.2) by $|b|^{p-2}b$ $(2 < p < \infty)$ and integrating over $\mathbb{R}^3$, using (1.4) and (2.1) and denoting $\phi := |b|^{\frac{p}{2}}$, after integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|b|^p dx + \frac{1}{2}\int_{\mathbb{R}^3}|b|^{p-2}|\nabla b|^2 dx + 4\frac{p-2}{p^2}\int_{\mathbb{R}^3}|\nabla |b|^{\frac{p}{2}}|^2 dx\\
&= \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot|b|^{p-2}b dx + \int_{\mathbb{R}^3}\frac{b\times \operatorname{curl} b}{\rho}\cdot \operatorname{curl}(|b|^{p-2}b) dx\\
&\le \|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3}\phi^2 dx + C\int_{\mathbb{R}^3}|\nabla b|^2|b|^{p-1} dx\\
&= \|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3}\phi^2 dx + C\int_{\mathbb{R}^3}|\nabla b|\cdot|b|^{\frac{p}{2}}\cdot|b|^{\frac{p}{2}-1}|\nabla b| dx\\
&\le \|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3}\phi^2 dx + C\|\nabla b\|_{L^q}^2\|\phi\|^2_{L^{\frac{2q}{q-2}}} + \frac{1}{4}\int_{\mathbb{R}^3}|b|^{p-2}|\nabla b|^2 dx\\
&\le \|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3}\phi^2 dx + C\|\nabla b\|_{L^q}^2\|\phi\|_{L^2}^{2\left(1-\frac{3}{q}\right)}\|\nabla\phi\|_{L^2}^{\frac{6}{q}} + \frac{1}{4}\int_{\mathbb{R}^3}|b|^{p-2}|\nabla b|^2 dx\\
&\le 2\frac{p-2}{p^2}\int_{\mathbb{R}^3}|\nabla|b|^{\frac{p}{2}}|^2 dx + \|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3}\phi^2 dx + C\|\nabla b\|_{L^q}^{\frac{2q}{q-3}}\|\phi\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}|b|^{p-2}|\nabla b|^2 dx,
\end{aligned}$$

which gives

$$\|b\|_{L^\infty(0,T;L^p)} \le C \quad \text{with} \quad 2 < p < \infty, \tag{2.4}$$

$$\|b \cdot \nabla b\|_{L^2(0,T;L^2)} \le C \quad \text{as} \quad p = 4. \tag{2.5}$$

Multiplying (1.1) by $\partial_t u$ and doing integration on $\mathbb{R}^3$, using (1.3), (1.4), (2.1) and (2.5), after integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \rho|\partial_t u|^2 dx \\
&= \int_{\mathbb{R}^3} (b \cdot \nabla) b \cdot \partial_t u dx - \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot \partial_t u dx \\
&\le \|b \cdot \nabla b\|_{L^2}\|\partial_t u\|_{L^2} + \|\sqrt{\rho}\partial_t u\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2} \\
&\le \frac{1}{2}\int \rho|\partial_t u|^2 dx + C\|b \cdot \nabla b\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|u\|_{L^\infty(0,T;H^1)} \le C, \tag{2.6}$$

$$\|\partial_t u\|_{L^2(0,T;L^2)} \le C. \tag{2.7}$$

On the other hand, since (u, π) is a solution of the Stokes system:

$$-\Delta u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) = f := b \cdot \nabla b - \rho\partial_t u - \rho u \cdot \nabla u, \tag{2.8}$$

thanks to the $\dot{H}^2$-theory of the Stokes system, we have

$$\|\nabla^2 u\|_{L^2} \le C\|f\|_{L^2} \le C\|b \cdot \nabla b\|_{L^2} + C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|u\|_{L^\infty}\|\nabla u\|_{L^2},$$

which yields

$$\|u\|_{L^2(0,T;H^2)} \le C. \tag{2.9}$$

Taking ∇ to (1.3), then multiplying it by $|\nabla\rho|^{m-2}\nabla\rho$ $(2 \le m < \infty)$, after integration, we find that

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla\rho|^m dx \le C\|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3} |\nabla\rho|^m dx,$$

which leads to

$$\|\nabla\rho\|_{L^\infty(0,T;L^m)} \le C \quad (2 \le m < \infty). \tag{2.10}$$

Multiplying (1.2) by $-\Delta b$ and integrating over the whole space, after integration by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla b|^2 + \int_{\mathbb{R}^3} |\Delta b|^2 dx \\
&= \int_{\mathbb{R}^3} (u \cdot \nabla b - b \cdot \nabla u)\Delta b dx + \int_{\mathbb{R}^3} \frac{1}{\rho}(\operatorname{curl} b \times b)\operatorname{curl} \Delta b dx \\
&=: I_1 + I_2.
\end{aligned} \tag{2.11}$$

Using (2.6) and (2.4), we bound I_1 as follows:

$$\begin{aligned}
I_1 &\le (\|u\|_{L^6}\|\nabla b\|_{L^3} + \|b\|_{L^6}\|\nabla u\|_{L^3})\|\Delta b\|_{L^2}\\
&\le C(\|\nabla b\|_{L^3} + \|\nabla u\|_{L^3})\|\Delta b\|_{L^2}\\
&\le \frac{1}{4}\|\Delta b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Using (2.1), (2.4) and (2.10), I_2 can be estimated as:

$$\begin{aligned}
I_2 &= -\sum_i \int_{\mathbb{R}^3} \frac{1}{\rho}(\operatorname{curl} b \times \partial_i b)\partial_i \operatorname{curl} b dx + \sum_i \int_{\mathbb{R}^3} \frac{\partial_i \rho}{\rho^2}(\operatorname{curl} b \times b)\partial_i \operatorname{curl} b dx\\
&\le C\|\nabla b\|_{L^q}\|\nabla b\|_{L^{\frac{2q}{q-2}}}\|\Delta b\|_{L^2} + C\|\nabla\rho\|_{L^{12}}\|b\|_{L^{12}}\|\nabla b\|_{L^3}\|\Delta b\|_{L^2}\\
&\le C\|\nabla b\|_{L^q}\|\nabla b\|_{L^2}^{1-\frac{3}{q}}\|\Delta b\|_{L^2}^{1+\frac{3}{q}} + C\|\nabla b\|_{L^3}\|\Delta b\|_{L^2}\\
&\le \frac{1}{4}\|\Delta b\|_{L^2}^2 + C\|\nabla b\|_{L^q}^{\frac{2q}{q-3}}\|\nabla b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (2.11) and using the Gronwall inequality, due to condition (1.7), we have

$$\|b\|_{L^\infty(0,T;H^1)} + \|b\|_{L^2(0,T;H^2)} \le C. \tag{2.12}$$

Applying the operator ∂_t to (1.1), we see that

$$\begin{aligned}
&\rho\partial_t^2 u + \rho u\cdot\nabla\partial_t u - \Delta\partial_t u + \nabla\partial_t\left(\pi + \frac{1}{2}|b|^2\right)\\
&= \operatorname{div}\partial_t(b\otimes b) - \partial_t\rho(\partial_t u + u\cdot\nabla u) - \rho\partial_t u\cdot\nabla u.
\end{aligned}$$

Multiplying the above equation by $\partial_t u$ and using (1.3), (1.4), (2.1), (2.6) and (2.7), after integration by parts, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \rho|\partial_t u|^2 dx + \int_{\mathbb{R}^3} |\nabla\partial_t u|^2 dx\\
&\le C\int_{\mathbb{R}^3} |b||\partial_t b||\nabla\partial_t u| dx + \left|\int_{\mathbb{R}^3} \rho u\cdot\nabla[(\partial_t u + u\cdot\nabla u)\partial_t u]dx\right|\\
&\quad + C\|\nabla u\|_{L^2}\|\partial_t u\|_{L^4}^2\\
&\le C\|b\|_{L^\infty}\|\partial_t b\|_{L^2}\|\nabla\partial_t u\|_{L^2} + C\|u_t\|_{L^3}\|u\|_{L^6}\|\nabla\partial_t u\|_{L^2}\\
&\quad + C\|u\|_{L^6}\|\nabla u\|_{L^6}^2\|\partial_t u\|_{L^2} + C\|u\|_{L^6}^2\|\Delta u\|\|\partial_t u\|_{L^6}\\
&\quad + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla\partial_t u\|_{L^2} + C\|\partial_t u\|_{L^4}^2\\
&\le \frac{1}{2}\|\nabla\partial_t u\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\partial_t b\|_{L^2}^2 + C\|\partial_t u\|_{L^2}^2\\
&\quad + C\|\Delta u\|_{L^2}^2\|\partial_t u\|_{L^2} + C\|\Delta u\|_{L^2}^2.
\end{aligned} \tag{2.13}$$

Applying the operator ∂_t to (1.2), then multiplying it by $\partial_t b$, and using (1.3), (1.4), (2.1), (2.4), (2.6), (2.7), (2.10) and (2.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_t b|^2dx+\int_{\mathbb{R}^3}|\nabla\partial_t b|^2dx\\
&=\int_{\mathbb{R}^3}\partial_t(u\times b)\cdot\operatorname{curl}\partial_t b dx-\int_{\mathbb{R}^3}\frac{1}{\rho}\partial_t(\operatorname{curl}b\times b)\cdot\partial_t\operatorname{curl}b dx\\
&\quad+\int_{\mathbb{R}^3}\frac{1}{\rho^2}\partial_t\rho(\operatorname{curl}b\times b)\cdot\partial_t\operatorname{curl}b dx\\
&\le C\big(\|\partial_t u\|_{L^2}\|b\|_{L^\infty}+\|u\|_{L^6}\|\partial_t b\|_{L^3}\big)\|\nabla\partial_t b\|_{L^2}\\
&\quad+C\|\nabla b\|_{L^q}\|\partial_t b\|_{L^{\frac{2q}{q-2}}}\|\nabla\partial_t b\|_{L^2}\\
&\quad+C\|u\|_{L^6}\|\nabla\rho\|_{L^{12}}\|\operatorname{curl}b\|_{L^6}\|b\|_{L^{12}}\|\nabla\partial_t b\|_{L^2}\\
&\le\frac{1}{4}\|\nabla\partial_t b\|_{L^2}^2+C\|\partial_t b\|_{L^2}^2+C\|b\|_{L^\infty}^2\|\partial_t u\|_{L^2}^2\\
&\quad+C\|\nabla b\|_{L^q}^{\frac{2q}{q-3}}\|\partial_t b\|_{L^2}^2+C\|\Delta b\|_{L^2}^2.
\end{aligned}\tag{2.14}$$

Combining (2.13) and (2.14) and using (2.9), (2.12), and the Gronwall inequality, we have

$$\|\partial_t u\|_{L^\infty(0,T;L^2)}+\|\partial_t u\|_{L^2(0,T;H^1)}\le C.\tag{2.15}$$

and

$$\|\partial_t b\|_{L^\infty(0,T;L^2)}+\|\partial_t b\|_{L^2(0,T;H^1)}\le C.$$

It follows from (2.4), (2.8), (2.9), (2.12) and (2.15) that

$$\|\Delta u\|_{L^2(0,T;L^3)}\le C.\tag{2.16}$$

Applying Δ to (1.3), then multiplying it by $\Delta\rho$ and integrating on $\mathbb{R}^3$, using (1.4), (2.10) and (2.16), we get

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}|\Delta\rho|^2dx&\le C\|\nabla u\|_{L^\infty}\int_{\mathbb{R}^3}|\Delta\rho|^2dx+C\|\Delta u\|_{L^3}\|\nabla\rho\|_{L^6}\|\Delta\rho\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}\|\Delta\rho\|_{L^2}^2+C\|\Delta u\|_{L^3}\|\Delta\rho\|_{L^2},
\end{aligned}$$

which gives

$$\|\nabla\rho\|_{L^\infty(0,T;H^1)}\le C.\tag{2.17}$$

Taking Δ to (1.2), then multiplying it by Δb, using (2.1), (2.4), (2.10) and (2.17), after integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\Delta b|^2dx+\int_{\mathbb{R}^3}|\nabla\Delta b|^2dx\\
&=\int_{\mathbb{R}^3}\Delta\frac{b\times\operatorname{curl}b}{\rho}\cdot\Delta\operatorname{curl}b\,dx+\int_{\mathbb{R}^3}\Delta(u\times b)\cdot\Delta\operatorname{curl}b\,dx\\
&=\int_{\mathbb{R}^3}\frac{1}{\rho}(\Delta b\times\operatorname{curl}b+2\sum_i\partial_i b\times\partial_i\operatorname{curl}b)\Delta\operatorname{curl}b\,dx\\
&\quad+\int_{\mathbb{R}^3}\Delta\frac{1}{\rho}\cdot(b\times\operatorname{curl}b)\cdot\Delta\operatorname{curl}b\,dx+2\sum_i\int_{\mathbb{R}^3}\partial_i\frac{1}{\rho}\cdot\partial_i(b\times\operatorname{curl}b)\cdot\Delta\operatorname{curl}b\,dx\\
&\quad+\int_{\mathbb{R}^3}\Delta(u\times b)\cdot\Delta\operatorname{curl}b\,dx\\
&\le C\|\nabla b\|_{L^q}\|\Delta b\|_{L^{\frac{2q}{q-2}}}\|\nabla\Delta b\|_{L^2}+C\|\Delta\rho\|_{L^2}\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|\nabla\Delta b\|_{L^2}\\
&\quad+C\|\nabla\rho\|_{L^4}^2\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|\nabla\Delta b\|_{L^2}+C\|\nabla\rho\|_{L^{12}}\|b\|_{L^{12}}\|\Delta b\|_{L^3}\|\nabla\Delta b\|_{L^2}\\
&\quad+C\|\nabla\rho\|_{L^6}\|\nabla b\|_{L^6}^2\|\nabla\Delta b\|_{L^2}\\
&\quad+C\big(\|\Delta u\|_{L^2}\|b\|_{L^\infty}+\|u\|_{L^\infty}\|\Delta b\|_{L^2}+\|\nabla u\|_{L^6}\|\nabla b\|_{L^3}\big)\|\nabla\Delta b\|_{L^2}\\
&\le\frac{1}{2}\|\nabla\Delta b\|_{L^2}^2+C\|\nabla b\|_{L^q}^{\frac{2q}{q-3}}\|\Delta b\|_{L^2}^2+C\|u\|_{H^2}^2\|b\|_{H^2}^2+C\|\Delta b\|_{L^2}^4+C\|\Delta b\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\|b\|_{L^\infty(0,T;H^2)}+\|b\|_{L^2(0,T;H^3)}\le C. \tag{2.18}$$

Here we used the Gagliardo-Nirenberg inequalities:

$$\|b\|_{L^\infty}^2\le C\|b\|_{L^6}\|\Delta b\|_{L^2},\quad \|\nabla b\|_{L^\infty}^2\le C\|\Delta b\|_{L^2}\|\nabla\Delta b\|_{L^2}.$$

It follows from (2.8), (2.15) and (2.18) that

$$\|u\|_{L^\infty(0,T;H^2)}+\|u\|_{L^2(0,T;W^{2,6})}\le C.$$

This completes the proof. □

Acknowledgement. The authors are indebted to the referee for careful reading of the manuscript and nice suggestions. This project is funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under Grant No. 27-130-35-HiCi. The authors, therefore acknowledge technical and financial support of KAU.

References

[1] Abidi, H. and Hmidi, T., Résultats d'existence dans des espaces critiques pour le système de la MHD inhomogène (in French). *Ann. Math. Blaise Pascal* 14 (2007), 103 – 148.

[2] Acheritogaray, M., Degond, P., Frouvelle, A. and Liu, J.-G., Kinetic fomulation and global existence for the Hall-magnetohydrodynamics system. *Kinetic Related Models* 4 (2011), 901 – 918.

[3] Chae, D., Degond, P., and Liu, J.-G., Well-posedness for Hall-magnetohydrodynamics. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 31 (2014), 555 – 565.

[4] Chae, D. and Lee, J., On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. *J. Diff. Equ.* 256 (2014), 3835 – 3858.

[5] Chae, D. and Schonbek, M., On the temporal decay for the Hall-magnetohydrodynamic equations. *J. Diff. Equ.* 255 (2013), 3971 – 3982.

[6] Chae, D. and Weng, S., Singularity formation for the incompressible Hall-MHD equations without resistivity. arXiv: 1312.5519 v1 [math. AP] 19 Dec 2013.

[7] Dreher, J. Runban, V. and Grauer, R., Axisymmetric flows in Hall-MHD: a tendency towards finite-time singularity formation. *Physica Scripta* 72 (2005), 451 – 455.

[8] Dumas, E. and Sueur, F., On the weak solutions to the Maxwell-Landau-Lifshitz equations and to the Hall-magnetohydrodynamic equations. *Comm. Math. Phys.* 330 (2014), 1179 – 1225.

[9] Fan, J., Alsaedi, A., Hayat, T., Nakamura, G. and Zhou, Y., On strong solutions to the compressible Hall-magnetohydrodynamic system. *Nonlin. Anal. Real World Appl.* 22 (2015), 423 – 434.

[10] Fan, J., Huang, S. and Nakamura, G., Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations. *Appl. Math. Lett.* 26 (2013), 963 – 967.

[11] Fan, J., Li, F., Nakamura, G. and Tan, Z., Regularity criteria for the three-dimensional magnetohydrodynamic equations. *J. Diff. Equ.* 256 (2014), 2858 – 2875.

[12] Fan, J. and Ozawa, T., Regularity criteria for Hall-magnetohydrodynamics and the space-time monopole equation in Lorenz gauge. *Contemp. Math.* 612 (2014), 81 – 89.

[13] Wu, H., Strong solution to the incompressible MHD equations with vacuum. *Comput. Math. Appl.* 61 (2011), 2742 – 2753.

Received May 12, 2014; revised November 15, 2014