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A Fourth-Order Dispersive Flow into Kähler Manifolds

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Abstract. We discuss a short-time existence theorem of solutions to the initial value problem for a fourth-order dispersive flow for curves parametrized by the real line into a compact Kähler manifold. Our equations geometrically generalize a physical model describing the motion of a vortex filament or the continuum limit of the Heisenberg spin chain system. Our results are proved by using so-called the energy method. We introduce a bounded gauge transform on the pullback bundle, and make use of local smoothing effect of the dispersive flow a little.

Keywords. Dispersive flow, geometric analysis, gauge transform, energy method, smoothing effect

Mathematics Subject Classification (2010). Primary 53C44, secondary 47G30

1. Introduction

Let (N, J, g) be a compact Kähler manifold of real dimension $2n$ with a complex structure J and a Kähler metric g. In the present paper we study the initial value problem for a mapping $\mathbb{R} \times \mathbb{R} \ni (t, x) \mapsto u(t, x) \in N$ of the form

$$\begin{aligned} u_t &= aJ(u)\nabla_x^3 u_x + \{1 + bg_u(u_x, u_x)\}J(u)\nabla_x u_x \\ &\quad + cg_u(\nabla_x u_x, u_x)J(u)u_x \end{aligned} \quad \text{in } \mathbb{R} \times \mathbb{R}, \tag{1}$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}, \tag{2}$$

where $u_t = du(\frac{\partial}{\partial t})$, $u_x = du(\frac{\partial}{\partial x})$, du is the differential of the mapping u, ∇ is the induced connection for the Levi-Civita connection ∇^N of (N, J, g), $a \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$ are constants, and $u_0 : \mathbb{R} \to N$ is a given initial curve on N. $u(t, \cdot)$ is a

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curve on N for any fixed $t \in \mathbb{R}$, and u describes the motion of a curve subject to the equation (1). If $a, b, c = 0$, then (1) is reduced to the one-dimensional Schrödinger map equation of the form

$$u_t = J(u)\nabla_x u_x,$$

and its stationary solutions are geodesics on N, see, e.g., [2,4,6,11,16] and references therein for the physical background and the mathematical study of the Schrödinger map equation.

Here we present local expression of the covariant derivative ∇_x. Let $y^1, \dots, y^{2n}$ be local coordinates of N. We denote by $\Gamma^\alpha_{\beta\gamma}$, $\alpha, \beta, \gamma = 1, \dots, 2n$, the Christoffel symbol of (N, J, g). For a smooth curve $u : \mathbb{R} \to N$, $\Gamma(u^{-1}TN)$ is the set of all smooth sections of the pullback bundle $u^{-1}TN$. If we express $V \in \Gamma(u^{-1}TN)$ as

$$V(x) = \sum_{\alpha=1}^{2n} V^\alpha(x) \left(\frac{\partial}{\partial y^\alpha}\right)_u,$$

then, $\nabla_x V$ is given by

$$\nabla_x V(x) = \sum_{\alpha=1}^{2n} \left\{ \frac{\partial V^\alpha}{\partial x}(x) + \sum_{\beta,\gamma=1}^{2n} \Gamma^\alpha_{\beta\gamma}\bigl(u(x)\bigr) V^\beta(x) \frac{\partial u^\gamma}{\partial x}(x) \right\} \left(\frac{\partial}{\partial y^\alpha}\right)_u.$$

Here we recall the physical background of (1). The equation generalizes a model system arising in classical mechanics of the form

$$\vec{u}_t = \vec{u} \times \bigl\{a\vec{u}_{xxxx} + \vec{u}_{xx} + b\langle \vec{u}_x, \vec{u}_x\rangle \vec{u}_{xx} + c\langle \vec{u}_{xx}, \vec{u}_x\rangle \vec{u}_x\bigr\}, \tag{3}$$

where $\mathbb{R} \times \mathbb{R} \ni (t, x) \mapsto \vec{u}(t, x) \in \mathbb{S}^2$, $\mathbb{S}^2$ is the two-dimensional unit sphere in $\mathbb{R}^3$, and $\vec{\xi} \times \vec{\eta}$ and $\langle \vec{\xi}, \vec{\eta}\rangle$ are the vector and the inner products of $\vec{\xi} \in \mathbb{R}^3$ and $\vec{\eta} \in \mathbb{R}^3$ respectively. The equation (3) describes the motion of a vortex filament or the continuum limit of the Heisenberg spin chain system. In [24], Porsezian, Daniel and Lakshmanan formulated the continuum limit of the Heisenberg spin chain system as (3) with $3a + c = 2b$ and $a \neq 0$, and showed that if $c = 0$ which is equivalent to $b = \frac{3a}{2}$, then (3) is completely integrable. In [7], Fukumoto studied the motion of vortex filament and obtained (3) with with $3a + c = 2b$ and $a \neq 0$. More precisely, in [7], x describes the length of a curve $\vec{\gamma}(t, x)$, and $\vec{u}(t, x) = \vec{\gamma}_x(t, x)$.

It seems to be natural and fundamental to study the initial value problem (1), (2). In particular, it is important to know the relationship between the geometric settings and the existence theorems of time-local and time-global solutions. The main difficulty of solving (1), (2) is the loss of derivative of order one occurring in (1) even in the case $N = \mathbb{S}^2$. If (N, J, g) is a compact almost Hermitian manifold, then the situation becomes more difficult. Indeed, the loss of derivative of order three may happen since $\nabla^N J$ does not necessarily vanish.

The known results on (1), (2) are limited. In particular, all the preceding results are concerned only with the case of $N = \mathbb{S}^2$. By the stereographic projection of

$\mathbb{S}^2 \setminus \{(0,0,1)\}$ onto $\mathbb{C}$, the equation (3) becomes a complex-valued fourth-order semilinear dispersive partial differential equation. The time-local existence for this equation was established, see Huo and Jia [13–15], and Segata [25, 26]. The idea of proof of these results are based on making use of so-called the local smoothing effect of $\exp(\sqrt{-1}t\frac{\partial^4}{\partial x^4})$ via Bourgain's Fourier restriction norm method, see [1] for this. The stereographic projection requires somewhat restriction on the range of the maps. For this reason, these results are not necessarily concerned with the study of maps to $\mathbb{S}^2$. On the other hand, Guo, Zeng and Su ([10]) studied time-local existence of weak solutions for the model equation (3) with $a \neq 0$, $c = 0$ and $b = \frac{3a}{2}$ for $x \in \mathbb{R}/\mathbb{Z}$. In this case no smoothing effect can be expected since the source of the maps $\mathbb{R}/\mathbb{Z}$ is compact and singularities of solutions come back periodically. Fortunately, however, the integrability condition works well to overcome the loss of derivative of order one.

The purpose of the present paper is to establish the short-time existence theorem for the initial value problem (1), (2) from the point of view of geometric analysis [8, 9, 12, 20] and higher order linear dispersive partial differential equations [18, 27]. The present paper is a continuation of our geometric analysis of dispersive flows [4, 5, 21–23]. The point of view of geometric analysis sometimes offers deep insights into the structure of dispersive systems, and leads one to discoveries. For example, Koiso [16] proved that some curvature condition on the target manifold guarantees the time-global existence for the Schrödinger flow for closed curves. Onodera [21] also established time-global existence theorem for a third-order dispersive flow for closed curves under some curvature condition on the target manifold. Moreover, Chihara [4] came to understand the relationship between the Kähler condition of the target manifold and the structure of the Schrödinger map equation of a closed Riemannian manifold to a compact almost Hermitian manifold.

Here we introduce some function spaces of mappings. For $k = 0, 1, 2, 3, \dots$, we denote by $H^{k+1}(\mathbb{R}; TN)$ the set of all continuous mappings of $\mathbb{R}$ to N whose derivatives up to $k+1$ are all square integrable:

$$\|u_x\|_{H^k}^2 = \sum_{l=0}^{k} \int_{\mathbb{R}} g_{u(x)}\big(\nabla_x^l u_x, \nabla_x^l u_x\big) dx < \infty.$$

The standard k-th order Sobolev space of $\mathbb{R}^d$-valued functions $\vec{z}$ on $\mathbb{R}$ is denoted by $H^k(\mathbb{R}; \mathbb{R}^d)$, and its norm is defined by

$$\|U\|_{H^k(\mathbb{R};\mathbb{R}^d)}^2 = \sum_{l=0}^{k} \int_{\mathbb{R}} \left| \frac{\partial^l U}{\partial x^l} \right|^2 dx,$$

where $|U| = \sqrt{\langle U, U \rangle}$, and $\langle U, V \rangle$ is the standard inner product for $U, V \in \mathbb{R}^d$. Set $H^k(\mathbb{R}) = H^k(\mathbb{R}; \mathbb{C})$ and $L^2(\mathbb{R}) = H^0(\mathbb{R}; \mathbb{C})$ for short. Let I be an interval in $\mathbb{R}$, and let X be an appropriate function space. We denote by $C(I; X)$ the set of all X-valued continuous functions on I, and by $L^\infty(I; X)$ the set of all X-valued essentially bounded functions on I.

The main results of the present paper are the following.

Theorem 1.1. *Let k be an integer greater than or equal to six. For any initial mapping $u_0 \in H^{k+1}(\mathbb{R};TN)$, there exists a positive number T depending only on $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that the initial value problem* (1), (2) *has a unique solution $u \in C\big([-T,T];H^{k+1}(\mathbb{R};TN)\big)$.*

Here we remark about the order of the smoothness of solutions required in Theorem 1.1. The condition $k \geqslant 6$ is determined by the Sobolev embedding $H^1(\mathbb{R}) \subset C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In our construction of solutions, we need $k \geqslant 4$ for the boundedness of the both hand sides of the equation (1). Our proof of uniqueness of solutions requires $k \geqslant 6$ for the boundedness of the second derivative of (1).

We shall prove Theorem 1.1 by using so-called the parabolic regularization and the uniform energy estimates of solutions to the regularized problems. In the latter part we introduce a gauge transform on the pullback bundle to overcome loss of derivative of order one. In other words, we slightly make use of local smoothing effect of dispersive equations via the gauge transform. In terms of pseudodifferential calculus, this consists of the identity and a pseudodifferential operator of order -1, The commutator between the lower order term of the gauge transform and the principal part of the equation becomes a second-order elliptic operator absorbing the loss of one derivative. This idea was actually applied to solving a third-order dispersive flow in [5] and [23].

The plan of the present paper is as follows. In Section 2 we pick up a one-dimensional fourth-order linear dispersive partial differential equation, and illustrate the local smoothing effect and the gauge transform. Section 3 is concerned with solving regularized problem. Section 4 is devoted to the construction of a time-local solution. Finally, we shall prove the uniqueness of solution and recover the continuity in time of solution in Section 5.

2. An auxiliary linear problem

The purpose of this section is to illustrate our idea of proof of Theorem 1.1. Consider the initial value problem of the form

$$u_t = \sqrt{-1}au_{xxxx} + \sqrt{-1}\{\beta(t,x)u_x\}_x + \gamma(t,x)u_x \quad \text{in } \mathbb{R}\times\mathbb{R}, \tag{4}$$
$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}, \tag{5}$$

where $u(t,x)$ is a complex-valued unknown function of $(t,x) \in \mathbb{R}\times\mathbb{R}$, $a \in \mathbb{R}\setminus\{0\}$ is a constant, $\beta(t,x) \in C\big(\mathbb{R};\mathscr{B}^\infty(\mathbb{R})\big)$ is real-valued, $\gamma(t,x) \in C\big(\mathbb{R};\mathscr{B}^\infty(\mathbb{R})\big)$ is complex-valued, $\mathscr{B}^\infty(\mathbb{R})$ is the set of all bounded smooth functions on $\mathbb{R}$ whose derivatives of any order are all bounded, and $u_0(x)$ is an initial data. We shall prove the following.

Proposition 2.1. *Suppose that there exists a function* $\phi(x) \in \mathscr{B}^\infty(\mathbb{R})$ *such that*

$$|\operatorname{Im}\gamma(t,x)| \leqslant \phi(x), \ (t,x) \in \mathbb{R}^2, \qquad \int_{\mathbb{R}} \phi(x)dx < \infty.$$

Then, the initial value problem (4), (5) *is* L^2*-well-posed, that is, for any* $u_0 \in L^2(\mathbb{R})$*,* (4), (5) *has a unique solution* $u \in C\big(\mathbb{R}; L^2(\mathbb{R})\big)$.

Roughly speaking, Proposition 2.1 says that if $\operatorname{Im}\gamma(t,x)$ is integrable in $x \in \mathbb{R}$ uniformly for $t \in \mathbb{R}$, one can solve the initial value problem (4), (5). In other words, $\operatorname{Im}\gamma(t,x)$ is seemingly an obstruction to the L^2-well-posedness, but can be resolved by the local smoothing effect of $\exp(\sqrt{-1}at\frac{\partial^4}{\partial x^4})$ described as

$$\left\| (1+x^2)^{-\frac{\delta}{4}} \left(-\frac{\partial^2}{\partial x^2}\right)^{\frac{3}{4}} \exp\left(\sqrt{-1}at\frac{\partial^4}{\partial x^4}\right) u_0 \right\|_{L^2(\mathbb{R}^2)} \leqslant C\|u_0\|_{L^2(\mathbb{R})}, \tag{6}$$

where $\delta > 1$ is a constant. This shows that solutions to the initial value problem for $u_t = \sqrt{-1}au_{xxxx}$ gains extra smoothness of order $\frac{3}{2}$ in x, see [3] for instance. We do not need to make full use of (6) for proving Proposition 2.1. We have only to use the gain of smoothness of order one for this. This means that Proposition 2.1 is never sharp, and that the condition given there is too strong. In fact, Mizuhara [18] and Tarama [27] studied the necessary and sufficient conditions for L^2-well-posedness of the initial value problem for more general higher-order linear dispersive partial differential equations in one space dimension. Under the condition in Proposition 2.1, one can eliminate $\operatorname{Im}\gamma(t,x)$ in (4) exactly by using a gauge transform. However, we shall actually make the local smoothing effect visible by using another gauge transform with $\phi(x)$. This will be convenient for applying our idea to (1) since the corresponding problematic terms in (1) are very complicated.

Proof of Proposition 2.1. We shall give only the outline of the energy estimates. Here we make use of elementary pseudodifferential calculus, see [17] and [28] for instance. Let $r > 0$ be a sufficiently large constant. Pick up $\varphi(\xi) \in C^\infty(\mathbb{R})$ such that

$$\varphi(\xi) = 1 \ \ (|\xi| \geqslant r+1), \quad \varphi(\xi) = 0 \ \ (|\xi| \leqslant r).$$

Here we introduce a pseudodifferential operator $\Lambda = I + \tilde{\Lambda}$ of order zero, where I is the identity operator, and

$$\tilde{\Lambda}v(x) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{\sqrt{-1}(x-y)\xi}\tilde{\lambda}(x,\xi)v(y)dyd\xi,$$
$$\tilde{\lambda}(x,\xi) = \Phi(x)\frac{\varphi(\xi)}{4a\xi}, \quad \Phi(x) = \int_0^x \phi(y)dy.$$

Denote the set of bounded linear operator of $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ by $\mathscr{L}\big(L^2(\mathbb{R})\big)$. It is easy to see that Λ is in $\mathscr{L}\big(L^2(\mathbb{R})\big)$, and invertible since $\Lambda = I + \mathcal{O}(r^{-1})$ provided that r is sufficiently large. Indeed the Neumann series $I+\sum_{l=1}^{\infty}(-\tilde{\Lambda})^l$ gives the inverse operator of Λ. Hence Λ is an automorphism on $L^2(\mathbb{R})$.

Set $v(t,x)=\Lambda u(t,x)$, i.e., $u=\Lambda^{-1}v$. Apply Λ to (4). Set $D_x=-\sqrt{-1}\frac{\partial}{\partial x}$ for short. Here we denote by $\mathscr{L}$ the set of all L^2-bounded operators on $\mathbb{R}$. In what follows, different positive constants are denoted by the same letter C, and different operators in $C(\mathbb{R};\mathscr{L})$ are denoted by the same notation $P(t)$. Then we have

$$
\begin{aligned}
\Lambda u_t &= v_t,\\
\Lambda\sqrt{-1}au_{xxxx} &= \sqrt{-1}a(I+\tilde{\Lambda})D_x^4(I-\tilde{\Lambda}+\tilde{\Lambda}^2-\tilde{\Lambda}^3+\cdots)v\\
&= \sqrt{-1}aD_x^4(I-\tilde{\Lambda}+\tilde{\Lambda}^2-\tilde{\Lambda}^3+\cdots)v\\
&\quad +\sqrt{-1}a\tilde{\Lambda}D_x^4(I-\tilde{\Lambda}+\tilde{\Lambda}^2-\tilde{\Lambda}^3+\cdots)v\\
&= \sqrt{-1}aD_x^4v-\sqrt{-1}aD_x^4\tilde{\Lambda}(I-\tilde{\Lambda}+\tilde{\Lambda}^2-\tilde{\Lambda}^3+\cdots)v\\
&\quad +\sqrt{-1}a\tilde{\Lambda}D_x^4(I-\tilde{\Lambda}+\tilde{\Lambda}^2-\tilde{\Lambda}^3+\cdots)v\\
&= \sqrt{-1}aD_x^4v+\sqrt{-1}a\big[\tilde{\Lambda},D_x^4\big](I-\tilde{\Lambda}+\tilde{\Lambda}^2-\tilde{\Lambda}^3+\cdots)v\\
&= \sqrt{-1}aD_x^4v+\sqrt{-1}a\big[\tilde{\Lambda},D_x^4\big]v-\sqrt{-1}a\big[\tilde{\Lambda},D_x^4\big]\tilde{\Lambda}v+P(t)v,\\
\Lambda\sqrt{-1}\{\beta(t,x)u_x\}_x &= \sqrt{-1}\{\beta(t,x)v_x\}_x+P(t)v,\\
\Lambda\gamma(t,x)u_x &= \gamma(t,x)v_x+P(t)v.
\end{aligned}
$$

Since $\sqrt{-1}a\big[\tilde{\Lambda},D_x^4\big]=\phi(x)\frac{\partial^2}{\partial x^2}+\frac{3}{2}\phi'(x)\frac{\partial}{\partial x}+P(t)$, we have

$$
\begin{aligned}
\Lambda\sqrt{-1}au_{xxxx} &= \sqrt{-1}aD_x^4v+\phi(x)v_{xx}+\frac{3}{2}\phi'(x)v_x-\sqrt{-1}\frac{\phi(x)\Phi(x)}{4a}v_x+P(t)v\\
&= \sqrt{-1}aD_x^4v+\{\phi(x)v_x\}_x+\frac{1}{2}\phi'(x)v_x-\sqrt{-1}\frac{\phi(x)\Phi(x)}{4a}v_x+P(t)v.
\end{aligned}
$$

Combining the above, we obtain

$$
\begin{aligned}
v_t &= \sqrt{-1}av_{xxxx}+\{\phi(x)v_x\}_x+\sqrt{-1}\{\beta(t,x)v_x\}_x\\
&\quad +\left\{\operatorname{Re}\gamma(t,x)+\frac{\phi'(x)}{2}\right\}v_x+\sqrt{-1}\left\{\operatorname{Im}\gamma(t,x)-\frac{\phi(x)\Phi(x)}{4a}\right\}v_x+P(t)v.
\end{aligned}
$$

Using this, we deduce

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}|v|^2dx &= -2\int_{\mathbb{R}}\phi(x)|v_x|^2dx+\sqrt{-1}\int_{\mathbb{R}}\gamma_1(t,x)(v_x\overline{v}-v\overline{v_x})dx+\\
&\quad +2\operatorname{Re}\int_{\mathbb{R}}\{P(t)v\}\overline{v}dx,\\
\gamma_1(t,x) &= \operatorname{Im}\gamma(t,x)-\frac{\phi(x)\Phi(x)}{4a}.
\end{aligned}
$$

Since $\gamma_1(t,x) = \mathcal{O}\big(\phi(x)\big)$, the Schwarz inequality implies that

$$
\begin{aligned}
\left|\int_{\mathbb{R}} \gamma_1(t,x)(v_x\overline{v} - v\overline{v_x})dx\right| &\leqslant C\left\{\int_{\mathbb{R}} \phi(x)|v_x|^2dx\right\}^{\frac{1}{2}}\left\{\int_{\mathbb{R}} \phi(x)|v|^2dx\right\}^{\frac{1}{2}} \\
&\leqslant C\left\{\int_{\mathbb{R}} \phi(x)|v_x|^2dx\right\}^{\frac{1}{2}}\left\{\int_{\mathbb{R}} |v|^2dx\right\}^{\frac{1}{2}} \\
&\leqslant \int_{\mathbb{R}} \phi(x)|v_x|^2dx + C\int_{\mathbb{R}} |v|^2dx.
\end{aligned}
$$

Hence for any $T > 0$ there exists a constant $C_T > 0$ depending on $T > 0$ such that

$$
\frac{d}{dt}\int_{\mathbb{R}} |v|^2dx + \int_{\mathbb{R}} \phi(x)|v_x|^2dx \leqslant C_T\int_{\mathbb{R}} |v|^2dx
$$

for $t \in [0,T]$. This implies that for $t \in [0,T]$,

$$
\int_{\mathbb{R}} |v(t,x)|^2dx + \int_0^t e^{C_T(t-s)}\left(\int_{\mathbb{R}} \phi(x)|v_x(t,x)|^2dx\right)ds \leqslant e^{C_Tt}\int_{\mathbb{R}} |v(0,x)|^2dx.
$$

The same inequality holds for the negative direction of t. Using these energy estimates, we can prove Proposition 2.1. We omit the details. $\square$

3. Parabolic regularization

In this section we shall solve the initial value problem for a regularized equation of the form

$$
\begin{aligned}
u_t = &\big(-\varepsilon + aJ(u)\big)\nabla_x^3 u_x \\
&+ \{1 + bg_u(u_x,u_x)\}J(u)\nabla_x u_x \quad \text{in } (0,\infty)\times\mathbb{R}, \qquad (7)\\
&+ cg_u(\nabla_x u_x, u_x)J(u)u_x \\
u(0,x) = &\, u_0(x) \qquad \text{in } \mathbb{R}, \qquad (8)
\end{aligned}
$$

where ε is a positive parameter. We shall use a sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ of the solutions to (7), (8) to construct a solution to (1), (2) in the next section. The results of this section are the following.

Lemma 3.1. *Let k be an integer not smaller than four. For any initial mapping $u_0 \in H^{k+1}(\mathbb{R};TN)$, there exists a positive number T_ε depending only on $\varepsilon > 0$ and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that the initial value problem* (7), (8) *has a unique solution $u^\varepsilon \in C\big([0,T_\varepsilon];H^{k+1}(\mathbb{R};TN)\big)$.*

The equation (7) corresponds to quasilinear parabolic partial differential equations. It is possible to solve (7)-(8) directly by using the fundamental solution of the differential operator

$$\frac{\partial}{\partial t} - \bigl(-\varepsilon + aJ(u)\bigr)\Bigl(\nabla^N_{du(\frac{\partial}{\partial x})}\Bigr)^3 \frac{\partial}{\partial x}$$

and so-called the Leray-Schauder fixed point theorem. We embed (7) in an appropriate Euclidean space $\mathbb{R}^d$ by the Nash isometric embedding $w : N \to \mathbb{R}^d$ (see e.g., [8, 9]), and deal with it as a system of quasilinear partial differential equations. In particular, we need to take care of the range of solutions to the embedded equation. Unfortunately, however, the above mentioned approach to (7) seems to be very complicated. For this reason, we consider more regularized initial value problem of the form

$$\begin{aligned} u_t = {} & \delta\nabla_x^5 u_x + \bigl(-\varepsilon + aJ(u)\bigr)\nabla_x^3 u_x \\ & + \{1 + bg_u(u_x, u_x)\}J(u)\nabla_x u_x \qquad \text{in } (0,\infty)\times\mathbb{R}, \\ & + cg_u(\nabla_x u_x, u_x)J(u)u_x \end{aligned} \tag{9}$$

$$u(0,x) = u_0(x) \qquad \text{in } \mathbb{R}, \tag{10}$$

and we shall prove Lemma 3.1 elementally. Here δ is a positive parameter. Let $\{u^{\varepsilon,\delta}\}_{\delta>0}$ be a sequence of solutions to (9), (10) for any fixed $\varepsilon > 0$. We will get a sequence of solutions $\{u^{\varepsilon}\}_{\varepsilon>0}$ to (7), (8) by the standard compactness arguments as $\delta \downarrow 0$. We first prove the following.

Lemma 3.2. *Let k be an integer not smaller than four. For any initial mapping $u_0 \in H^{k+1}(\mathbb{R}; TN)$, there exists a positive number $T_{\varepsilon,\delta}$ depending only on $\varepsilon > 0$, $\delta > 0$ and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that the initial value problem* (9)*,* (10) *has a unique solution $u^{\varepsilon,\delta} \in C\bigl([0, T_{\varepsilon,\delta}]; H^{k+1}(\mathbb{R}; TN)\bigr)$.*

To prove Lemma 3.2, we embed (9), (10) in an appropriate Euclidean space $\mathbb{R}^d$. Let $w \in C^\infty(N; \mathbb{R}^d)$ be the Nash isometric embedding, and let $w_r(N)$ be a tubular neighborhood of $w(N)$ with sufficiently small $r > 0$ defined by

$$w_r(N) = \{v_0 + v_1 \in \mathbb{R}^d \mid v_0 \in w(N),\ v_1 \in T_{v_0}w(N)^\perp,\ |v_1| < r\},$$

where $T_{v_0}w(N)^\perp$ is the orthogonal complement of $T_{v_0}w(N)$ in $T_{v_0}\mathbb{R}^d \simeq \mathbb{R}^d$. The mapping

$$\Pi : w_r(N) \ni v_0 + v_1 \mapsto v_0 \in w(N)$$

is the natural projection. Set $\rho(v) = v - \Pi(v)$ for $v \in w_r(N)$.

Let u be a solution to (9), (10). Set $v = w(u)$ for short. Then we have

$$\begin{aligned} v_t &= dw(u_t) \\ &= dw\Bigl(\delta\nabla_x^5 u_x + \bigl(-\varepsilon + aJ(u)\bigr)\nabla_x^3 u_x + \cdots\Bigr) \\ &= \delta v_{xxxxxxx} + F(v, v_x, v_{xx}, v_{xxx}, v_{xxxx}, v_{xxxxx}), \end{aligned} \tag{11}$$

where $F \in C^\infty(\mathbb{R}^{6d};\mathbb{R}^d)$ is an appropriate function. Set

$$\tilde{F}(v) = F(v, v_x, v_{xx}, v_{xxx}, v_{xxxx}, v_{xxxxx})$$

for short. The initial value problem for (11) with $v(0,x) = w\bigl(u_0(x)\bigr)$ is equivalent to an integral equation of the form

$$v(t) = \exp\left(t\delta\frac{\partial^6}{\partial x^6}\right) w(u_0) + \int_0^t \exp\left((t-s)\delta\frac{\partial^6}{\partial x^6}\right)\tilde{F}\bigl(v(s)\bigr)ds. \tag{12}$$

We shall solve (12) without considering the range of v, that is, we shall deal with

$$v(t) = \exp\left(t\delta\frac{\partial^6}{\partial x^6}\right) w(u_0) + \int_0^t \exp\left((t-s)\delta\frac{\partial^6}{\partial x^6}\right)\tilde{F}\Bigl(\Pi\bigl(v(s)\bigr)\Bigr)ds. \tag{13}$$

If a solution v to (13) satisfies $\Pi(v) = v$, that is, $\rho(v) = 0$, then v solves (12) and $u = w^{-1}(v)$ is a solution to (9), (10). Note that for any $\alpha > 0$, there exists a constant $C_{\delta,\alpha} > 0$ depending only on δ and α such that

$$\left\|\exp\left(t\delta\frac{\partial^6}{\partial x^6}\right)f\right\|_{H^{6-\alpha}(\mathbb{R})} \leqslant \frac{C_{\delta,\alpha}}{t^{(6-\alpha)/6}}\|f\|_{L^2(\mathbb{R})}, \quad t > 0.$$

Combining this and the contraction mapping theorem, we can prove the following.

Lemma 3.3. *Let k be an integer not smaller than four. For any initial mapping $u_0 \in H^{k+1}(\mathbb{R};TN)$, there exists a positive number $T_{\varepsilon,\delta}$ depending only on $\varepsilon > 0$, $\delta > 0$ and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that the integral equation* (13) *has a unique solution v satisfying*

$$v \in C\bigl([0,T_{\varepsilon,\delta}] \times \mathbb{R}; w_r(N)\bigr), \quad v_x \in C\bigl([0,T_{\varepsilon,\delta}]; H^k(\mathbb{R};\mathbb{R}^d)\bigr). \tag{14}$$

We omit the detail of the proof of Lemma 3.3. Here we will complete the proof of Lemma 3.2.

Proof of Lemma 3.2. Let v be a unique solution to (13) in Lemma 3.3. We have only to show that $\rho(v) = v - \Pi(v) = 0$. Since $\Pi \in \mathscr{B}^\infty(\mathbb{R}^d;\mathbb{R}^d)$, the smoothness of v given in (14) implies that

$$\rho(v) \in \mathscr{B}\bigl([0,T_{\varepsilon,\delta}] \times \mathbb{R};\mathbb{R}^d\bigr), \tag{15}$$

$$\rho(v)_x = v_x - \frac{\partial\Pi}{\partial v}(v)v_x \in C\bigl([0,T_{\varepsilon,\delta}]; H^4(\mathbb{R};\mathbb{R}^d)\bigr), \tag{16}$$

$$\rho(v)_{xx}, \rho(v)_{xxx} \in C\bigl([0,T_{\varepsilon,\delta}]; H^4(\mathbb{R};\mathbb{R}^d)\bigr), \tag{17}$$

where $\mathscr{B}$ denotes the set of all bounded continuous function. We shall evaluate $|\rho(v)|^2 = \langle\rho(v),\rho(v)\rangle$. Fix arbitrary $\eta \in (0,1]$. A simple computation gives

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} e^{-\eta x^2}|\rho(v)|^2 dx = \int_{\mathbb{R}} e^{-\eta x^2}\langle\rho(v)_t,\rho(v)\rangle dx. \tag{18}$$

Since $\Pi(v)_t = \frac{\partial \Pi}{\partial v}(v) v_t \in T_{\Pi(v)} w(N)$, $\rho(v) \in T_{\Pi(v)} w(N)^{\perp}$, we have

$$\langle \Pi(v)_t, \rho(v) \rangle = 0, \quad \langle \rho(v)_t, \rho(v) \rangle = \Big\langle \big\{ \rho(v)_t + \Pi(v)_t \big\}, \rho(v) \Big\rangle = \langle v_t, \rho(v) \rangle.$$

Substituting this into (18), we get

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} e^{-\eta x^2} |\rho(v)|^2 dx = \int_{\mathbb{R}} e^{-\eta x^2} \langle v_t, \rho(v) \rangle dx. \tag{19}$$

Set $u = w^{-1}\big(\Pi(v)\big)$ for short. Here we remark that

$$\begin{aligned} v_t &= \delta v_{xxxxxx} + \tilde{F}\big(\Pi(v)\big) \\ &= \delta \rho(v)_{xxxxxx} + \big\{ \delta \Pi(v)_{xxxxxx} + \tilde{F}\big(\Pi(v)\big) \big\} \\ &= \delta \rho(v)_{xxxxxx} + dw \Big(\delta \nabla_x^5 u_x + \big(-\varepsilon + aJ(u) \big) \nabla_x^3 u_x + \cdots \Big), \end{aligned} \tag{20}$$

and the second term of the right hand side above belongs to $T_{\Pi(v)} w(N)$. Substituting (20) into (19), we have

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} e^{-\eta x^2} |\rho(v)|^2 dx = \delta \int_{\mathbb{R}} e^{-\eta x^2} \langle \rho(v)_{xxxxxx}, \rho(v) \rangle dx. \tag{21}$$

In view of the integration by parts, (21) becomes

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} e^{-\eta x^2} |\rho(v)|^2 dx = -\delta \int_{\mathbb{R}} \Big\langle \rho(v)_{xxx}, \big\{ e^{-\eta x^2} \rho(v) \big\}_{xxx} \Big\rangle dx. \tag{22}$$

Substituting

$$\begin{aligned} \big\{ e^{-\eta x^2} \rho(v) \big\}_{xxx} &= e^{-\eta x^2} \rho(v)_{xxx} + 3 \big\{ e^{-\eta x^2} \big\}_x \rho(v)_{xx} + 3 \big\{ e^{-\eta x^2} \big\}_{xx} \rho(v)_x + \big\{ e^{-\eta x^2} \big\}_{xxx} \rho(v), \\ \big\{ e^{-\eta x^2} \big\}_x &= -2\eta x e^{-\eta x^2}, \\ \big\{ e^{-\eta x^2} \big\}_{xx} &= \big(-2\eta + 4\eta^2 x^2 \big) e^{-\eta x^2}, \\ \big\{ e^{-\eta x^2} \big\}_{xxx} &= \big(12\eta^2 x - 8\eta^3 x^3 \big) e^{-\eta x^2} \end{aligned}$$

into (22), we have

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} e^{-\eta x^2} |\rho(v)|^2 dx = &-\delta \int_{\mathbb{R}} e^{-\eta x^2} |\rho(v)_{xxx}|^2 dx \\ &+ \delta \int_{\mathbb{R}} \big(-12\eta^2 x + 8\eta^3 x^3 \big) e^{-\eta x^2} \big\langle \rho(v)_{xxx}, \rho(v) \big\rangle dx \qquad (23) \\ &+ \delta \int_{\mathbb{R}} \big(6\eta - 12\eta^2 x^2 \big) e^{-\eta x^2} \big\langle \rho(v)_{xxx}, \rho(v)_x \big\rangle dx \qquad (24) \\ &+ \delta \int_{\mathbb{R}} 6\eta x e^{-\eta x^2} \big\langle \rho(v)_{xxx}, \rho(v)_{xx} \big\rangle dx. \qquad (25) \end{aligned}$$

Here we recall the smoothness of v as in (15)–(17). Combining $\eta \in (0,1]$, the Schwarz inequality for the integration on $\mathbb{R}$ and the change of variable $y = \eta^{\frac{1}{2}}x$, we deduce that

$$\begin{aligned}
|(23)| &\leqslant 12\delta \int_{\mathbb{R}} \bigl(\eta^2|x| + \eta^3|x|^3\bigr) e^{-\eta x^2} |\rho(v)_{xxx}||\rho(v)| dx \\
&\leqslant C\delta\eta^{\frac{3}{2}} \int_{\mathbb{R}} \bigl(\eta^{\frac{1}{2}}|x| + \eta^{\frac{3}{2}}|x|^3\bigr) e^{-\eta x^2} |\rho(v)_{xxx}| dx \\
&\leqslant C\delta\eta^{\frac{3}{2}} \int_{\mathbb{R}} \bigl(1 + \eta x^2\bigr)^{\frac{3}{2}} e^{-\eta x^2} |\rho(v)_{xxx}| dx \\
&\leqslant C\delta\eta^{\frac{3}{2}} \left\{ \int_{\mathbb{R}} (1+\eta x^2)^3 e^{-2\eta x^2} dx \right\}^{\frac{1}{2}} \\
&\leqslant C\delta\eta^{\frac{5}{4}} \left\{ \int_{\mathbb{R}} (1+y^2)^3 e^{-2y^2} dy \right\}^{\frac{1}{2}} \\
&= C\delta\eta^{\frac{5}{4}}, && (26) \\
|(24)| &\leqslant 12\delta\eta \int_{\mathbb{R}} \bigl(1+\eta x^2\bigr) e^{-\eta x^2} |\rho(v)_{xxx}||\rho(v)_x| dx \\
&\leqslant 12\delta\eta \left\{ \sup_{y\in\mathbb{R}} (1+y^2) e^{-y^2} \right\} \int_{\mathbb{R}} |\rho(v)_{xxx}||\rho(v)_x| dx \\
&= C\delta\eta && (27) \\
|(25)| &\leqslant 6\delta\eta^{\frac{1}{2}} \int_{\mathbb{R}} \bigl(\eta x^2\bigr)^{\frac{1}{2}} e^{-\eta x^2} |\rho(v)_{xxx}||\rho(v)_{xx}| dx \\
&\leqslant 6\delta\eta^{\frac{1}{2}} \left\{ \sup_{y\in\mathbb{R}} |y| e^{-y^2} \right\} \int_{\mathbb{R}} |\rho(v)_{xxx}||\rho(v)_{xx}| dx \\
&= C\delta\eta^{\frac{1}{2}}, && (28)
\end{aligned}$$

where C is a positive constant which is independent of η. Combining (23)–(28), we deduce that there exists a positive constant C_0 which is independent of η such that

$$\frac{d}{dt} \int_{\mathbb{R}} e^{-\eta x^2} |\rho(v)|^2 dx \leqslant C_0 \delta \eta^{\frac{1}{2}}, \quad \eta \in (0,1],\ t \in [0, T_{\varepsilon,\delta}].$$

Since $\rho\bigl(v(0,\cdot)\bigr) = \rho\bigl(w(u_0)\bigr) = 0$, we have $\int_{\mathbb{R}} e^{-\eta x^2} \bigl|\rho\bigl(v(t,x)\bigr)\bigr|^2 dx \leqslant C_0\delta\eta^{\frac{1}{2}} t$. Applying the Beppo-Levi theorem to this, we deduce that for any fixed $t \in [0, T_{\varepsilon,\delta}]$

$$0 \leqslant \int_{\mathbb{R}} \bigl|\rho\bigl(v(t,x)\bigr)\bigr|^2 dx = \lim_{\eta \downarrow 0} \int_{\mathbb{R}} e^{-\eta x^2} \bigl|\rho\bigl(v(t,x)\bigr)\bigr|^2 dx = 0.$$

This implies that $\rho\bigl(v(t,\cdot)\bigr)$ belongs to $L^2(\mathbb{R};\mathbb{R}^d)$ and $\rho\bigl(v(t,x)\bigr) = 0$ a.e. $x \in \mathbb{R}$ for any $t \in [0, T_{\varepsilon,\delta}]$. Thus we deduce that $\rho\bigl(v(t,x)\bigr) = 0$ for any $(t,x) \in [0, T_{\varepsilon.\delta}] \times \mathbb{R}$ since $v \in C\bigl([0, T_{\varepsilon.\delta}] \times \mathbb{R}; \mathbb{R}^d\bigr)$. $\square$

We will conclude the present section with the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $\{u^\delta\}_{\delta\in(0,1]}$ be a sequence of solutions to (9), (10) with a fixed parameter ε. We shall show that there exists T_ε depending only on ε and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that $\{u^\delta\}_{\delta\in(0,1]}$ is bounded in $L^\infty\big(0,T_\varepsilon;H^{k+1}(\mathbb{R};TN)\big)$. If this is true, then the standard compact arguments imply that there exists a mapping u such that u satisfies

$$u\in C\big([0,T_\varepsilon];H^k(\mathbb{R};TN)\big)\bigcap L^\infty\big(0,T_\varepsilon;H^{k+1}(\mathbb{R};TN)\big)$$

and solves (7)-(8) provided that $\delta\downarrow 0$. The key of this uniform estimates is also the smoothing property of the parabolic operator $\frac{\partial}{\partial t}+\varepsilon\frac{\partial^4}{\partial x^4}$. By using this property, we can also prove the uniqueness and the continuity in the time variable, but we omit the details.

We abbreviate $J(u^\delta)$ and $g_{u^\delta}(\cdot,\cdot)$ by J and $g(\cdot,\cdot)$ respectively. We will evaluate

$$\sum_{l=0}^{k}\int_{\mathbb{R}} g\big(\nabla_x^l u_x^\delta,\nabla_x^l u_x^\delta\big)dx.$$

We apply ∇_x^{l+1}, $l=0,1,2,\dots,k$ to

$$\begin{aligned} u_t^\delta &= \delta\nabla_x^5 u_x^\delta+\big(-\varepsilon+aJ\big)\nabla_x^3 u_x^\delta \\ &\quad+\{1+bg(u_x^\delta,u_x^\delta)\}J\nabla_x u_x^\delta+cg(\nabla_x u_x^\delta,u_x^\delta)Ju_x^\delta \quad \text{in } (0,T_{\varepsilon,\delta})\times\mathbb{R}. \end{aligned}\tag{29}$$

We compute this term by term. Let R be the Riemann curvature tensor of (N,J,g). Note that for the left hand side of (29),

$$\nabla_t\big(u_x^\delta\big)=\nabla_x u_t^\delta,\tag{30}$$

$$\nabla_t\big(\nabla_x^l u_x^\delta\big)=\nabla_x^{l+1}u_t^\delta+\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{R(u_t^\delta,u_x^\delta)\nabla_x^m u_x^\delta\Big\},\quad l=1,2,3,\dots\tag{31}$$

$$\begin{aligned} &\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{R(u_t^\delta,u_x^\delta)\nabla_x^m u_x^\delta\Big\}\\ &=\delta\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{R\big(\nabla_x^5 u_x^\delta,u_x^\delta\big)\nabla_x^m u_x^\delta\Big\}-\varepsilon\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{R\big(\nabla_x^3 u_x^\delta,u_x^\delta\big)\nabla_x^m u_x^\delta\Big\}\\ &\quad+a\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{R\big(J(u^\delta)\nabla_x^3 u_x^\delta,u_x^\delta\big)\nabla_x^m u_x^\delta\Big\}\\ &\quad+\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{\big(1+bg(u_x^\delta,u_x^\delta)\big)R\big(J(u^\delta)\nabla_x u_x^\delta,u_x^\delta\big)\nabla_x^m u_x^\delta\Big\}\\ &\quad+c\sum_{m=0}^{l-1}\nabla_x^{l-1-m}\Big\{g\big(\nabla_x u_x^\delta,u_x^\delta\big)R\big(J(u^\delta)u_x^\delta,u_x^\delta\big)\nabla_x^m u_x^\delta\Big\}. \end{aligned}$$

Hence,

$$
\begin{aligned}
\sum_{m=0}^{l-1} \nabla_x^{l-1-m}\Big\{R(u_t^\delta, u_x^\delta)\nabla_x^m u_x^\delta\Big\} = {}& \delta\nabla_x\Big\{R\big(\nabla_x^{l+3}u_x^\delta, u_x^\delta\big)u_x^\delta\Big\} \\
& + \delta\mathcal{O}\Big(g\big(\nabla_x^{l+3}u_x^\delta, \nabla_x^{l+3}u_x^\delta\big)^{\frac{1}{2}}\Big) \\
& + \mathcal{O}\left(\sum_{\alpha=1}^{2} g\big(\nabla_x^{l+\alpha}u_x^\delta, \nabla_x^{l+\alpha}u_x^\delta\big)^{\frac{1}{2}}\right) \\
& + \mathcal{O}\left(\sum_{m=0}^{l} g\big(\nabla_x^{m}u_x^\delta, \nabla_x^{m}u_x^\delta\big)^{\frac{1}{2}}\right).
\end{aligned}
\tag{32}
$$

Here we used the Sobolev embedding. On the other hand, we have

$$
\begin{aligned}
\nabla_x^{l+1}u_t^\delta = {}& \delta\nabla_x^6\big(\nabla_x^l u_x^\delta\big) + \big(-\varepsilon + aJ(u^\delta)\big)\nabla_x^4\big(\nabla_x^l u_x^\delta\big) + J(u^\delta)\nabla_x^2\big(\nabla_x^l u_x^\delta\big) \\
& + b\sum_{\mu+\nu=0}^{l+1} \frac{(l+1)!}{\mu!\nu!(l+1-\mu-\nu)!} g\big(\nabla_x^\mu u_x^\delta, \nabla_x^\nu u_x^\delta\big)J(u^\delta)\nabla_x^{l+2-\mu-\nu}u_x^\delta \\
& + c\sum_{\mu+\nu=0}^{l+1} \frac{(l+1)!}{\mu!\nu!(l+1-\mu-\nu)!} g\big(\nabla_x^{\mu+1} u_x^\delta, \nabla_x^\nu u_x^\delta\big)J(u^\delta)\nabla_x^{l+1-\mu-\nu}u_x^\delta \\
= {}& \delta\nabla_x^6\big(\nabla_x^l u_x^\delta\big) + \big(-\varepsilon + aJ(u^\delta)\big)\nabla_x^4\big(\nabla_x^l u_x^\delta\big) \\
& + J(u^\delta)\nabla_x^2\big(\nabla_x^l u_x^\delta\big) + bQ_1 + cQ_2.
\end{aligned}
\tag{33}
$$

We modify Q_1 and Q_2 a little for the sake of convenience for our energy estimates:

$$
\begin{aligned}
Q_1 = {}& \sum_{\mu+\nu=0}^{l+1} \frac{(l+1)!}{\mu!\nu!(l+1-\mu-\nu)!} g\big(\nabla_x^\mu u_x^\delta, \nabla_x^\nu u_x^\delta\big)J(u^\delta)\nabla_x^{l+2-\mu-\nu}u_x^\delta \\
= {}& g(u_x^\delta, u_x^\delta)J(u^\delta)\nabla_x^2\big(\nabla_x^l u_x^\delta\big) + 2g\Big(\nabla_x\big(\nabla_x^l u_x^\delta\big), u_x^\delta\Big)J(u^\delta)\nabla_x u_x^\delta \\
& + 2(l+1)g\big(\nabla_x u_x^\delta, u_x^\delta\big)J(u^\delta)\nabla_x\big(\nabla_x^l u^\delta\big) \\
& + \sum_{\substack{\mu+\nu=2 \\ \mu,\nu\leqslant l}}^{l+1} \frac{(l+1)!}{\mu!\nu!(l+1-\mu-\nu)!} g\big(\nabla_x^\mu u_x^\delta, \nabla_x^\nu u_x^\delta\big)J(u^\delta)\nabla_x^{l+2-\mu-\nu}u_x^\delta \\
= {}& \nabla_x\Big\{g(u_x^\delta, u_x^\delta)J(u^\delta)\nabla_x\big(\nabla_x^l u_x^\delta\big)\Big\} + 2g\Big(\nabla_x\big(\nabla_x^l u_x^\delta\big), u_x^\delta\Big)J(u^\delta)\nabla_x u_x^\delta \\
& + 2lg\big(\nabla_x u_x^\delta, u_x^\delta\big)J(u^\delta)\nabla_x\big(\nabla_x^l u^\delta\big) \\
& + \sum_{\substack{\mu+\nu=2 \\ \mu,\nu\leqslant l}}^{l+1} \frac{(l+1)!}{\mu!\nu!(l+1-\mu-\nu)!} g\big(\nabla_x^\mu u_x^\delta, \nabla_x^\nu u_x^\delta\big)J(u^\delta)\nabla_x^{l+2-\mu-\nu}u_x^\delta
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
Q_2 &= \sum_{\mu+\nu=0}^{l+1} \frac{(l+1)!}{\mu!\nu!(l+1-\mu-\nu)!} g\bigl(\nabla_x^{\mu+1}u_x^\delta, \nabla_x^\nu u_x^\delta\bigr) J(u^\delta) \nabla_x^{l+1-\mu-\nu} u_x^\delta \\
&= g\bigl(\nabla_x u_x^\delta, u_x^\delta\bigr) J(u^\delta)\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr) + g\Bigl(\nabla_x^2\bigl(\nabla_x^l u_x^\delta\bigr), u_x^\delta\Bigr) J(u^\delta) u_x^\delta \\
&\quad + (l+1) g\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr), \nabla_x u_x^\delta\Bigr) J(u^\delta) u_x^\delta \\
&\quad + (l+1) g\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr), u_x^\delta\Bigr) J(u^\delta) \nabla_x u_x^\delta \\
&\quad + g\Bigl(\nabla_x u_x^\delta, \nabla_x\bigl(\nabla_x^l u_x^\delta\bigr)\Bigr) J(u^\delta) u_x^\delta \\
&\quad + \sum_{\substack{\mu+\nu+\rho=l+1 \\ \mu\leqslant l-1,\ \nu,\rho\leqslant l}} \frac{(l+1)!}{\mu!\nu!\rho!} g\bigl(\nabla_x^{\mu+1}u_x^\delta, \nabla_x^\nu u_x^\delta\bigr) J(u^\delta)\nabla_x^\rho u_x^\delta \\
&= \nabla_x\Bigl\{ g\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr), u_x^\delta\Bigr) J(u^\delta) u_x^\delta \Bigr\} + g\bigl(\nabla_x u_x^\delta, u_x^\delta\bigr) J(u^\delta)\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr) \\
&\quad + (l+1) g\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr), \nabla_x u_x^\delta\Bigr) J(u^\delta) u_x^\delta \\
&\quad + l g\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\delta\bigr), u_x^\delta\Bigr) J(u^\delta) \nabla_x u_x^\delta \\
&\quad + \sum_{\substack{\mu+\nu+\rho=l+1 \\ \mu\leqslant l-1,\ \nu,\rho\leqslant l}} \frac{(l+1)!}{\mu!\nu!\rho!} g\bigl(\nabla_x^{\mu+1}u_x^\delta, \nabla_x^\nu u_x^\delta\bigr) J(u^\delta)\nabla_x^\rho u_x^\delta.
\end{aligned} \tag{35}
$$

Combining (30)–(35), we have

$$
\begin{aligned}
\nabla_t\bigl(\nabla_x^l u_x^\delta\bigr) &= \delta\nabla_x^6\bigl(\nabla_x^l u_x^\delta\bigr) + \bigl(-\varepsilon + J(u^\delta)\bigr)\nabla_x^4\bigl(\nabla_x^l u_x^\delta\bigr) \\
&\quad + \delta\nabla_x\Bigl\{R\bigl(\nabla_x^{l+3}u_x^\delta, u_x^\delta\bigr)u_x^\delta\Bigr\} + \delta\mathcal{O}\Bigl(g\bigl(\nabla_x^{l+3}u_x^\delta, \nabla_x^{l+3}u_x^\delta\bigr)^{\frac{1}{2}}\Bigr) \\
&\quad + \mathcal{O}\left(\sum_{\alpha=1}^{2} g\bigl(\nabla_x^{l+\alpha}u_x^\delta, \nabla_x^{l+\alpha}u_x^\delta\bigr)^{\frac{1}{2}}\right) \\
&\quad + \mathcal{O}\left(\sum_{m=0}^{l} g\bigl(\nabla_x^{m}u_x^\delta, \nabla_x^{m}u_x^\delta\bigr)^{\frac{1}{2}}\right).
\end{aligned} \tag{36}
$$

Now we compute

$$
\frac{d}{dt}\frac{1}{2}\sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)dx = \sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_t\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)dx.
$$

Substitute (36) into this. Using the integration by parts and the property of J, we have

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)dx \\
&= -\delta\sum_{l=0}^{k}\int_{\mathbb{R}} g\Bigl(\nabla_x^3\bigl(\nabla_x^l u_x^\delta\bigr), \nabla_x^3\bigl(\nabla_x^l u_x^\delta\bigr)\Bigr)dx - \varepsilon\sum_{l=0}^{k}\int_{\mathbb{R}} g\Bigl(\nabla_x^2\bigl(\nabla_x^l u_x^\delta\bigr), \nabla_x^2\bigl(\nabla_x^l u_x^\delta\bigr)\Bigr)dx \\
&\quad -\delta\sum_{l=0}^{k}\int_{\mathbb{R}} g\Bigl(R\bigl(\nabla_x^{l+3}u_x^\delta, u_x^\delta\bigr)u_x^\delta, \nabla_x\bigl(\nabla_x^l u_x^\delta\bigr)\Bigr)dx \\
&\quad +\delta\sum_{l=0}^{k}\int_{\mathbb{R}} \mathcal{O}\Bigl(g\bigl(\nabla_x^{l+3}u_x^\delta, \nabla_x^{l+3}u_x^\delta\bigr)^{\frac{1}{2}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)^{\frac{1}{2}}\Bigr)dx \\
&\quad +\sum_{l=0}^{k}\int_{\mathbb{R}} \mathcal{O}\Bigl(\sum_{\alpha=1}^{2} g\bigl(\nabla_x^{l+\alpha}u_x^\delta, \nabla_x^{l+\alpha}u_x^\delta\bigr)^{\frac{1}{2}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)^{\frac{1}{2}}\Bigr)dx \\
&\quad +\sum_{l=0}^{k}\int_{\mathbb{R}} \mathcal{O}\Bigl(g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)\Bigr)dx \\
&= -\delta\sum_{l=0}^{k}\int_{\mathbb{R}} g\Bigl(\nabla_x^3\bigl(\nabla_x^l u_x^\delta\bigr), \nabla_x^3\bigl(\nabla_x^l u_x^\delta\bigr)\Bigr)dx - \varepsilon\sum_{l=0}^{k}\int_{\mathbb{R}} g\Bigl(\nabla_x^2\bigl(\nabla_x^l u_x^\delta\bigr), \nabla_x^2\bigl(\nabla_x^l u_x^\delta\bigr)\Bigr)dx \\
&\quad +\delta\sum_{l=0}^{k}\int_{\mathbb{R}} \mathcal{O}\Bigl(\sum_{\beta=0}^{1} g\bigl(\nabla_x^{l+3}u_x^\delta, \nabla_x^{l+3}u_x^\delta\bigr)^{\frac{1}{2}} g\bigl(\nabla_x^{l+\beta} u_x^\delta, \nabla_x^{l+\beta} u_x^\delta\bigr)^{\frac{1}{2}}\Bigr)dx \\
&\quad +\sum_{l=0}^{k}\int_{\mathbb{R}} \mathcal{O}\Bigl(\sum_{\alpha=1}^{2} g\bigl(\nabla_x^{l+\alpha}u_x^\delta, \nabla_x^{l+\alpha}u_x^\delta\bigr)^{\frac{1}{2}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)^{\frac{1}{2}}\Bigr)dx \\
&\quad +\sum_{l=0}^{k}\int_{\mathbb{R}} \mathcal{O}\Bigl(g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)\Bigr)dx. \qquad (37)
\end{aligned}
$$

Integration by parts, the Schwarz inequality and an elementary inequality $2ab \leqslant a^2 + b^2$ for $a, b > 0$ give

$$
\begin{aligned}
&\sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^{l+1}u_x^\delta, \nabla_x^{l+1}u_x^\delta\bigr)dx \\
&= -\sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^{l+2}u_x^\delta, \nabla_x^l u_x^\delta\bigr)dx \\
&\leqslant \sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^{l+2}u_x^\delta, \nabla_x^{l+2}u_x^\delta\bigr)^{\frac{1}{2}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)^{\frac{1}{2}}dx \\
&\leqslant \frac{\varepsilon}{2}\sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^{l+2}u_x^\delta, \nabla_x^{l+2}u_x^\delta\bigr)dx + \frac{1}{2\varepsilon}\sum_{l=0}^{k}\int_{\mathbb{R}} g\bigl(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\bigr)dx,
\end{aligned}
$$

$$\begin{aligned}
&\sum_{l=0}^{k}\int_{\mathbb{R}}\mathcal{O}\Big(g\big(\nabla_x^{l+2}u_x^\delta,\nabla_x^{l+2}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)^{\frac12}\Big)dx\\
&\leqslant C\sum_{l=0}^{k}\int_{\mathbb{R}}\Big(g\big(\nabla_x^{l+2}u_x^\delta,\nabla_x^{l+2}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)^{\frac12}dx\\
&\leqslant \frac{\varepsilon}{2}\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+2}u_x^\delta,\nabla_x^{l+2}u_x^\delta\big)dx+\frac{C^2}{2\varepsilon}\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)dx.
\end{aligned}$$

Using this and an elementary inequality $2ab\leqslant a^2+b^2$ for $a,b>0$ again, we obtain

$$\begin{aligned}
&\left|\delta\sum_{l=0}^{k}\int_{\mathbb{R}}\mathcal{O}\Big(\sum_{\beta=0}^{1}g\big(\nabla_x^{l+3}u_x^\delta,\nabla_x^{l+3}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l+\beta}u_x^\delta,\nabla_x^{l+\beta}u_x^\delta\big)^{\frac12}\Big)dx\right|\\
&\leqslant \frac{\delta}{2}\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+3}u_x^\delta,\nabla_x^{l+3}u_x^\delta\big)dx\\
&\quad+C\delta\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+1}u_x^\delta,\nabla_x^{l+1}u_x^\delta\big)dx+C\delta\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)dx\\
&\leqslant \frac{\delta}{2}\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+3}u_x^\delta,\nabla_x^{l+3}u_x^\delta\big)dx+\frac{\varepsilon}{2}\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+2}u_x^\delta,\nabla_x^{l+2}u_x^\delta\big)dx\\
&\quad+\left(\frac{C^2\delta^2}{2\varepsilon}+C\delta\right)\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)dx, \qquad (38)\\
&\left|\sum_{l=0}^{k}\int_{\mathbb{R}}\mathcal{O}\Big(\sum_{\alpha=1}^{2}g\big(\nabla_x^{l+\alpha}u_x^\delta,\nabla_x^{l+\alpha}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)^{\frac12}\Big)dx\right|\\
&\leqslant C\sum_{l=0}^{k}\int_{\mathbb{R}}\sum_{\alpha=1}^{2}g\big(\nabla_x^{l+\alpha}u_x^\delta,\nabla_x^{l+\alpha}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)^{\frac12}dx\\
&=C\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+2}u_x^\delta,\nabla_x^{l+2}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)^{\frac12}dx\\
&\quad+C\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+1}u_x^\delta,\nabla_x^{l+1}u_x^\delta\big)^{\frac12}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)^{\frac12}dx\\
&\leqslant\left(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right)\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l+2}u_x^\delta,\nabla_x^{l+2}u_x^\delta\big)dx\\
&\quad+\left(\frac{C^2}{\varepsilon}+\frac{C^2}{2\varepsilon}\right)\sum_{l=0}^{k}\int_{\mathbb{R}}g\big(\nabla_x^{l}u_x^\delta,\nabla_x^{l}u_x^\delta\big)dx. \qquad (39)
\end{aligned}$$

Substitute (38), (39) into (37). We deduce that there exists a positive constant $C(\varepsilon)$ depending only on $\varepsilon \in (0,1]$ and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that

$$\frac{1}{2}\frac{d}{dt}\sum_{l=0}^{k}\int_{\mathbb{R}} g\big(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\big)dx \leqslant C(\varepsilon)\int_{\mathbb{R}} g\big(\nabla_x^l u_x^\delta, \nabla_x^l u_x^\delta\big)dx, \quad t \in [0, T_{\varepsilon,\delta}].$$

This implies that there exists $T_\varepsilon > 0$ depending only on $\varepsilon \in (0,1]$ and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that $\{u^\delta\}_{\delta\in(0,1]}$ is bounded in $L^\infty(0,T_\varepsilon; H^k(\mathbb{R};TN))$. The standard compactness arguments shows the existence of solution to (7), (8). The uniqueness and the continuity in time of solutions can be proved by the same energy method. We omit the details. $\square$

4. Uniform energy estimates

In this section we shall obtain uniform energy estimates of $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$, and construct a time-local solution to (1), (2) by the standard compactness argument. More precisely we shall show that there exists $T > 0$ which is independent of ε such that $T_\varepsilon \geqslant T$ and that $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty\big(0,T; H^{k+1}(\mathbb{R};TN)\big)$. For this purpose, we need to overcome the loss of one derivative and introduce a gauge transform of sections of the pullback bundle $(u^\varepsilon)^{-1}TN$ of the form

$$\begin{aligned} V_k^\varepsilon &= \nabla_x^k u_x^\varepsilon + \frac{M}{4a}\Phi^\varepsilon(t,x)J(u^\varepsilon)\nabla_x^{k-1}u_x^\varepsilon, \\ \Phi^\varepsilon(t,x) &= \int_{-\infty}^{x} g\big(u_y^\varepsilon(t,y), u_y^\varepsilon(t,y)\big)dy, \end{aligned} \tag{40}$$

where M is a positive constant determined later. The second term of the right hand side of (40) corresponds to the pseudodifferential operator $\tilde{\Lambda}$ introduced in Section 2. We will obtain the uniform bounds of

$$\mathscr{N}(u^\varepsilon)^2 = \int_{\mathbb{R}}\left\{g(V_k^\varepsilon, V_k^\varepsilon) + \sum_{l=0}^{k-1} g\big(\nabla_x^l u_x^\varepsilon, \nabla_x^l u_x^\varepsilon\big)\right\}dx.$$

In view of the Sobolev embedding with $k \geqslant 4$, it is easy to see that $\{\mathscr{N}(u^\varepsilon)\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T)$ if and only if $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^k(\mathbb{R};TN))$. Hence we shall show that there exists $T > 0$ which is independent of $\varepsilon \in (0,1]$ such that $T_\varepsilon \geqslant T$ and that $\{\mathscr{N}(u^\varepsilon)\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T)$.

Proof of existence. Let $k \geqslant 4$. Firstly we compute the energy estimates for $\nabla_x^l u_x^\varepsilon$ with $l = 0,1,\ldots,k-1$. In the same way as the previous section we have for

$l = 0, 1, \dots, k$

$$\begin{aligned}\nabla_t \nabla_x^l u_x^\varepsilon &= \bigl(-\varepsilon + aJ(u^\varepsilon)\bigr)\nabla_x^4\bigl(\nabla_x^l u_x^\varepsilon\bigr) + J(u^\varepsilon)\nabla_x^2\bigl(\nabla_x^l u_x^\varepsilon\bigr) \\ &\quad + \nabla_x\Bigl\{R\Bigl(\bigl(-\varepsilon + aJ(u^\varepsilon)\bigr)\nabla_x\bigl(\nabla_x^l u_x^\varepsilon\bigr), u_x\Bigr)u_x^\varepsilon \\ &\quad + bg(u_x^\varepsilon, u_x^\varepsilon)J(u^\varepsilon)\nabla_x\bigl(\nabla_x^l u_x^\varepsilon\bigr) + cg\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\varepsilon\bigr), u_x^\varepsilon\Bigr)J(u^\varepsilon)u_x^\varepsilon\Bigr\} \\ &\quad + \mathcal{O}\Bigl(g\Bigl(\nabla_x\bigl(\nabla_x^l u_x^\varepsilon\bigr), \nabla_x\bigl(\nabla_x^l u_x^\varepsilon\bigr)\Bigr)^{\frac12} g\bigl(\nabla_x u_x^\varepsilon, \nabla_x u_x^\varepsilon\bigr)^{\frac12} g(u_x^\varepsilon, u_x^\varepsilon)^{\frac12}\Bigr) \\ &\quad + \mathcal{O}\left(\sum_{j=0}^{l} g\bigl(\nabla_x^j u_x^\varepsilon, \nabla_x^j u_x^\varepsilon\bigr)^{\frac12}\right). \end{aligned} \tag{41}$$

Applying the integration by parts to the second term of the right hand side of the above for $l = 0, 1, \dots, k-1$, we deduce that

$$\begin{aligned}&\frac{d}{dt}\sum_{l=0}^{k-1}\int_{\mathbb{R}} g\bigl(\nabla_x^l u_x^\varepsilon, \nabla_x^l u_x^\varepsilon\bigr)dx \\ &\leqslant -2\varepsilon\sum_{l=0}^{k-1}\int_{\mathbb{R}} g\bigl(\nabla_x^{l+2} u_x^\varepsilon, \nabla_x^{l+2} u_x^\varepsilon\bigr)dx + C_1\mathscr{N}(u^\varepsilon)^2,\end{aligned} \tag{42}$$

where C_1 is a positive constant which is independent of $\varepsilon \in (0, 1]$.

Secondly we consider the energy estimates for V_k^ε. For this purpose we shall obtain the partial differential equation satisfied by V_k. The principal part of V_k^ε is $\nabla_x^k u_x^\varepsilon$ and satisfies (41) with $l = k$. We shall obtain the equation for the lower order term of V_k^ε. We begin with

$$\begin{aligned}&\nabla_t\left(\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_x^{k-1}u_x^\varepsilon\right) \\ &= \frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_t\nabla_x^{k-1}u_x^\varepsilon + \left\{\frac{M}{2a}\int_{-\infty}^{x} g\bigl(\nabla_t u_x^\varepsilon, u_x^\varepsilon\bigr)dy\right\}\nabla_x^{k-1}u_x^\varepsilon.\end{aligned} \tag{43}$$

Substitute (41) into the right hand side of (43). The first term of it becomes

$$\begin{aligned}&\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_t\nabla_x^{k-1}u_x^\varepsilon \\ &= \frac{M}{4a}\Phi^\varepsilon(t,x)\Biggl[\bigl(-\varepsilon + aJ(u^\varepsilon)\bigr)\nabla_x^4\bigl(\nabla_x^{k-1}u_x^\varepsilon\bigr) + J(u^\varepsilon)\nabla_x^2\bigl(\nabla_x^{k-1}u_x^\varepsilon\bigr) \\ &\quad + \nabla_x\Bigl\{R\Bigl(\bigl(-\varepsilon + aJ(u^\varepsilon)\nabla_x^k u_x^\varepsilon, u_x\bigr)\Bigr)u_x^\varepsilon + bg(u_x^\varepsilon, u_x^\varepsilon)J(u^\varepsilon)\nabla_x^k u_x^\varepsilon \\ &\quad + cg\Bigl(\nabla_x^k u_x^\varepsilon, u_x^\varepsilon\Bigr)J(u^\varepsilon)u_x^\varepsilon\Bigr\} + \mathcal{O}\left(\sum_{l=0}^{k} g\bigl(\nabla_x^l u_x^\varepsilon, \nabla_x^l u_x^\varepsilon\bigr)^{\frac12}\right)\Biggr]\end{aligned}$$

$$
\begin{aligned}
&= \bigl(-\varepsilon + aJ(u^\varepsilon)\bigr)\nabla_x^4 \left\{\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_x^{k-1}u_x^\varepsilon\right\} J(u^\varepsilon)\nabla_x^2 \left\{\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_x^{k-1}u_x^\varepsilon\right\} \\
&\quad - \bigl(-\varepsilon + aJ(u^\varepsilon)\bigr)\sum_{j=1}^{4}\frac{4!}{j!(4-j)!}\left\{\frac{\partial^j}{\partial x^j}\Phi^\varepsilon(t,x)\right\}\frac{M}{4a}\nabla_x^{k+3-j}u_x^\varepsilon \\
&\quad - J(u^\varepsilon)\sum_{j=1}^{2}\frac{2!}{j!(2-j)!}\left\{\frac{\partial^j}{\partial x^j}\Phi^\varepsilon(t,x)\right\}\frac{M}{4a}\nabla_x^{k+1-j}u_x^\varepsilon \\
&\quad + \mathcal{O}\Bigl(g\bigl(\nabla_x^{k+1}u_x^\varepsilon, \nabla_x^{k+1}u_x^\varepsilon\bigr)^{\frac12} g(u_x^\varepsilon, u_x^\varepsilon)^{\frac12}\Bigr) \\
&\quad + \mathcal{O}\left(\sum_{l=0}^{k} g\bigl(\nabla_x^l u_x^\varepsilon, \nabla_x^l u_x^\varepsilon\bigr)^{\frac12}\right).
\end{aligned}
\tag{44}
$$

Since $\{\Phi^\varepsilon\}_x = g(u_x^\varepsilon, u_x^\varepsilon)$,

$$
\begin{aligned}
&\sum_{j=1}^{4}\frac{4!}{j!(4-j)!}\left\{\frac{\partial^j}{\partial x^j}\Phi^\varepsilon(t,x)\right\}\frac{M}{4a}\nabla_x^{k+3-j}u_x^\varepsilon \\
&= \frac{M}{a}\nabla_x\Bigl\{g(u_x^\varepsilon, u_x^\varepsilon)\nabla_x^{k+1}u_x^\varepsilon\Bigr\} \\
&\quad + \mathcal{O}\Bigl(g\bigl(\nabla_x^{k+1}u_x^\varepsilon, \nabla_x^{k+1}u_x^\varepsilon\bigr)^{\frac12} g(u_x^\varepsilon, u_x^\varepsilon)^{\frac12}\Bigr) + \mathcal{O}\left(\sum_{l=0}^{k} g\bigl(\nabla_x^l u_x^\varepsilon, \nabla_x^l u_x^\varepsilon\bigr)^{\frac12}\right).
\end{aligned}
$$

Substituting this into (44), we have

$$
\begin{aligned}
&\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_t\nabla_x^{k-1}u_x^\varepsilon \\
&= \bigl(-\varepsilon + aJ(u^\varepsilon)\bigr)\nabla_x^4 \left\{\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_x^{k-1}u_x^\varepsilon\right\} \\
&\quad + J(u^\varepsilon)\nabla_x^2 \left\{\frac{M}{4a}\Phi^\varepsilon(t,x)\nabla_x^{k-1}u_x^\varepsilon\right\} \\
&\quad - M\left(-\frac{\varepsilon}{a} + J(u^\varepsilon)\right)\nabla_x\Bigl\{g(u_x^\varepsilon, u_x^\varepsilon)\nabla_x^{k+1}u_x^\varepsilon\Bigr\} \\
&\quad + \mathcal{O}\Bigl(g\bigl(\nabla_x^{k+1}u_x^\varepsilon, \nabla_x^{k+1}u_x^\varepsilon\bigr)^{\frac12} g(u_x^\varepsilon, u_x^\varepsilon)^{\frac12}\Bigr) + \mathcal{O}\left(\sum_{l=0}^{k} g\bigl(\nabla_x^l u_x^\varepsilon, \nabla_x^l u_x^\varepsilon\bigr)^{\frac12}\right).
\end{aligned}
\tag{45}
$$

On the other hand,

$$
\begin{aligned}
\left\{\frac{M}{2a}\int_{-\infty}^{x} g\bigl(\nabla_t u_x^\varepsilon, u_x^\varepsilon\bigr)dy\right\}\nabla_x^{k-1} &= \left\{\frac{M}{2a}\int_{-\infty}^{x} g\bigl(-\varepsilon\nabla_x^4 u_x^\varepsilon + \cdots, u_x^\varepsilon\bigr)dy\right\}\nabla_x^{k-1} \\
&= \mathcal{O}\Bigl(\|u_x^\varepsilon\|_{H^4}^2\Bigr)\nabla_x^{k-1}.
\end{aligned}
\tag{46}
$$

Multiply (43) by $J(u^\varepsilon)$, and substitute (45) and (46) into it. We deduce

$$
\begin{aligned}
&\nabla_t\left(\frac{M}{4a}\Phi^\varepsilon(t,x)J(u^\varepsilon)\nabla_x^{k-1}u_x^\varepsilon\right)\\
&=\bigl(-\varepsilon+aJ(u^\varepsilon)\bigr)\nabla_x^4\left\{\frac{M}{4a}\Phi^\varepsilon(t,x)J(u^\varepsilon)\nabla_x^{k-1}u_x^\varepsilon\right\}\\
&\quad+J(u^\varepsilon)\nabla_x^2\left\{\frac{M}{4a}\Phi^\varepsilon(t,x)J(u^\varepsilon)\nabla_x^{k-1}u_x^\varepsilon\right\}\\
&\quad+M\left(1+\frac{\varepsilon}{a}J(u^\varepsilon)\right)\nabla_x\Bigl\{g(u_x^\varepsilon,u_x^\varepsilon)\nabla_x^{k+1}u_x^\varepsilon\Bigr\}\\
&\quad+\mathcal{O}\biggl(g\bigl(\nabla_x^{k+1}u_x^\varepsilon,\nabla_x^{k+1}u_x^\varepsilon\bigr)^{\frac12}g(u_x^\varepsilon,u_x^\varepsilon)^{\frac12}\biggr)+\mathcal{O}\left(\sum_{l=0}^{k}g\bigl(\nabla_x^l u_x^\varepsilon,\nabla_x^l u_x^\varepsilon\bigr)^{\frac12}\right). \qquad (47)
\end{aligned}
$$

Combining (41) with $l=k$ and (47), we obtain

$$
\begin{aligned}
\nabla_t V_k=&\bigl(-\varepsilon+aJ(u^\varepsilon)\bigr)\nabla_x^4V_k^\varepsilon+J(u^\varepsilon)\nabla_x^2V_k^\varepsilon\\
&+M\left(1+\frac{\varepsilon}{a}J(u^\varepsilon)\right)\nabla_x\Bigl\{g(u_x^\varepsilon,u_x^\varepsilon)\nabla_xV_k^\varepsilon\Bigr\}\\
&+\nabla_x\Bigl\{R\Bigl(\bigl(-\varepsilon+aJ(u^\varepsilon)\bigr)\nabla_x\bigl(\nabla_x^ku_x^\varepsilon\bigr),u_x\Bigr)u_x^\varepsilon\\
&+bg(u_x^\varepsilon,u_x^\varepsilon)J(u^\varepsilon)\nabla_x\bigl(\nabla_x^ku_x^\varepsilon\bigr)+cg\Bigl(\nabla_x\bigl(\nabla_x^ku_x^\varepsilon\bigr),u_x^\varepsilon\Bigr)J(u^\varepsilon)u_x^\varepsilon\Bigr\}\\
&+\mathcal{O}\biggl(g\bigl(\nabla_x^{k+1}u_x^\varepsilon,\nabla_x^{k+1}u_x^\varepsilon\bigr)^{\frac12}g(u_x^\varepsilon,u_x^\varepsilon)^{\frac12}\biggr)+\mathcal{O}\left(\sum_{l=0}^{k}g\bigl(\nabla_x^l u_x^\varepsilon,\nabla_x^l u_x^\varepsilon\bigr)^{\frac12}\right). \qquad (48)
\end{aligned}
$$

By using (48) and integration by parts, we deduce that there exists positive constants C_2 and C_3 which are independent of $\varepsilon\in(0,1]$ such that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}g(V_k^\varepsilon,V_k^\varepsilon)dx&=2\int_{\mathbb{R}}g\bigl(\nabla_tV_k^\varepsilon,V_k^\varepsilon\bigr)dx\\
&\leqslant-2\varepsilon\int_{\mathbb{R}}g\bigl(\nabla_x^2V_k^\varepsilon,\nabla_x^2V_k^\varepsilon\bigr)dx+C_3\mathscr{N}(u^\varepsilon)^2\\
&\quad-(2M-C_2)\int_{\mathbb{R}}g(u_x^\varepsilon,u_x^\varepsilon)g\bigl(\nabla_xV_k^\varepsilon,\nabla_xV_k^\varepsilon\bigr)dx. \qquad (49)
\end{aligned}
$$

Combining (42) and (49), we deduce that there exists a positive constant C_4 which is independent of $\varepsilon\in(0,1]$ such that

$$
\frac{d}{dt}\mathscr{N}(u^\varepsilon)\leqslant C_4\mathscr{N}(u^\varepsilon)\quad\text{and}\quad\mathscr{N}\bigl(u^\varepsilon(t)\bigr)\leqslant\mathscr{N}(u_0)e^{C_4t},\quad t\in[0,T_\varepsilon]
$$

provided that M is sufficiently large. This shows that there exists a positive time T such that $T\leqslant T_\varepsilon$ and that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^{k+1}(\mathbb{R};TN))$.

Thus by using the standard compactness argument, we can deduce that there exists a function $u \in L^\infty(0,T;H^{k+1}(\mathbb{R};TN))$ solving (1), (2). Moreover, the lower semi-continuity of the norm shows that

$$\mathscr{N}\bigl(u(t)\bigr) \leqslant \mathscr{N}(u_0)e^{C_4 t}, \quad t \in [0,T], \tag{50}$$

which will be used in the next section. □

5. Uniqueness and continuity in time

Finally in this section we prove the uniqueness of solution and recover the continuity of the unique solution in time.

Proof of uniqueness. Let $u, v \in C\bigl([0,T]\times\mathbb{R};N\bigr) \cap L^\infty\bigl(0,T;H^7(\mathbb{R};TN)\bigr)$ be solutions to (1), (2). Set

$$\begin{aligned}
&U = w(u), \quad \tilde{U} = dw(\tilde{u}), \quad \tilde{u} = \nabla_x u_x + \frac{M}{4a}\Phi(t,x)J(u)u_x, \\
&V = w(v), \quad \tilde{V} = dw(\tilde{v}), \quad \tilde{v} = \nabla_x v_x + \frac{M}{4a}\Phi(t,x)J(v)v_x, \\
\phi(t,x) &= \sum_{l=0}^{2}\Bigl\{g_u\bigl(\nabla_x^l u_x(t,x), \nabla_x^l u_x(t,x)\bigr) + g_v\bigl(\nabla_x^l v_x(t,x), \nabla_x^l v_x(t,x)\bigr)\Bigr\}, \\
\Phi(t,x) &= \int_{-\infty}^{x}\phi(t,y)dy.
\end{aligned}$$

It suffices to show that $U(t,x) = V(t,x)$ on $[0,T]\times\mathbb{R}$ for sufficiently small $T>0$. For this reason, we may assume that $U(t,x)$ and $V(t,x)$ are in the same local coordinate patch on $w(N)$ for each $(t,x) \in [0,T]\times\mathbb{R}$. Note that

$$U_t = dw(u_t), \quad U_x = dw(u_x), \quad V_t = dw(v_t), \quad V_x = dw(v_x).$$

We shall obtain the partial differential equations satisfied by $Z = U - V$, $Z_x = U_x - V_x$ and $\tilde{Z} = \tilde{U} - \tilde{V}$, and evaluate

$$D(t)^2 = \frac{1}{2}\int_{\mathbb{R}}\bigl\{|Z(t,x)|^2 + |Z_x(t,x)|^2 + |\tilde{Z}(t,x)|^2\bigr\}dx.$$

Note that $D(0) = 0$. In the same way as the uniform energy estimates in the previous section, the highest order derivative $\tilde{Z}$ gains extra smoothness to obtain the energy estimates. In what follows different positive constants are denoted by the same letter C.

First we compute the image of (1) by dw for the estimate of Z and Z_x. Let $\nu_j(U)$, ($j = 2n+1, \dots, d$) be local expression of an orthonormal basis of $T_U w(N)^\perp$. Let $P(U) : T_U\mathbb{R}^d \to T_U w(N)$ be the orthogonal projection. Set

$$\tilde{J}(U)\xi = dw\Bigl(J\bigl(w^{-1}(U)\bigr)dw^{-1}(\xi)\Bigr)$$

for $\xi \in T_U w(N)$. We shall obtain the expression of $dw\bigl(J(u)\nabla_x^3 u_x\bigr)$. By using $\nabla_x J(u) = 0$ and $dw(\nabla_x \cdots) = P(U)\{dw(\cdots)\}_x$, we deduce that

$$
\begin{aligned}
dw\bigl(J(u)\nabla_x^3 u_x\bigr) &= dw\Bigl(\nabla_x^2\bigl(J(u)\nabla_x u_x\bigr)\Bigr) \\
&= P(U)\Bigl[P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x\Bigr]_x \\
&= \Bigl[P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x\Bigr]_x \\
&\quad - \sum_{j=2n+1}^{d} \Bigl\langle \Bigl[P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x\Bigr]_x, \nu_j(U)\Bigr\rangle \nu_j(U). \qquad (51)
\end{aligned}
$$

Since $\langle P(U)\cdots, \nu_j(U)\rangle = 0$, $j = 2n+1, \dots, d$, we have

$$
\Bigl\langle \Bigl[P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x\Bigr]_x, \nu_j(U)\Bigr\rangle = -\Bigl\langle P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x, \frac{\partial \nu_j}{\partial U}(U)U_x\Bigr\rangle.
$$

Then, the equality (51) becomes

$$
\begin{aligned}
dw\bigl(J(u)\nabla_x^3 u_x\bigr) &= \Bigl[P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x\Bigr]_x \\
&\quad + \sum_{j=2n+1}^{d} \left\langle P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x, \frac{\partial \nu_j}{\partial U}(U)U_x \right\rangle \nu_j(U).
\end{aligned}
$$

In the same way, we obtain

$$
\begin{aligned}
dw\bigl(J(u)\nabla_x^3 u_x\bigr) &= dw\Bigl(\nabla_x\bigl(J(u)\nabla_x^2 u_x\bigr)\Bigr) \\
&= P(U)\Bigl[\tilde{J}(U)P(U)\bigl\{P(U)U_{xx}\bigr\}_x\Bigr]_x.
\end{aligned}
$$

It is relatively easy to compute the image of the second and the third terms of the right hand side of (1) by dw. Thus we obtain

$$
\begin{aligned}
U_t &= a\Bigl[P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x\Bigr]_x \\
&\quad + a\sum_{j=2n+1}^{d} \left\langle P(U)\bigl\{\tilde{J}(U)P(U)U_{xx}\bigr\}_x, \frac{\partial \nu_j}{\partial U}(U)U_x \right\rangle \nu_j(U) \\
&\quad + \bigl\{1 + b\langle U_x, U_x\rangle\bigr\}\tilde{J}(U)P(U)U_{xx} + c\bigl\langle P(U)U_{xx}, U_x\bigr\rangle \tilde{J}(U)U_x \qquad (52) \\
&= aP(U)\Bigl[\tilde{J}(U)P(U)\bigl\{P(U)U_{xx}\bigr\}_x\Bigr]_x \\
&\quad + \bigl\{1 + b\langle U_x, U_x\rangle\bigr\}\tilde{J}(U)P(U)U_{xx} + c\bigl\langle P(U)U_{xx}, U_x\bigr\rangle \tilde{J}(U)U_x. \qquad (53)
\end{aligned}
$$

Using (52), we have

$$
\begin{aligned}
Z_t &= a\Big[P(U)\big\{\tilde{J}(U)P(U)Z_{xx}\big\}_x\Big]_x \\
&\quad + a\Big[P(U)\big\{\tilde{J}(U)P(U)V_{xx}\big\}_x - P(V)\big\{\tilde{J}(V)P(V)V_{xx}\big\}_x\Big]_x \\
&\quad + a\sum_{j=2n+1}^{d}\left\langle P(U)\big\{\tilde{J}(U)P(U)Z_{xx}\big\}_x, \frac{\partial\nu_j}{\partial U}(U)U_x\right\rangle \nu_j(U) \\
&\quad + a\sum_{j=2n+1}^{d}\Bigg[\left\langle P(U)\big\{\tilde{J}(U)P(U)V_{xx}\big\}_x, \frac{\partial\nu_j}{\partial U}(U)U_x\right\rangle \nu_j(U) \\
&\quad - \left\langle P(V)\big\{\tilde{J}(V)P(V)V_{xx}\big\}_x, \frac{\partial\nu_j}{\partial V}(V)V_x\right\rangle \nu_j(V)\Bigg] \\
&\quad + \mathcal{O}\big(|Z_{xx}|+|Z_x|+|Z|\big) \\
&= a\Big[P(U)\big\{\tilde{J}(U)P(U)Z_{xx}\big\}_x\Big]_x + \mathcal{O}\big(|U_x||Z_{xxx}|+|Z_{xx}|+|Z_x|+|Z|\big).
\end{aligned} \tag{54}
$$

Similarly, using (53), we have

$$
Z_t = aP(U)\Big[\tilde{J}(U)P(U)\big\{P(U)Z_{xx}\big\}_x\Big]_x + \mathcal{O}\big(|Z_{xx}|+|Z_x|+|Z|\big). \tag{55}
$$

It follows that $Z \in C^1\big([0,T];L^2(\mathbb{R};\mathbb{R}^d)\big)$ from $Z_t \in C\big([0,T];L^2(\mathbb{R};\mathbb{R}^d)\big)$ and $Z(0)=0 \in L^2(\mathbb{R};\mathbb{R}^d)$. In particular, $Z(t) \in L^2(\mathbb{R};\mathbb{R}^d)$ for all $t \in [0,T]$ is guaranteed. By using (54) and integration by parts, we deduce

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}|Z|^2dx &= \int_{\mathbb{R}}\langle Z_t, Z\rangle dx \\
&= a\int_{\mathbb{R}}\left\langle \Big[P(U)\big\{\tilde{J}(U)P(U)Z_{xx}\big\}_x\Big]_x, Z\right\rangle dx \\
&\quad + \int_{\mathbb{R}}\left\langle \mathcal{O}\big(|Z_{xxx}|+|Z_{xx}|+|Z_x|+|Z|\big), Z\right\rangle dx \\
&= -a\int_{\mathbb{R}}\left\langle P(U)\big\{\tilde{J}(U)P(U)Z_{xx}\big\}_x, Z_x\right\rangle dx \\
&\quad + \mathcal{O}\left(\int_{\mathbb{R}}\big(|Z_{xx}|^2+|Z_x|^2+|Z|^2\big)dx\right) \\
&= a\int_{\mathbb{R}}\left\langle \tilde{J}(U)P(U)Z_{xx}, \big\{P(U)Z_x\big\}_x\right\rangle dx \\
&\quad + \mathcal{O}\left(\int_{\mathbb{R}}\big(|Z_{xx}|^2+|Z_x|^2+|Z|^2\big)dx\right) \\
&\leqslant CD(t)^2.
\end{aligned} \tag{56}
$$

By using (55), we deduce

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}|Z_x|^2dx &= \int_{\mathbb{R}}\langle Z_{xt}, Z_x\rangle dx \\
&= -\int_{\mathbb{R}}\langle Z_t, Z_{xx}\rangle dx \\
&= -a\int_{\mathbb{R}}\Bigl\langle P(U)\Bigl[\tilde{J}(U)P(U)\bigl\{P(U)Z_{xx}\bigr\}_x\Bigr]_x, Z_{xx}\Bigr\rangle dx + \mathcal{O}\bigl(D(t)^2\bigr) \\
&= a\int_{\mathbb{R}}\Bigl\langle \tilde{J}(U)P(U)\bigl\{P(U)Z_{xx}\bigr\}_x, \bigl\{P(U)Z_{xx}\bigr\}_x\Bigr\rangle dx + \mathcal{O}\bigl(D(t)^2\bigr) \\
&= \mathcal{O}\bigl(D(t)^2\bigr) \\
&\leqslant CD(t)^2.
\end{aligned} \tag{57}$$

Next we shall obtain the equation for $\tilde{Z}$ and evaluate it. The equation for $\tilde{u}$ is

$$\begin{aligned}
\nabla_t\tilde{u} = {} & aJ(u)\nabla_x^4\tilde{u} + J(u)\nabla_x^2\tilde{u} + M\nabla_x\bigl\{\phi(t,x)\nabla_x\tilde{u}\bigr\} \\
& + \nabla_x\Bigl[R\bigl(J(u)\nabla_x\tilde{u}, u_x\bigr)u_x + bg_u(u_x, u_x)J(u)\nabla_x\tilde{u} + cg_u(\nabla_x\tilde{u}, u_x)J(u)u_x\Bigr] \\
& + \mathcal{O}\bigl(|u_x||\nabla_x\tilde{u}||\tilde{u}| + |\tilde{u}| + |u_x| + |u|\bigr).
\end{aligned} \tag{58}$$

Set $\xi(u)$ as the right hand side of (1) for short. We have

$$dw\bigl(\xi(u)\bigr) = \mathcal{O}\Bigl(|\tilde{U}_{xx}| + |\tilde{U}_x| + |\tilde{U}| + |U_x|\Bigr).$$

We compute the image of (58) by dw. Since $\bigl\langle\tilde{U}, \nu_j(U)\bigr\rangle = 0$, the left hand side of (58) becomes

$$\begin{aligned}
dw(\nabla_t\tilde{u}) &= \tilde{U}_t - \sum_{j=2n+1}^{d}\bigl\langle\tilde{U}_t, \nu_j(U)\bigr\rangle\nu_j(U) \\
&= \tilde{U}_t + \sum_{j=2n+1}^{d}\Bigl\langle\tilde{U}, \frac{\partial\nu_j}{\partial U}(U)U_t\Bigr\rangle\nu_j(U) \\
&= \tilde{U}_t + \sum_{j=2n+1}^{d}\Bigl\langle\tilde{U}, \frac{\partial\nu_j}{\partial U}(U)dw\bigl(\xi(u)\bigr)\Bigr\rangle\nu_j(U) \\
&= \tilde{U}_t + \sum_{j=2n+1}^{d}\Bigl\langle\tilde{U}, \mathcal{O}\Bigl(|\tilde{U}_{xx}| + |\tilde{U}_x| + |\tilde{U}| + |U_x|\Bigr)\Bigr\rangle\nu_j(U).
\end{aligned} \tag{59}$$

In the same way as (53) together with (58) and (59), we deduce that

$$\begin{aligned}\tilde{U}_t = {} & aP(U)\Big[P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{U}_x\big)_x\big\}_x\Big]_x \\ & + P(U)\big\{\tilde{J}(U)P(U)\tilde{U}_x\big\}_x + M\big(\phi(t,x)\tilde{U}_x\big)_x \\ & + \sum_{j=2n+1}^{d}\Big\langle \tilde{U}, \mathcal{O}\Big(|\tilde{U}_{xx}| + |\tilde{U}_x| + |\tilde{U}| + |U_x|\Big)\Big\rangle \nu_j(U) \\ & + \mathcal{O}\Big(\big(\phi(t,x)\tilde{U}_x\big)_x\Big) + \mathcal{O}\Big(\phi(t,x)^{\frac{1}{2}}|\tilde{U}_x| + |\tilde{U}| + |U_x|\Big).\end{aligned}$$

Taking the difference between the equations for $\tilde{U}$ and $\tilde{V}$, we obtain

$$\begin{aligned}\tilde{Z}_t = {} & aP(U)\Big[P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x\Big]_x \\ & + P(U)\big\{\tilde{J}(U)P(U)\tilde{Z}_x\big\}_x + M\big(\phi(t,x)\tilde{Z}_x\big)_x + \sum_{j=2n+1}^{d}\Big\langle \tilde{U}, \mathcal{O}\big(|\tilde{Z}_{xx}|\big)\Big\rangle \nu_j(U) \\ & + \mathcal{O}\Big(\big(\phi(t,x)\tilde{Z}_x\big)_x\Big) + \mathcal{O}\Big(\phi(t,x)^{\frac{1}{2}}|\tilde{Z}_x| + |\tilde{Z}| + |Z_x| + |Z|\Big).\end{aligned}$$

Using this and integration by parts, we deduce that there exists a positive constant C_1 such that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}\langle \tilde{Z}, \tilde{Z}\rangle dx = {} & \int_{\mathbb{R}}\langle \tilde{Z}_t, \tilde{Z}\rangle dx \\ \leqslant {} & a\int_{\mathbb{R}}\Big\langle P(U)\Big[P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x\Big]_x, \tilde{Z}\Big\rangle dx \qquad (60) \\ & + \int_{\mathbb{R}}\Big\langle P(U)\big\{\tilde{J}(U)P(U)\tilde{Z}_x\big\}_x, \tilde{Z}\Big\rangle dx \qquad (61) \\ & + \sum_{j=2n+1}^{d}\int_{\mathbb{R}}\mathcal{O}\Big(\phi(t,x)^{\frac{1}{2}}|\tilde{Z}_{xx}|\Big)\langle \nu_j(U), \tilde{Z}\rangle dx \qquad (62) \\ & - (M - C_1)\int_{\mathbb{R}}\phi(t,x)|\tilde{Z}_x|^2 dx + CD(t)^2.\end{aligned}$$

We will evaluate (60), (61) and (62), respectively. First we remark that

$$\langle \nu_j(U), \tilde{Z}\rangle = \langle \nu_j(U), \tilde{U}-\tilde{V}\rangle = -\langle \nu_j(U), \tilde{V}\rangle = -\langle \nu_j(U)-\nu_j(V), \tilde{V}\rangle = \mathcal{O}\big(\phi(t,x)^{\frac{1}{2}}Z\big) \qquad (63)$$

since $\big\langle \nu_j(U), \tilde{U}\big\rangle = 0$ and $\big\langle \nu_j(V), \tilde{V}\big\rangle = 0$. Applying (63) and integration by parts to (62), we have

$$(62) \leqslant CD(t)^2. \qquad (64)$$

Here we note that

$$\begin{aligned}& \Big\langle \Big[P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x\Big]_x, \nu_j(U)\Big\rangle \\ & = -\Big\langle P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x, \frac{\partial \nu_j}{\partial U}(U)U_x\Big\rangle\end{aligned} \qquad (65)$$

since $\Big\langle P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x, \nu_j(U)\Big\rangle = 0$. Applying integration by parts, (63) and (65) to (60), we have

$$\begin{aligned}
(60) &= a\int_{\mathbb{R}}\Big\langle\Big[P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x\Big]_x, \tilde{Z}\Big\rangle dx \\
&\quad - a\sum_{j=2n+1}^{d}\int_{\mathbb{R}}\Big\langle\Big[P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x\Big]_x, \nu_j(U)\Big\rangle\langle\nu_j(U), \tilde{Z}\rangle dx \\
&= -a\int_{\mathbb{R}}\Big\langle\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x, P(U)\tilde{Z}_x\Big\rangle dx \\
&\quad + a\sum_{j=2n+1}^{d}\int_{\mathbb{R}}\Big\langle P(U)\big\{\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x\big\}_x, \frac{\partial\nu_j}{\partial U}(U)U_x\Big\rangle\langle\nu_j(U), \tilde{Z}\rangle dx \\
&= a\int_{\mathbb{R}}\Big\langle\tilde{J}(U)P(U)\big(P(U)\tilde{Z}_x\big)_x, \big(P(U)\tilde{Z}_x\big)_x\Big\rangle dx \\
&\quad + \int_{\mathbb{R}}\mathcal{O}\Big(\phi(t,x)^{\frac{1}{2}}\big(|\tilde{Z}_{xxx}| + |\tilde{Z}_{xx}| + |\tilde{Z}_x|\big)\Big)\mathcal{O}\Big(\phi(t,x)^{\frac{1}{2}}Z\Big)dx \\
&= \int_{\mathbb{R}}\mathcal{O}\Big(\phi(t,x)|\tilde{Z}_x|^2 + |\tilde{Z}|^2 + |Z_x|^2 + |Z|^2\Big)dx \\
&\leqslant C_2\int_{\mathbb{R}}\phi(t,x)|\tilde{Z}_x|^2 dx + CD(t)^2. \qquad (66)
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
(61) &= -\int_{\mathbb{R}}\langle\tilde{J}(U)P(U)\tilde{Z}_x, P(U)\tilde{Z}_x\rangle - \int_{\mathbb{R}}\Big\langle\tilde{J}(U)P(U)\tilde{Z}_x, \frac{\partial P}{\partial U}(U)U_x\tilde{Z}\Big\rangle \\
&= \int_{\mathbb{R}}\mathcal{O}\Big(\phi(t,x)^{\frac{1}{2}}|\tilde{Z}_x||\tilde{Z}|\Big)dx \\
&\leqslant C_3\int_{\mathbb{R}}\phi(t,x)|\tilde{Z}_x|^2 dx + CD(t)^2. \qquad (67)
\end{aligned}$$

Combining (64), (66) and (67), we obtain

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}|\tilde{Z}|^2 dx \leqslant -(M - C_1 - C_2 - C_3)\int_{\mathbb{R}}\phi(t,x)|\tilde{Z}_x|^2 + CD(t)^2 \leqslant CD(t)^2 \quad (68)$$

provided that M is sufficiently large. Combining (56), (57) and (68), we have

$$\frac{d}{dt}D(t)^2 \leqslant CD(t)^2, \quad D(0)^2 = 0,$$

which implies that $D(t)^2 \equiv 0$. This completes the proof of the uniqueness of solutions. □

Finally, we shall prove the continuity of the unique solution in time.

Proof of continuity in time. Let k be an integer not smaller than six, and let u be a unique solution to the initial value problem satisfying $u \in L^\infty(0,T;H^{k+1}(\mathbb{R};TN))$. By using the equation (1), we deduce that $u_t \in L^\infty(0,T;H^{k-3}(\mathbb{R};TN))$ and $u \in C([0,T];H^{k-3}(\mathbb{R};TN))$. Moreover, by using the interpolation technique, we can deduce that $u \in C([0,T];H^k(\mathbb{R};TN))$ and that u is H^{k+1}-valued continuous weakly. Set

$$\begin{aligned} V_k &= \nabla_x^k u_x + \frac{M}{4a}\Phi(t,x)J(u)\nabla_x^{k-1}u_x, \\ \Phi(t,x) &= \int_{-\infty}^x g\bigl(u_y(t,y),u_y(t,y)\bigr)dy, \\ \mathscr{N}(u)^2 &= \int_{\mathbb{R}}\left\{g(V_k,V_k)+\sum_{l=0}^{k-1}g\bigl(\nabla_x^l u_x,\nabla_x^l u_x\bigr)\right\}dx, \end{aligned}$$

where M is a positive constant determined in Section 4. It suffices to show that V_k is $L^2(\mathbb{R};TN)$-valued continuous at $t=0$. It is easy to see that V_k is $L^2(\mathbb{R};TN)$-valued continuous weakly and that

$$\int_{\mathbb{R}} g(V_k(0),V_k(0))dx \leqslant \liminf_{t\downarrow 0}\int_{\mathbb{R}} g(V_k(t),V_k(t))dx.$$

By using (50) and $u \in C([0,T];H^k(\mathbb{R};TN))$, we deduce

$$\limsup_{t\downarrow 0}\int_{\mathbb{R}} g(V_k(t),V_k(t))dx \leqslant \int_{\mathbb{R}} g(V_k(0),V_k(0))dx.$$

Hence

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} g(V_k(t),V_k(t))dx = \int_{\mathbb{R}} g(V_k(0),V_k(0))dx.$$

Combining this and weak continuity, we deduce that V_k is $L^2(\mathbb{R};TN)$-valued continuous at $t=0$. □

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