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Variational-Hemivariational Approach to a Quasistatic Viscoelastic Problem with Normal Compliance, Friction and Material Damage

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Abstract. This work studies a model for quasistatic frictional contact between a viscoelastic body and a reactive foundation. The constitutive law is assumed to be nonlinear and contains damage effects modeled by a parabolic differential inclusion. Contact is described by the normal compliance condition and a subdifferential frictional condition. A variational-hemivariational formulation of the problem is provided and the existence and uniqueness of its solution is proved. The proof is based on a surjectivity result for pseudomonotone coercive operators and a fixed point argument.

Keywords. Quasistatic contact, nonlinear viscoelastic material, normal compliance, subdifferential friction condition, damage, variational-hemivariational inequality

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1. Introduction

We study a mathematical model for the process of quasistatic evolution in which a viscoelastic body or component comes in frictional contact with a reactive foundation, when material damage may develop. It extends the model that has been developed and analyzed in [18] by replacing the Coulomb frictional contact condition with a more general subdifferential condition. Thus, it may be

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applied to a wider range of friction conditions. We establish, under reasonably general assumptions on the problem data, the existence of the unique variational solution of the model. The novelty here is that the boundary conditions are multivalued of the general subdifferential type.

Recent models for mechanical damage derived from thermomechanical considerations can be found in [15, 27] (see also [8] for additional information and references). Material damage refers to the phenomena in a solid body in which microcracks and microcavities open and grow as a result of the internal strains and stresses. This leads to the gradual or possibly rapid decrease in the load carrying capacity of the body, leading eventually to the breaking of the system. The concept of material damage is closely related to that of fatigue, but is more general. Indeed, fatigue in a mechanical system is associated with many cycles of loading and unloading, or forcing cycles, while damage includes it, but also allows for rapid deterioration of the mechanical integrity of the system. Recent mathematical results for problems with material damage can be found in [9, 17, 20–22, 26, 29] and the many references therein. Mathematical analysis of one-dimensional problems with damage appeared in [1, 2, 11, 16, 19]. Modeling and analysis of contact problems with adhesion and friction can be found in [5, 6, 8]. For some other contact effects we refer to [3, 7, 10, 13, 25, 28].

Models and mathematical results on various aspects of contact can be found in the monographs [15, 23, 27, 29] and the many references therein.

The model we analyze consists of a quasistatic system that is nonlinear for the mechanical behavior of the body, a parabolic inclusion for the development of material damage, the normal compliance contact condition, and a general subdifferential condition of friction that includes, as a special case, Coulomb's law of dry friction. We establish the existence and uniqueness of the weak or variational solution for the model. The general idea of the proof comes from [18], where fixed point arguments for operators that satisfy certain integral constraints were used. Since in this work the boundary condition is multivalued, we use in addition a surjectivity result for pseudomonotone operators.

The rest of the paper is structured as follows. The following section recalls various mathematical notions and tools needed later. The model and the mathematical problem are introduced in Section 3. Its variational formulation is described in Section 4, where the assumptions on the problem data are listed. The proofs of the existence of the unique solution to the system of two coupled inequalities, the variational-hemivariational inequality for displacement and the variational inequality for material damage, can be found in Section 5, and summarized in Theorem 5.1. The proof is conducted in steps in which auxiliary problems are introduced and solved by applying various fixed point arguments.

2. Preliminaries

In this section we provide various mathematical definitions, notation, and results used in the paper.

If X is a reflexive Banach space, we denote by X^* its topological dual and $\langle \cdot, \cdot \rangle_{X^* \times X}$ denotes the duality pairing of X and X^*. A mapping $A \colon X \to X^*$ is called *bounded* if A maps bounded sets of X into bounded sets of X^*. It is called *monotone* if $\langle Au - Az, u - z \rangle_{X^* \times X} \geqslant 0$ for all u, $z \in X$. It is *maximal monotone* if it is monotone, and if $z \in X$ and $w \in X^*$ satisfy $\langle Au - w, u - z \rangle_{X^* \times X} \geqslant 0$ for all $u \in X$, then we have $w = Az$. Moreover, the operator $A \colon X \to X^*$ is called to be *pseudomonotone* if it is bounded and if $u_n \to u$ weakly in X and $\limsup \langle Au_n, u_n - u \rangle_{X^* \times X} \leqslant 0$ imply

$$\langle Au, u - v \rangle_{X^* \times X} \leqslant \liminf \langle Au_n, u_n - v \rangle_{X^* \times X}, \quad \text{for all } v \in X.$$

An equivalent definition of pseudomonotonicity is: A mapping $A \colon X \to X^*$ is pseudomonotone if it is bounded and if $u_n \to u$ weakly in X and

$$\limsup \langle Au_n, u_n - u \rangle_{X^* \times X} \leqslant 0,$$

imply $\lim \langle Au_n, u_n - u \rangle_{X^* \times X} = 0$ and $Au_n \to Au$ weakly in X^*.

Definition 2.1. Let X be a reflexive Banach space and $A \colon D(A) \subset X \to 2^{X^*}$ be a multivalued operator. We say that the operator A is

(a) *monotone* if

$$\langle u^* - v^*, u - v \rangle_{X^* \times X} \geqslant 0 \quad \text{for all } u, v \in D(A),\ u^* \in Au,\ v^* \in Av.$$

(b) *maximal monotone* if it is monotone and it has a maximal graph (in the sense of inclusion among all monotone operators), i.e., if

$$\langle u^* - w, u - v \rangle_{X^* \times X} \geqslant 0 \quad \text{for all } u \in D(A),\ u^* \in Au,$$

implies that $v \in D(A)$ and $w \in Av$.

Definition 2.2. Let Y, Z be two Hausdorff topological spaces and let $F \colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ be a multifunction. We say that F is upper semicontinuous, if for any closed set $C \subseteq Z$, the set $F^-(C) = \{y \in Y :\ F(y) \cap C \neq \emptyset\}$ is closed in Y.

Definition 2.3. Let X be a reflexive Banach space. We say that a multivalued operator $A \colon X \to 2^{X^*}$ is *pseudomonotone* if

(a) for every $u \in X$, the set $Au \subset X^*$ is nonempty, closed and convex;
(b) A is upper semicontinuous from each finite dimensional subspace of X into X^* endowed with its weak topology;
(c) for every sequences $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \to u$ weakly in X, $u_n^* \in Au_n$ for all $n \geqslant 1$, and $\limsup \langle u_n^*, u_n - u \rangle_{X^* \times X} \leqslant 0$, we have that for every $v \in X$, there exists $u^*(v) \in Au$ such that

$$\langle u^*(v), u - v \rangle_{X^* \times X} \leqslant \liminf \langle u_n^*, u_n - v \rangle_{X^* \times X}.$$

The following surjectivity result is due to Naniewicz-Panagiotopoulos [24, Theorem 2.12] cast in a form used in the proof of the existence theorem.

Theorem 2.4. *If X is a real reflexive Banach space, $\overline{T}\colon X \to 2^{X^*}$ is a maximal monotone operator, $T\colon X \to 2^{X^*}$ is a multivalued bounded pseudomonotone operator which is coercive in the following sense*

$$\langle u^*, u\rangle_{X^*\times X} \;\geqslant\; c(\|u\|_X)\|u\|_X \quad \text{for all } u \in D(T),\ u^* \in T(u),$$

where $c\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a function such that $c(r) \longrightarrow +\infty$ as $r \to +\infty$, then $\overline{T} + T$ is surjective.

We turn to some basic tools from convex analysis and nonsmooth analysis, cf. Clarke [12].

Definition 2.5. Let X be a Banach space and let $\varphi\colon X \to \mathbb{R}$ be a locally Lipschitz function. The *generalized directional derivative, in the sense of Clarke,* of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{\varphi(y+\lambda v) - \varphi(y)}{\lambda}$$

and the generalized gradient (subdifferential) of φ at x, denoted by $\partial\varphi(x)$, is a subset of a dual space X^* given by

$$\partial\varphi(x) = \{\, \zeta \in X^* \mid \varphi^0(x; v) \geqslant \langle \zeta, v\rangle_{X^*\times X} \text{ for all } v \in X \,\}.$$

Proposition 2.6. *If $\varphi\colon X \to \mathbb{R}$ is a locally Lipschitz function on a Banach space X, then for every $x \in X$ the generalized gradient $\partial\varphi(x)$ is a nonempty, convex, and weak* compact subset of X^*, and the graph of the generalized gradient $\partial\varphi$ is closed in $X\times$ $(w^*$–$X^*)$-topology, i.e., if $\{x_n\} \subset X$ and $\{\zeta_n\} \subset X^*$ are sequences such that $\zeta_n \in \partial\varphi(x_n)$ and $x_n \to x$ in X, $\zeta_n \to \zeta$ weak* in X^*, then $\zeta \in \partial\varphi(x)$.*

Given a convex, lower-semicontinuous (l.s.c.) function $\varphi\colon X \to (-\infty, +\infty]$ on a Banach space X, we recall that φ is proper if it is not identically $+\infty$. The effective domain of φ is denoted by $\operatorname{dom}\varphi = \{x \in X \mid \varphi(x) < +\infty\}$.

Definition 2.7. Let X be a Banach space and let $\varphi\colon X \to (-\infty, +\infty]$ be a proper, l.s.c. and convex function. The mapping $\partial\varphi\colon X \to 2^{X^*}$ defined by

$$\partial\varphi(x) = \{x^* \in X^* \mid \langle x^*, v - x\rangle_{X^*\times X} \leqslant \varphi(v) - \varphi(x) \text{ for all } v \in X\}$$

for $x \in X$ with $\varphi(x) < +\infty$ and by $\partial\varphi(x) = \emptyset$ for $x \in X$ with $\varphi(x) = +\infty$, is called the *subdifferential* of φ. An element $x^* \in \partial\varphi(x)$ (if any) is called a *subgradient* of φ at x.

The following important result is due to Denkowski-Migórski-Papageorgiou [14, Theorem 6.3.19].

Theorem 2.8. *Let X be a real Banach space and let φ be a proper, l.s.c. and convex function on X. Then $\partial\varphi$ is a maximal monotone operator from X to X^*.*

The following fixed point argument will be needed below.

Theorem 2.9. *If X is a real Banach space and $\Lambda\colon C([0,T];X) \longrightarrow C([0,T];X)$ is an operator for which there exist $k \in \mathbb{N}_+$ and $c > 0$ such that for all $t \in [0,T]$, we have*

$$\|(\Lambda u)(t) - (\Lambda v)(t)\|_X^k \leqslant c \int_0^t \|u(s) - v(s)\|_X^k ds \quad \text{for all } u, v \in C([0,T];X),$$

then Λ has a unique fixed point in $C([0,T];X)$.

The proof of this theorem is similar to that in Migórski-Ochal-Sofonea [23, pp. 107–108].

3. The model

We consider a viscoelastic body that occupies a domain $\Omega \subseteq \mathbb{R}^d$ ($d = 2, 3$ in applications) with a Lipschitz continuous boundary Γ. We assume that Γ has outward normal $\boldsymbol{\nu}$ and consists of three sets $\overline{\Gamma}_D$, $\overline{\Gamma}_N$ and $\overline{\Gamma}_C$ such that Γ_D, Γ_N and Γ_C are pairwise disjoint, and meas$(\Gamma_D) > 0$. The body is held fixed on Γ_D and surface tractions of density $\boldsymbol{f}_N$ act on Γ_N. The potential contact surface is Γ_C and the gap g between it and the deformable foundation is measured along the normal $\boldsymbol{\nu}$.

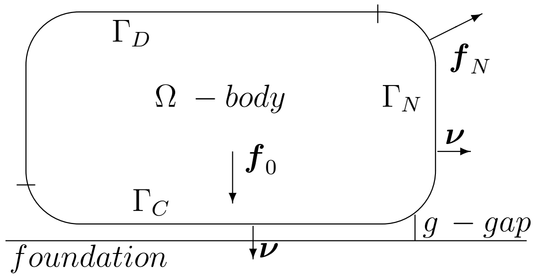

Figure 1: Γ_C is the contact surface

The volume forces $\boldsymbol{f}_0$ that act in Ω and the tractions $\boldsymbol{f}_N$ are assumed to vary slowly in time so the process is quasistatic, as the accelerations in the system are negligible. We describe the contact process with the normal compliance condition and a subdifferential friction condition. We use a viscoelastic constitutive law with damage effects, which are described by the damage function β, the growth of which is governed by a parabolic differential inclusion and satisfies a homogeneous Neumann boundary condition.

We denote by $[0,T]$ the time interval of interest, with $T>0$, and use the notation

$$Q = \Omega \times (0,T).$$

The classical model for the process is as follows.

Problem P: Find a displacement field $\boldsymbol{u}\colon Q \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon Q \longrightarrow \mathbb{S}^d$ and a damage function $\beta\colon Q \longrightarrow \mathbb{R}$, such that for all $t \in (0,T)$,

$$\boldsymbol{\sigma}(t) \;=\; \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))) + \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\beta(t)) \quad \text{in } \Omega, \tag{1}$$

$$\dot{\beta}(t) - \kappa\Delta\beta(t) + \partial\psi_{[0,1]}(\beta(t)) \;\ni\; \phi(t,\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\beta(t)) \quad \text{in } \Omega, \tag{2}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \;=\; \boldsymbol{0} \quad \text{in } \Omega, \tag{3}$$

$$\frac{\partial\beta(t)}{\partial\boldsymbol{\nu}} \;=\; 0 \quad \text{on } \Gamma, \tag{4}$$

$$\boldsymbol{u}(t) \;=\; \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} \;=\; \boldsymbol{f}_N(t) \quad \text{on } \Gamma_N, \tag{6}$$

$$-\sigma_\nu(t) \;=\; k_\nu p_\nu(u_\nu(t) - g) \quad \text{on } \Gamma_C, \tag{7}$$

$$-\boldsymbol{\sigma}_\tau(t) \;\in\; k_\tau \partial j_\tau(\dot{\boldsymbol{u}}_\tau(t)) \quad \text{on } \Gamma_C, \tag{8}$$

$$\boldsymbol{u}(0) \;=\; \boldsymbol{u}_0, \;\; \beta(0) \;=\; \beta_0 \quad \text{in } \Omega. \tag{9}$$

Here, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$, $\frac{\partial\beta}{\partial\nu}$ is the normal derivative on Γ, Div denotes the divergence operator, σ_ν and $\boldsymbol{\sigma}_\tau$ stand for the normal and tangential traces of $\boldsymbol{\sigma}$, respectively, u_ν and $\dot{\boldsymbol{u}}_\tau$ are the normal and tangential components of displacement $\boldsymbol{u}$ and velocity $\dot{\boldsymbol{u}}$, respectively. By $\partial\psi_{[0,1]}$ we denote the convex subdifferential of ψ, the indicator function of $[0,1]$ and by ∂j_τ we mean the Clarke subdifferential of j_τ with respect to the last variable.

The viscoelastic constitutive law (1) depends linearly on the velocity and is nonlinear in the elastic part, which includes the effects of material damage. The evolution of the damage variable β is described by the parabolic inclusion (2) with the damage source function ϕ and boundary condition (4). Since we assume that the process is quasistatic, we use the steady state equation (3) to describe the evolutions of the mechanical state of the body. Equations (5) and (6) represent the displacement and traction boundary conditions, respectively. Relation (7) is the so-called *normal compliance condition* and the inclusion (8) is the subdifferential frictional condition, which generalizes Coulomb's dry friction. The initial conditions for displacement and damage are given by (9).

The damage process, as was noted above, is associated with the opening and growth of microcracks and microcavities in the material as a result of the internal strains. As these grow the strength and load carrying capacity of the system decrease. The *damage function* β is chosen as

$$\beta = \frac{Y_{eff}}{Y_0},$$

where Y_0 is the Young modulus of the original material and Y_{eff} is the current effective modulus. Thus, β measures the fractional decrease in the mechanical stiffness of the system, so it satisfies the condition $0 \leqslant \beta \leqslant 1$. When $\beta = 1$ the material is undamaged, i.e., in its original form, when $\beta = 0$ the material is completely damaged and cannot support any load, and when $0 < \beta < 1$ there is fractional reduction in strength. The term $\partial\psi_{[0,1]}(\beta(t))$ in (2) guarantees that $0 \leq \beta \leq 1$, thus preserving the interpretation of β as a fraction. We assume that the damage source function depends on the indicated variables and examples can be found in [15,18,27]. The derivation of (2) from thermodynamic considerations can be found in [15, 27].

4. Variational-hemivariational formulation

We first introduce the notation and concepts needed in the sequel, describe the assumptions imposed on the problem data as well as the variational formulation of the problem. In what follows the indices i and j run between 1 and d and the summation convention over repeated indices is used. Also, an index following a comma indicates a partial derivative.

In $\mathbb{R}^d$ we use the inner product $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$, and the norm $\|\boldsymbol{v}\|_{\mathbb{R}^d} = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$, for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$. By $\mathbb{S}^d$ we denote the space of second order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d, with the inner product $\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\boldsymbol{\tau}_{ij}$ and the norm $\|\boldsymbol{\tau}\|_{\mathbb{S}^d} = \sqrt{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}$, for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$.

Next, we introduce the following spaces

$$\begin{aligned}
H &= \big\{\boldsymbol{v} = (v_i) \mid v_i \in L^2(\Omega)\big\} = L^2(\Omega;\mathbb{R}^d),\\
H_1 &= \big\{\boldsymbol{v} = (v_i) \mid v_i \in H^1(\Omega)\big\} = H^1(\Omega;\mathbb{R}^d),\\
\mathscr{H} &= \big\{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\big\} = L^2(\Omega;\mathbb{S}^d),\\
\mathscr{H}_1 &= \big\{\boldsymbol{\tau} \in \mathscr{H} \mid (\tau_{ij,j}) \in H\big\},
\end{aligned}$$

which are real Hilbert spaces endowed with the inner products

$$\begin{aligned}\langle \boldsymbol{u}, \boldsymbol{v}\rangle_H &= \int_\Omega u_i v_i \, dx, \quad \langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,\\ \langle \boldsymbol{u}, \boldsymbol{v}\rangle_{H_1} &= \langle \boldsymbol{u}, \boldsymbol{v}\rangle_H + \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}},\\ \langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_{\mathscr{H}_1} &= \langle \boldsymbol{\sigma}, \boldsymbol{\tau}\rangle_{\mathscr{H}} + \langle \mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau}\rangle_H,\end{aligned}$$

and with the respective norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathscr{H}}$, $\|\cdot\|_{\mathscr{H}_1}$. Here $\boldsymbol{\varepsilon}\colon H_1 \longrightarrow \mathscr{H}$ and $\mathrm{Div}\colon \mathscr{H}_1 \longrightarrow H$ are the *deformation* and *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

For an element $\boldsymbol{v} \in H_1$ we denote by $\boldsymbol{v}$ its trace on Γ and by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$ its *normal* and *tangential* components on the boundary. For an element $\boldsymbol{\sigma} \in \mathscr{H}_1$, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the *normal* and *tangential* traces of $\boldsymbol{\sigma}$. If $\boldsymbol{\sigma}$ is smooth then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}.$$

The following Green formula holds

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}} + \langle \mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{v}\rangle_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\, d\Gamma \quad \text{for all } \boldsymbol{v} \in H_1 \text{ and } \boldsymbol{\sigma} \in \mathscr{H}_1. \tag{10}$$

Let V be the closed subspace of H_1, given by

$$V = \big\{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_D\big\}.$$

Since $\mathrm{meas}\,(\Gamma_D) > 0$ and Γ is Lipschitz continuous, the Korn inequality holds

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathscr{H}} \geqslant c\|\boldsymbol{v}\|_{H_1} \quad \text{for all } \boldsymbol{v} \in V, \tag{11}$$

where here and below c represents a positive constant which may change from line to line and may depend on the data. We define the inner product on V by

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_V = \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V. \tag{12}$$

It follows from (11) and (12) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Moreover, we denote by $\gamma\colon V \longrightarrow L^2(\Gamma;\mathbb{R}^d)$ the trace operator, by $\|\gamma\|$ its norm in $\mathscr{L}(V; L^2(\Gamma;\mathbb{R}^d))$ and by $\gamma^*\colon L^2(\Gamma;\mathbb{R}^d) \longrightarrow V^*$ the adjoint operator to γ.

Note that the assumption on the Lipschitz continuity of the boundary Γ ensures that the outward normal vector $\boldsymbol{\nu}$ is defined a.e. on Γ, the normal and tangential components of various functions make sense, and the Korn inequality holds.

Finally for a real Banach space $(X, \|\cdot\|_X)$ we use the standard notation for Bochner-Lebesgue spaces $L^p(0,T;X)$ (with $1 \leqslant p \leqslant +\infty$), Bochner-Sobolev spaces $H^k(0,T;X)$ (with $k \in \mathbb{N}$), space of vector-valued continuous functions $C([0,T];X)$ and space of vector-valued continuously differentiable functions $C^1([0,T];X)$. Moreover, if X_1 and X_2 are two real Hilbert spaces then $X_1 \times X_2$ denotes the product space endowed with the canonical inner product $\langle\cdot,\cdot\rangle_{X_1\times X_2}$ and norm $\|\cdot\|_{X_1\times X_2}$.

We assume the following on the problem data.

$\underline{H(\mathcal{A})}$: The *viscosity operator* $\mathcal{A}\colon \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies:

(a) $\mathcal{A}(\cdot,\boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$;
(b) $\mathcal{A}(\boldsymbol{x},\cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $\boldsymbol{x} \in \Omega$;
(c) there exist $a_0 \in L^2(\Omega)$, $a_0 \geqslant 0$ and $a_1 > 0$ such that for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$,
$$\|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon})\|_{\mathbb{S}^d} \leqslant a_0(\boldsymbol{x}) + a_1\|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d};$$
(d) there exists $m_{\mathcal{A}} > 0$ such that for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$
$$(\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d};$$
(e) $\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{0}$ for a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\mathcal{G})}$: The *elasticity operator* $\mathcal{G}\colon \Omega \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{S}^d$ is such that

(a) $\mathcal{G}(\cdot,\boldsymbol{\varepsilon},\beta)$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\beta \in \mathbb{R}$;
(b) there exists $L_{\mathcal{G}} > 0$ such that
$$\|\mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}_1,\beta_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}_2,\beta_2)\|_{\mathbb{S}^d} \leqslant L_{\mathcal{G}}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} + |\beta_1 - \beta_2|)$$
for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, $\beta_1, \beta_2 \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Omega$;
(c) $\mathcal{G}(\boldsymbol{x},\boldsymbol{0},0) \in \mathscr{H}$ for a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\phi)}$: The *damage source function* $\phi\colon Q \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies:

(a) $\phi(\cdot,\cdot,\boldsymbol{\varepsilon},\beta)$ is measurable on Q for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\beta \in \mathbb{R}$;
(b) there exists $L_\phi > 0$ such that
$$|\phi(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1,\beta_1) - \phi(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2,\beta_2)| \leqslant L_\phi(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} + |\beta_1 - \beta_2|)$$
for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, $\beta_1, \beta_2 \in \mathbb{R}$ and a.e. $(\boldsymbol{x},t) \in Q$;
(c) $\phi(\boldsymbol{x},\cdot,\boldsymbol{\varepsilon},\beta)$ is continuous on $[0,T]$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\beta \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Omega$;
(d) $\phi(\cdot,\cdot,\boldsymbol{0},0) \in L^2(Q)$.

$\underline{H(p_\nu)}$: The *normal compliance function* $p_\nu\colon \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+$ is such that

(a) $p_\nu(\cdot, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$;

(b) there exists $L_\nu > 0$ such that for all $r_1, r_2 \in \mathbb{R}$ and a.e. $\boldsymbol{x} \in \Gamma_C$

$$|p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leqslant L_\nu |r_1 - r_2|;$$

(c) $p_\nu(\cdot, r) = 0$ for $r \leqslant 0$ on Γ_C.

$\underline{H(k_\nu)}$: The *surface contact stiffness coefficient* $k_\nu\colon \Gamma_C \longrightarrow \mathbb{R}_+$ is a measurable function such that $0 \leqslant k_\nu(\boldsymbol{x}) \leqslant \overline{k}_\nu$, for some $\overline{k}_\nu > 0$, a.e. $\boldsymbol{x} \in \Gamma_C$.

$\underline{H(j_\tau)}$: The *friction dissipation pseudopotential* $j_\tau\colon \Gamma_C \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies:

(a) $j_\tau(\cdot, \boldsymbol{\xi})$ is measurable on Γ_C for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and there exists $\boldsymbol{e} \in L^2(\Gamma_C; \mathbb{R}^d)$ such that $j_\tau(\cdot, \boldsymbol{e}(\cdot)) \in L^2(\Gamma_C)$;

(b) $j_\tau(\boldsymbol{x}, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x} \in \Gamma_C$;

(c) there exist $c_{0\tau}, c_{1\tau} > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and for a.e. $\boldsymbol{x} \in \Gamma_C$

$$\|\partial j_\tau(\boldsymbol{x}, \boldsymbol{\xi})\|_{\mathbb{R}^d} \leqslant c_{0\tau} + c_{1\tau} \|\boldsymbol{\xi}\|_{\mathbb{R}^d};$$

(d) there exists $c_{2\tau} \geqslant 0$ such that for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$ and for a.e. $\boldsymbol{x} \in \Gamma_C$

$$j_\tau^0(\boldsymbol{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j_\tau^0(\boldsymbol{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leqslant c_{2\tau} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathbb{R}^d}^2.$$

$\underline{H(k_\tau)}$: The *friction coefficient* $k_\tau\colon \Gamma_C \longrightarrow \mathbb{R}_+$ is a measurable function such that $0 \leqslant k_\tau(\boldsymbol{x}) \leqslant \overline{k}_\tau$ for some $\overline{k}_\tau > 0$ and for a.e. $\boldsymbol{x} \in \Gamma_C$.

The *volume force density, surface traction density, gap function, and initial functions*, respectively, are assumed to satisfy:

$\underline{H_0}$: (a) $\boldsymbol{f}_0 \in C([0,T]; H)$;

(b) $\boldsymbol{f}_N \in C([0,T]; L^2(\Gamma_N; \mathbb{R}^d))$;

(c) $g \in L^\infty(\Gamma_C)$, $g \geqslant 0$;

(d) $\boldsymbol{u}_0 \in V$;

(e) $\beta_0 \in H^1(\Omega)$ is such that $0 \leqslant \beta_0 \leqslant 1$ a.e. in Ω.

Finally, we have the following additional assumptions:

$\underline{H_1}$: (a) $c_{2\tau} \overline{k}_\tau \|\gamma\|^2 < m_{\mathcal{A}}$;

(b) at least one of the following two conditions holds:

(i) $m_{\mathcal{A}} > c_{1\tau} \overline{k}_\tau \sqrt{2} \|\gamma\|^2$;

(ii) there exists $d_\tau > 0$ such that $j_\tau^0(\boldsymbol{x}, \boldsymbol{\xi}; -\boldsymbol{\xi}) \leqslant d_\tau (1 + \|\boldsymbol{\xi}\|_{\mathbb{R}^d})$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Omega$.

Remark 4.1. Every convex function $j\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies $H(j_\tau)$(b) and (d) with $c_{2\tau} = 0$. Indeed, the convexity of j implies that $j^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) \leqslant j(\boldsymbol{\xi}_2) - j(\boldsymbol{\xi}_1)$ and $j^0(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leqslant j(\boldsymbol{\xi}_1) - j(\boldsymbol{\xi}_2)$, and adding these inequalities yields $j^0(\boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j^0(\boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leqslant 0$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$.

If the function $j\colon \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $j(r) = \int_0^r h(s)\, ds$, for $r \in \mathbb{R}$ with a continuous function $h\colon \mathbb{R} \longrightarrow \mathbb{R}$, then the condition $H(j_\tau)(d)$ is equivalent to each one of the following two conditions:

(i) $(h(s_1) - h(s_2))(s_2 - s_1) \leqslant c(s_1 - s_2)^2$ for all $s_1, s_2 \in \mathbb{R}$ and some $c > 0$;
(ii) the function $\mathbb{R} \ni s \longmapsto cs + h(s) \in \mathbb{R}$ is nondecreasing for some $c > 0$.

We now derive the variational formulation of Problem P. To that end, we consider the function for combined forces and tractions $\boldsymbol{f}$: $[0,T] \longrightarrow V^*$, given by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} = \langle \boldsymbol{f}_0(t), \boldsymbol{v}\rangle_H + \langle \boldsymbol{f}_N(t), \gamma\boldsymbol{v}\rangle_{L^2(\Gamma_N;\mathbb{R}^d)} \quad \text{for all } \boldsymbol{v}\in V,\ t\in[0,T], \tag{13}$$

and the set of admissible damage functions

$$\mathscr{K} = \big\{\zeta \in H^1(\Omega) :\ 0 \leqslant \zeta \leqslant 1 \text{ a.e. in } \Omega\big\}.$$

Assume that $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta\}$ are sufficiently smooth functions that solve (1)–(9), $\boldsymbol{v} \in V$, $\zeta \in \mathscr{K}$ and $t \in [0,T]$. First we use the steady state equation (3) and the Green formula (10) to obtain

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\rangle_{\mathscr{H}} = \langle \boldsymbol{f}_0(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\rangle_H + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, d\Gamma. \tag{14}$$

Taking into account the boundary condition (5), the fact that

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot\boldsymbol{v} = \sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t)\cdot\boldsymbol{v}_\tau,$$

the equality (7), and the definition of the Clarke subdifferential combined with (8), we obtain

$$\begin{aligned}
&\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, d\Gamma \\
&\geqslant \int_{\Gamma_N} \boldsymbol{f}_N(t)\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\, d\Gamma - \int_{\Gamma_C} k_\nu p_\nu(u_\nu(t) - g)(v_\nu - \dot{u}_\nu(t))\, d\Gamma \\
&\quad - \int_{\Gamma_C} k_\tau j_\tau^0(\dot{\boldsymbol{u}}_\tau(t); \boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau(t))\, d\Gamma.
\end{aligned} \tag{15}$$

Hence, using (15) and (13) in (14), we have

$$\begin{aligned}
&\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\rangle_{\mathscr{H}} \\
&+ \int_{\Gamma_C} k_\nu p_\nu(u_\nu(t) - g)(v_\nu - \dot{u}_\nu(t))\, d\Gamma + \int_{\Gamma_C} k_\tau j_\tau^0(\dot{\boldsymbol{u}}_\tau(t); \boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau(t))\, d\Gamma \\
&\geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\rangle_{V^*\times V}.
\end{aligned}$$

Next, using the definition of the subdifferential of the indicator function $\psi_{[0,1]}$ and integration by parts, we see that

$$0 \geqslant \langle \phi(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)) - \dot{\beta}(t),\ \zeta - \beta(t)\rangle_{L^2(\Omega)} - \kappa\langle \nabla\beta(t),\ \nabla\zeta - \nabla\beta(t)\rangle_{L^2(\Omega;\mathbb{R}^d)}.$$

Now collecting these relations and inequalities leads to the following *variational-hemivariational* formulation of Problem P.

Problem $\mathbf{P_V}$: Find $\boldsymbol{u} \in C^1([0,T];V)$, $\boldsymbol{\sigma} \in C([0,T];\mathscr{H}_1)$ and $\beta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))) + \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)) \quad \text{in } \Omega, \text{ for all } t \in [0,T] \tag{16}$$

$$\begin{aligned}&\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\rangle_{\mathscr{H}} + \int_{\Gamma_C} k_\nu p_\nu(u_\nu(t) - g)(v_\nu - \dot{u}_\nu(t))\, d\Gamma \\ &+ \int_{\Gamma_C} k_\tau j_\tau^0(\dot{\boldsymbol{u}}_\tau(t); \boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau(t))\, d\Gamma \\ &\geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T]\end{aligned} \tag{17}$$

$$\begin{aligned}&\langle \dot{\beta}(t), \zeta - \beta(t)\rangle_{L^2(\Omega)} + \kappa\langle \nabla\beta(t), \nabla\zeta - \nabla\beta(t)\rangle_{L^2(\Omega;\mathbb{R}^d)} \\ &\geqslant \langle \phi(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)),\ \zeta - \beta(t)\rangle_{L^2(\Omega)} \quad \text{for all } \zeta \in \mathscr{K} \text{ and all } t \in [0,T]\end{aligned} \tag{18}$$

$$\beta(t) \in \mathscr{K} \quad \text{for all } t \in [0,T] \tag{19}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0 \quad \text{in } \Omega. \tag{20}$$

5. Existence and uniqueness results

We now state and prove the main result of this work.

Theorem 5.1. *If hypotheses $H(\mathcal{A})$, $H(\mathcal{G})$, $H(\phi)$, $H(p_\nu)$, $H(k_\nu)$, $H(j_\tau)$, $H(k_\tau)$, H_0 and H_1 hold, then Problem P_V has a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$.*

The proof is done in steps in which auxiliary problems are analyzed. In the first step we assume that the elastic part of the stress $\boldsymbol{\eta} \in C([0,T];\mathscr{H})$ and the damage source function $\theta \in C([0,T];L^2(\Omega))$ are given and consider the following two auxiliary problems.

Problem $\mathbf{P}_\eta^1$: Find $\boldsymbol{u}_\eta \in C^1([0,T];V)$ and $\boldsymbol{\sigma}_\eta \in C([0,T];\mathscr{H}_1)$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t))) + \boldsymbol{\eta}(t) \quad \text{for all } t \in [0,T]$$

$$\begin{aligned}&\langle \boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t))\rangle_{\mathscr{H}} + \int_{\Gamma_C} k_\nu p_\nu(u_{\eta\nu}(t) - g)(v_\nu - \dot{u}_{\eta\nu}(t))\, d\Gamma \\ &+ \int_{\Gamma_C} k_\tau j_\tau^0(\dot{\boldsymbol{u}}_{\eta\tau}(t); \boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_{\eta\tau}(t))\, d\Gamma \\ &\geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T]\end{aligned} \tag{21}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{22}$$

Problem P_θ: Find $\beta_\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned}
&\langle \dot\beta_\theta(t), \zeta - \beta_\theta(t)\rangle_{L^2(\Omega)} + \kappa\langle \nabla\beta_\theta(t), \nabla\zeta - \nabla\beta_\theta(t)\rangle_{L^2(\Omega;\mathbb{R}^d)} \\
&\geqslant \langle \theta(t), \zeta - \beta_\theta(t)\rangle_{L^2(\Omega)} \quad \text{for all } \zeta \in \mathscr{K} \text{ and all } t \in [0,T] \qquad (23)\\
&\beta_\theta(t) \in \mathscr{K} \quad \text{for all } t \in [0,T]\\
&\beta_\theta(0) \;=\; \beta_0. \qquad (24)
\end{aligned}$$

Proposition 5.2. *Under the hypotheses of Theorem* 5.1*, for every fixed* $\boldsymbol{\eta} \in C([0,T];\mathscr{H})$*, Problem* P^1_η *admits a unique solution* $(\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta)$*.*

Proof. First, we formulate Problem P^1_η in terms of the velocity $\boldsymbol{w}_\eta = \dot{\boldsymbol{u}}_\eta$. Then,

$$\boldsymbol{u}_\eta(t) \;=\; \int_0^t \boldsymbol{w}_\eta(s)\,ds + \boldsymbol{u}_0 \quad \text{for all } t \in [0,T]. \qquad (25)$$

Problem P^1_η can be written in the following form.

Problem P^2_η: Find $\boldsymbol{w}_\eta \in C([0,T];V)$ and $\boldsymbol{\sigma}_\eta \in C([0,T];\mathscr{H}_1)$ such that

$$\begin{aligned}
&\boldsymbol{\sigma}_\eta(t) \;=\; \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_\eta(t))) + \boldsymbol{\eta}(t) \quad \text{for all } t \in [0,T]\\
&\langle \boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{w}_\eta(t))\rangle_{\mathscr{H}} + \int_{\Gamma_C} k_\nu p_\nu\Big(\int_0^t w_{\eta\nu}(s)\,ds + u_{0\nu} - g\Big)(v_\nu - w_{\eta\nu}(t))\,d\Gamma\\
&+ \int_{\Gamma_C} k_\tau j^0_\tau(\boldsymbol{w}_{\eta\tau}(t); \boldsymbol{v}_\tau - \boldsymbol{w}_{\eta\tau}(t))\,d\Gamma\\
&\geqslant \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}_\eta(t)\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T].
\end{aligned}$$

We need the following operators and functions. The operator $A\colon V \longrightarrow V^*$ is given by

$$\langle A(\boldsymbol{w}), \boldsymbol{v}\rangle_{V^*\times V} \;=\; \langle \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}} \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in V,$$

$\mathcal{R}\colon C([0,T];V) \longrightarrow C([0,T];L^2(\Gamma_C))$ is given by

$$(\mathcal{R}\boldsymbol{w})(t) \;=\; p_\nu\Big(\int_0^t w_\nu(s)\,ds + u_{0\nu} - g\Big) \quad \text{for all } \boldsymbol{w} \in C([0,T];V) \text{ and } t \in [0,T],$$

and the functions $\varphi, j\colon \Gamma_C \times \mathbb{R}^d \longrightarrow \mathbb{R}$, and $\widetilde{\boldsymbol{f}}\colon [0,T] \longrightarrow V^*$ are given by

$$\begin{aligned}
\varphi(\boldsymbol{x}, \boldsymbol{\xi}) \;&=\; k_\nu(\boldsymbol{x})\,\xi_\nu && \text{for all } \boldsymbol{x} \in \Gamma_C,\ \boldsymbol{\xi} \in \mathbb{R}^d,\\
j(\boldsymbol{x}, \boldsymbol{\xi}) \;&=\; k_\tau(\boldsymbol{x})\,j_\tau(\boldsymbol{x}, \boldsymbol{\xi}_\tau) && \text{for all } \boldsymbol{x} \in \Gamma_C,\ \boldsymbol{\xi} \in \mathbb{R}^d,\\
\langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v}\rangle_{V^*\times V} \;&=\; \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} - \langle \boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathscr{H}} && \text{for all } \boldsymbol{v} \in V,\ t \in [0,T].
\end{aligned}$$

Using this notation Problem P_η^2 takes the following form.

Problem $\mathbf{P}_\eta^3$: Find $\boldsymbol{w}_\eta \in C([0,T];V)$ such that

$$\begin{aligned}
&\langle A(\boldsymbol{w}_\eta(t)), \boldsymbol{v} - \boldsymbol{w}_\eta(t)\rangle_{V^*\times V} + \int_{\Gamma_C} (\mathcal{R}\boldsymbol{w}_\eta)(t)\big(\varphi(\gamma\boldsymbol{v}) - \varphi(\gamma\boldsymbol{w}_\eta(t))\big)\, d\Gamma \\
&+ \int_{\Gamma_C} j^0\big(\gamma\boldsymbol{w}_\eta(t); \gamma\boldsymbol{v} - \gamma\boldsymbol{w}_\eta(t)\big)\, d\Gamma \\
&\geqslant \langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v} - \boldsymbol{w}_\eta(t)\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T].
\end{aligned}$$

For given $\boldsymbol{\mu} \in C([0,T];V)$, let $\boldsymbol{z}_\mu = \mathcal{R}\boldsymbol{\mu}$ and consider the following problem.

Problem $\mathbf{P}_{\eta\mu}$: Find $\boldsymbol{w}_{\eta\mu} \in C([0,T];V)$ such that

$$\begin{aligned}
&\langle A(\boldsymbol{w}_{\eta\mu}(t)), \boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)\rangle_{V^*\times V} + \int_{\Gamma_C} \boldsymbol{z}_\mu(t)\big(\varphi(\gamma\boldsymbol{v}) - \varphi(\gamma\boldsymbol{w}_{\eta\mu}(t))\big)\, d\Gamma \\
&+ \int_{\Gamma_C} j^0\big(\gamma\boldsymbol{w}_{\eta\mu}(t); \gamma\boldsymbol{v} - \gamma\boldsymbol{w}_{\eta\mu}(t)\big)\, d\Gamma \\
&\geqslant \langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)\rangle_{V^*\times V} \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T]. \qquad (26)
\end{aligned}$$

Lemma 5.3. *Problem* $\mathrm{P}_{\eta\mu}$ *admits a unique solution* $\boldsymbol{w}_{\eta\mu}$.

Proof. Consider the functional $J\colon L^2(\Gamma_C;\mathbb{R}^d) \longrightarrow \mathbb{R}$ given by

$$J(\boldsymbol{v}) = \int_{\Gamma_C} j(\boldsymbol{v})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in L^2(\Gamma_C;\mathbb{R}^d),$$

the operator $B\colon V \longrightarrow 2^{V^*}$ given by

$$B(\boldsymbol{v}) = \gamma^*\partial J(\gamma\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V,$$

and the functional $\Phi\colon [0,T]\times V \longrightarrow \mathbb{R}$ given by

$$\Phi(t,\boldsymbol{v}) = \int_{\Gamma_C} \boldsymbol{z}_\mu(t)\,\varphi(\gamma\boldsymbol{v})\, d\Gamma \quad \text{for all } \boldsymbol{v} \in V \text{ and all } t \in [0,T].$$

From [23, Theorem 3.47], we know that the functional J is well defined, Lipschitz continuous on bounded subsets of $L^2(\Gamma_C;\mathbb{R}^d)$, and satisfies

$$J^0(\boldsymbol{u};\boldsymbol{v}) \leqslant \int_{\Gamma_C} j^0(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}); \boldsymbol{v}(\boldsymbol{x}))\, d\Gamma \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in L^2(\Gamma_C;\mathbb{R}^d) \qquad (27)$$

and

$$\|\partial J(\boldsymbol{u})\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leqslant \overline{c}_0 + \overline{c}_1\|\boldsymbol{u}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \quad \text{for all } \boldsymbol{u} \in L^2(\Gamma_C;\mathbb{R}^d), \qquad (28)$$

with $\overline{c}_0 = c_{0\tau}\overline{k}_\tau\sqrt{2\,\mathrm{meas}\,(\Gamma_C)}$ and $\overline{c}_1 = c_{1\tau}\overline{k}_\tau\sqrt{2}$. Also the values of ∂J are nonempty, convex and weakly compact subsets of $L^2(\Gamma_C;\mathbb{R}^d)$ (see, e.g., [23, Proposition 3.23(iv)]) and thus for every $\boldsymbol{v} \in V$, the set $B(\boldsymbol{v})$ is nonempty, closed and convex in V^*. Next, from (28), for every $\boldsymbol{v} \in V$, we have

$$\|B(\boldsymbol{v})\|_{V^*}\leqslant \|\gamma^*\|\|\partial J(\gamma\boldsymbol{v})\|_{L^2(\Gamma_C;\mathbb{R}^d)}\leqslant \overline{c}_0\|\gamma\|+\overline{c}_1\|\gamma\|^2\|\boldsymbol{v}\|_V, \tag{29}$$

so the operator B is bounded. From (29), for every $\boldsymbol{v} \in V$, we also have

$$\langle B(\boldsymbol{v}),\boldsymbol{v}\rangle_{V^*\times V} \geqslant -\overline{c}_1\|\gamma\|^2\|\boldsymbol{v}\|_V^2 - \overline{c}_0\|\gamma\|\|\boldsymbol{v}\|_V. \tag{30}$$

Next we show that the operator B is also generalized pseudomonotone. Let $\{\boldsymbol{v}_n\}_{n\geqslant 1}$ be a sequence of V such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in V, let $\{\boldsymbol{v}_n^*\}_{n\geqslant 1}$ be a sequence of V^* such that $\boldsymbol{v}_n^* \to \boldsymbol{v}^*$ weakly in V^*, $\boldsymbol{v}_n^* \in B(\boldsymbol{v}_n)$ for all $n \geqslant 1$ and $\limsup\langle \boldsymbol{v}_n^*, \boldsymbol{v}_n-\boldsymbol{v}\rangle_{V^*\times V} \leqslant 0$. From the definition of the operator B we have that $\boldsymbol{v}_n^* = \gamma^*\boldsymbol{\zeta}_n$ with $\boldsymbol{\zeta}_n \in \partial J(\gamma\boldsymbol{v}_n)$ for all $n \geqslant 1$. From (28) we see that the sequence $\{\boldsymbol{\zeta}_n\}_{n\geqslant 1}$ is bounded in $L^2(\Gamma_C;\mathbb{R}^d)$ and so, passing to a subsequence if necessary, we may assume that $\boldsymbol{\zeta}_n \to \boldsymbol{\zeta}$ weakly in $L^2(\Gamma_C;\mathbb{R}^d)$. Since the graph of ∂J is closed in $L^2(\Gamma_C;\mathbb{R}^d)\times(w-L^2(\Gamma_C;\mathbb{R}^d))$-topology and $\gamma\boldsymbol{v}_n \to \gamma\boldsymbol{v}$ in $L^2(\Gamma_C;\mathbb{R}^d)$, by the compactness of the trace operator we obtain that $\boldsymbol{\zeta} \in \partial J(\gamma\boldsymbol{v})$. Moreover, as $\boldsymbol{v}_n^* = \gamma^*\boldsymbol{\zeta}_n$, it follows that $\boldsymbol{v}^* = \gamma^*\boldsymbol{\zeta}$. Thus, $\boldsymbol{v}^* \in \gamma^*\partial J(\gamma\boldsymbol{v}) = B(\boldsymbol{v})$ and also $\langle \boldsymbol{v}_n^*, \boldsymbol{v}_n\rangle_{V^*\times V} = \langle \gamma^*\boldsymbol{\zeta}_n, \boldsymbol{v}_n\rangle_{V^*\times V} = \langle \boldsymbol{\zeta}_n, \gamma\boldsymbol{v}_n\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \to \langle \boldsymbol{\zeta}, \gamma\boldsymbol{v}\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} = \langle \gamma^*\boldsymbol{\zeta}, \boldsymbol{v}\rangle_{V^*\times V} = \langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{V^*\times V}$, which proves that the operator B is generalized pseudomonotone and therefore it is also pseudomonotone.

As for the functional Φ, first note that by hypothesis $H(k_\nu)$, we have $\varphi(\cdot,\gamma\boldsymbol{v}(\cdot)) \in L^2(\Gamma_C)$ for all $\boldsymbol{v} \in V$ and because $\boldsymbol{z}_\mu(t) \in L^2(\Gamma_C)$ for all $t \in [0,T]$, we have that $\Phi(t,\cdot)$ is well defined for all $t \in [0,T]$. Moreover, it is clear that $\mathrm{dom}\,\Phi(t,\cdot) = V$ for all $t \in [0,T]$. Next, since $\boldsymbol{z}_\mu(\boldsymbol{x},t) \geqslant 0$ (see hypothesis $H(p_\nu)$) and $\varphi(\boldsymbol{x},\cdot)$ is convex, we easily infer that $\Phi(t,\cdot)$ is convex for all $t \in [0,T]$. Also, it is proper and continuous. Therefore, by Theorem 2.8, we deduce that for all $t \in [0,T]$, the operator $\partial\Phi(t,\cdot)\colon V \longrightarrow 2^{V^*}$ is maximal monotone with $D(\partial\Phi(t,\cdot)) = V$.

Next, let us consider the operator A. First we check that it is bounded. Using hypothesis $H(\mathcal{A})$(c) and the Hölder inequality, for all $\boldsymbol{u},\boldsymbol{v} \in V$, we have

$$\begin{aligned}
|\langle A(\boldsymbol{u}),\boldsymbol{v}\rangle_{V^*\times V}| &\leqslant \int_\Omega \|\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathbb{S}^d}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathbb{S}^d}\,dx \\
&\leqslant \left(\int_\Omega 2(a_0^2(\boldsymbol{x}) + a_1^2\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\mathbb{S}^d}^2)\,dx\right)^{\frac{1}{2}}\|\boldsymbol{v}\|_V \\
&\leqslant \sqrt{2}(\|a_0\|_{L^2(\Omega)} + a_1\|\boldsymbol{u}\|_V)\|\boldsymbol{v}\|_V,
\end{aligned}$$

so for all $\boldsymbol{u} \in V$, we get

$$\|A(\boldsymbol{u})\|_{V^*} \leqslant \sqrt{2}(\|a_0\|_{L^2(\Omega)} + a_1\|\boldsymbol{u}\|_V),$$

which implies the boundedness of the operator A.

Next we show that the operator A is coercive. Using hypotheses $H(\mathcal{A})$(d) and (e), for all $\boldsymbol{u} \in V$, we have

$$\langle A\boldsymbol{u}, \boldsymbol{u} \rangle_{V^* \times V} = \int_\Omega \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u})\, dx \geqslant m_{\mathcal{A}} \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{\mathbb{S}^d}\, dx = m_{\mathcal{A}} \|\boldsymbol{u}\|^2_V, \quad (31)$$

so the operator A is coercive.

Next we show that the operator A is monotone and continuous. First using hypothesis $H(\mathcal{A})$(d) for all $\boldsymbol{u}, \boldsymbol{v} \in V$, we have

$$\langle A(\boldsymbol{u}) - A(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v} \rangle_{V^* \times V} = \int_\Omega (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u})) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{v}))) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{v}))dx \geqslant 0,$$

which shows the monotonicity of A. Next, let $\{\boldsymbol{u}_n\}_{n \geqslant 1}$ be a sequence in V such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in V, which implies that $\boldsymbol{\varepsilon}(\boldsymbol{u}_n) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ in $L^2(\Omega; \mathbb{S}^d)$. Passing to a subsequence if necessary, we may assume that $\boldsymbol{\varepsilon}(\boldsymbol{u}_n)(\boldsymbol{x}) \to \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})$ in $\mathbb{S}^d$ for almost all $\boldsymbol{x} \in \Omega$ and $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n)(\boldsymbol{x})\|_{\mathbb{S}^d} \leqslant w(\boldsymbol{x})$ for all $n \geqslant 1$ and almost all $\boldsymbol{x} \in \Omega$ with some $w \in L^2(\Omega)$. From the continuity of $\mathcal{A}(\boldsymbol{x}, \cdot)$ on $\mathbb{S}^d$ (see hypothesis $H(\mathcal{A})$(b)), we have $\|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}_n)(\boldsymbol{x})) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))\|^2_{\mathbb{S}^d} \to 0$ for almost all $\boldsymbol{x} \in \Omega$ and from hypothesis $H(\mathcal{A})$(c), we deduce that

$$\begin{aligned}
&\|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}_n)(\boldsymbol{x})) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))\|^2_{\mathbb{S}^d} \\
&\leqslant 2(a_0(\boldsymbol{x}) + a_1 \|\boldsymbol{\varepsilon}(\boldsymbol{u}_n)(\boldsymbol{x})\|_{\mathbb{S}^d})^2 + 2(a_0(\boldsymbol{x}) + a_1 \|\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\|_{\mathbb{S}^d})^2 \\
&\leqslant 8a_0^2(\boldsymbol{x}) + 4a_1^2(w(\boldsymbol{x})^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x})\|^2_{\mathbb{S}^d})
\end{aligned}$$

for almost all $\boldsymbol{x} \in \Omega$. Thus, by using the Lebesgue dominated convergence theorem we obtain

$$\|\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|^2_{\mathscr{H}} = \int_\Omega \|\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|^2_{\mathbb{S}^d}\, dx \to 0.$$

On the other hand, using the Hölder inequality, for every $\boldsymbol{v} \in V$, we get

$$\begin{aligned}
\langle A(\boldsymbol{u}_n) - A(\boldsymbol{u}), \boldsymbol{v} \rangle_{V^* \times V} &= \int_\Omega (\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}))) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \\
&\leqslant \|\mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathscr{H}} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathscr{H}},
\end{aligned}$$

so we conclude that $A(\boldsymbol{u}_n) \to A(\boldsymbol{u})$ in V^*. Thus, we conclude that A is continuous.

The operator A being bounded, monotone and hemicontinuous, is also pesudomonotone (see [23, Theorem 3.69(i)]). Since the class of multivalued pseudomonotone operators is closed under addition of mappings (see, e.g., [23, Proposition 3.59]), we deduce that $A + B$ is pseudomonotone.

Finally we establish the coercivity of the operator $A+B$. First assume that hypothesis H_1(b)(i) holds. Using (30) and (31), for all $\boldsymbol{v} \in V$, we have

$$\begin{aligned}\langle A(\boldsymbol{v}) + B(\boldsymbol{v}), \boldsymbol{v}\rangle_{V^*\times V} &= \langle A(\boldsymbol{v}), \boldsymbol{v}\rangle_{V^*\times V} + \langle B(\boldsymbol{v}), \boldsymbol{v}\rangle_{V^*\times V} \\ &\geqslant (m_{\mathcal{A}} - c_{1\tau}\overline{k}_\tau\sqrt{2}\|\gamma\|^2)\|\boldsymbol{v}\|_V^2 - \overline{c}_0\|\gamma\|\|\boldsymbol{v}\|_V.\end{aligned}$$

Since by hypothesis H_1(b)(i), we have $m_{\mathcal{A}} > c_{1\tau}\overline{k}_\tau\sqrt{2}\|\gamma\|^2$, thus the operator $A+B$ is coercive. Next assume that hypothesis H_1(b)(ii) holds. Then, using (27), we have

$$J^0(\boldsymbol{v}; -\boldsymbol{v}) \leqslant d_1(1 + \|\boldsymbol{v}\|_{L^2(\Gamma_C;\mathbb{R}^d)}) \quad \text{for all } \boldsymbol{v} \in L^2(\Gamma_C;\mathbb{R}^d),$$

for some $d_1 > 0$. Therefore for every $\boldsymbol{v} \in V$ and $\boldsymbol{\zeta} \in \partial J(\gamma\boldsymbol{v})$, we have

$$\langle \boldsymbol{\zeta}, \gamma\boldsymbol{v}\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} \geqslant -J^0(\gamma\boldsymbol{v}; -\gamma\boldsymbol{v}) \geqslant -d_1 - d_2\|\gamma\|\|\boldsymbol{v}\|_V,$$

for some $d_2 > 0$. So, from this and the coercivity of A (see (31)), we see that $A+B$ is coercive.

Since we have checked that under our hypotheses the operator $A+B$ is bounded, pseudomonotone and coercive and the operator $\partial\Phi(t,\cdot)\colon V \longrightarrow 2^{V^*}$ is maximal monotone with $D(\partial\Phi(t,\cdot)) = V$, for all $t \in [0,T]$, we can apply Theorem 2.4 and deduce that $A(\cdot) + B(\cdot) + \partial\Phi(t,\cdot)\colon V \longrightarrow 2^{V^*}$ is surjective and so for each $t \in [0,T]$ there exists $\boldsymbol{w}_{\eta\mu}(t) \in V$ such that

$$A(\boldsymbol{w}_{\eta\mu}(t)) + B(\boldsymbol{w}_{\eta\mu}(t)) + \partial\Phi(t, \boldsymbol{w}_{\eta\mu}(t)) \ni \widetilde{\boldsymbol{f}}(t).$$

This means that

$$A(\boldsymbol{w}_{\eta\mu}(t)) + \gamma^*\zeta_1(t) + \zeta_2(t) = \widetilde{\boldsymbol{f}}(t), \tag{32}$$

with $\zeta_1(t) \in \partial J(\gamma\boldsymbol{w}_{\eta\mu}(t))$ and $\zeta_2(t) \in \partial\Phi(t, \boldsymbol{w}_{\eta\mu}(t))$. Let $\boldsymbol{v} \in V$ and applying $\boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)$ to (32), we find

$$\begin{aligned}&\langle A(\boldsymbol{w}_{\eta\mu}(t)), \boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)\rangle_{V^*\times V} \\ &+\langle \zeta_1(t), \gamma\boldsymbol{v} - \gamma\boldsymbol{w}_{\eta\mu}(t)\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} + \langle \zeta_2(t), \boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)\rangle_{V^*\times V} \\ &= \langle \widetilde{\boldsymbol{f}}(t), \boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)\rangle_{V^*\times V}.\end{aligned}$$

Using the definitions and properties of the Clarke and convex subdifferentials, we get

$$\begin{aligned}\langle \zeta_1(t), \gamma\boldsymbol{v} - \gamma\boldsymbol{w}_{\eta\mu}(t)\rangle_{L^2(\Gamma_C;\mathbb{R}^d)} &\leqslant \int_{\Gamma_C} j^0(\gamma\boldsymbol{w}_{\eta\mu}(t); \gamma\boldsymbol{v} - \gamma\boldsymbol{w}_{\eta\mu}(t))\, d\Gamma, \\ \langle \zeta_2(t), \boldsymbol{v} - \boldsymbol{w}_{\eta\mu}(t)\rangle_{V^*\times V} &\leqslant \int_{\Gamma_C} \boldsymbol{z}_\mu(t)(\varphi(\gamma\boldsymbol{v}) - \varphi(\gamma\boldsymbol{w}_{\eta\mu}(t)))\, d\Gamma,\end{aligned}$$

and it follows that $\boldsymbol{w}_{\eta\mu}(t)$ is a solution of Problem $\mathrm{P}_{\eta\mu}$.

Next, we show the uniqueness of $\boldsymbol{w}_{\eta\mu}$, so for a fixed $t \in [0,T]$, assume that $\boldsymbol{w}_1(t), \boldsymbol{w}_2(t) \in V$ are two solutions of $\mathrm{P}_{\eta\mu}$. We write (26) for $\boldsymbol{w}_1(t)$ with $\boldsymbol{v} = \boldsymbol{w}_2(t)$ and then for $\boldsymbol{w}_2(t)$ with $\boldsymbol{v} = \boldsymbol{w}_1(t)$. Adding the resulting inequalities yields

$$\begin{aligned}
&\langle A(\boldsymbol{w}_1(t)) - A(\boldsymbol{w}_2(t)), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V} \\
&\leqslant \int_{\Gamma_C} \big(j^0\big(\gamma\boldsymbol{w}_1(t); \gamma\boldsymbol{w}_2(t) - \gamma\boldsymbol{w}_1(t)\big) + j^0\big(\gamma\boldsymbol{w}_2(t); \gamma\boldsymbol{w}_1(t) - \gamma\boldsymbol{w}_2(t)\big)\big)d\Gamma.
\end{aligned}$$

Then, using hypotheses $H(\mathcal{A})$(d) and $H(j_\tau)$(d), we obtain

$$m_{\mathcal{A}}\|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V^2 \;\leqslant\; c_{2\tau}\overline{k}_\tau\|\gamma\|^2\|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V^2,$$

so from hypothesis H_1(a), we deduce that $\boldsymbol{w}_1(t) = \boldsymbol{w}_2(t)$ for all $t \in [0,T]$.

To complete the proof of the lemma we show that the mapping $[0,T] \ni t \longmapsto \boldsymbol{w}_{\eta\mu}(t) \in V$ is continuous. Let $t_1, t_2 \in [0,T]$ and let us denote $\widehat{\boldsymbol{w}}_i = \boldsymbol{w}_{\eta\mu}(t_i)$, $\widehat{\boldsymbol{z}}_i = \boldsymbol{z}_\mu(t_i)$, $\widehat{\boldsymbol{f}}_i = \boldsymbol{f}(t_i)$, $\widehat{\boldsymbol{\eta}}_i = \boldsymbol{\eta}(t_i)$ for $i = 1,2$. We write (26) for $t = t_1$ with $\boldsymbol{v} = \widehat{\boldsymbol{w}}_2$ and then for $t = t_2$ with $\boldsymbol{v} = \widehat{\boldsymbol{w}}_1$. Adding the resulting inequalities yields

$$\begin{aligned}
&\langle A(\widehat{\boldsymbol{w}}_1) - A(\widehat{\boldsymbol{w}}_2),\ \widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\rangle_{V^*\times V} \\
&\leqslant \int_{\Gamma_C} (\widehat{\boldsymbol{z}}_2 - \widehat{\boldsymbol{z}}_1)(\varphi(\gamma\widehat{\boldsymbol{w}}_1) - \varphi(\gamma\widehat{\boldsymbol{w}}_2))d\Gamma \\
&\quad + \int_{\Gamma_C} \big(j^0\big(\gamma\widehat{\boldsymbol{w}}_1;\ \gamma\widehat{\boldsymbol{w}}_2 - \gamma\widehat{\boldsymbol{w}}_1\big) + j^0\big(\gamma\widehat{\boldsymbol{w}}_2;\ \gamma\widehat{\boldsymbol{w}}_1 - \gamma\widehat{\boldsymbol{w}}_2\big)\big)d\Gamma \\
&\quad + \langle \widehat{\boldsymbol{f}}_1 - \widehat{\boldsymbol{f}}_2,\ \widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\rangle_{V^*\times V} + \langle \widehat{\boldsymbol{\eta}}_2 - \widehat{\boldsymbol{\eta}}_1,\ \boldsymbol{\varepsilon}(\widehat{\boldsymbol{w}}_1) - \boldsymbol{\varepsilon}(\widehat{\boldsymbol{w}}_2)\rangle_{\mathscr{H}}.
\end{aligned}$$

Using hypotheses $H(\mathcal{A})$(d), $H(p_\nu)$, $H(k_\nu)$, $H(k_\tau)$, $H(j_\tau)$(d), we obtain

$$\begin{aligned}
&m_{\mathcal{A}}\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V^2 \\
&\leqslant \|\widehat{\boldsymbol{z}}_1 - \widehat{\boldsymbol{z}}_2\|_{L^2(\Gamma_C)} L_\nu \overline{k}_\nu \|\gamma\|\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V + \overline{k}_\tau c_{2\tau}\|\gamma\|^2\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V^2 \\
&\quad + \|\widehat{\boldsymbol{f}}_1 - \widehat{\boldsymbol{f}}_2\|_{V^*}\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V + \|\widehat{\boldsymbol{\eta}}_1 - \widehat{\boldsymbol{\eta}}_2\|_{\mathscr{H}}\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V.
\end{aligned}$$

Using hypothesis H_1(a), we find

$$\|\widehat{\boldsymbol{w}}_1 - \widehat{\boldsymbol{w}}_2\|_V \;\leqslant\; c\big(\|\widehat{\boldsymbol{z}}_1 - \widehat{\boldsymbol{z}}_2\|_{L^2(\Gamma_C)} + \|\widehat{\boldsymbol{f}}_1 - \widehat{\boldsymbol{f}}_2\|_{V^*} + \|\widehat{\boldsymbol{\eta}}_1 - \widehat{\boldsymbol{\eta}}_2\|_{\mathscr{H}}\big).$$

It now follows from the continuity of $\boldsymbol{z}_\mu$, $\boldsymbol{f}$, $\boldsymbol{\eta}$, that the function

$$[0,T] \ni t \;\longmapsto\; \boldsymbol{w}_{\eta\mu}(t) \in V$$

is continuous. The proof of the lemma is complete. □

Lemma 5.4. *Problem* P_η^3 *has a unique solution* $\boldsymbol{w}_\eta$.

Proof. In order to show the existence and uniqueness of the solution to Problem P_η^3 we use the fixed point argument of Theorem 2.9. Due to Lemma 5.3, we can define the operator $\Lambda_\eta\colon C([0,T];V) \longrightarrow C([0,T];V)$ by

$$\Lambda_\eta \boldsymbol{\mu} \;=\; \boldsymbol{w}_{\eta\mu} \quad \text{for all } \boldsymbol{\mu} \in C([0,T];V),$$

where $\boldsymbol{w}_{\eta\mu}$ is the unique solution to Problem $\mathrm{P}_{\eta\mu}$. We show that Λ_η has a unique fixed point $\boldsymbol{\mu}^* \in C([0,T];V)$. Let $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in C([0,T];V)$ and $\boldsymbol{z}_i = \boldsymbol{z}_{\mu_i} = \mathcal{R}\boldsymbol{\mu}_i \in C([0,T];L^2(\Gamma_C))$ for $i = 1,2$. Let $\boldsymbol{w}_i = \boldsymbol{w}_{\eta\mu_i}$ be the solutions to Problem $\mathrm{P}_{\eta\mu}$ for $\boldsymbol{\mu} = \boldsymbol{\mu}_i$ $(i = 1,2)$. Then

$$\|(\Lambda_\eta \boldsymbol{\mu}_1)(t) - (\Lambda_\eta \boldsymbol{\mu}_2)(t)\|_V \;=\; \|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V \quad \text{for all } t \in [0,T]. \tag{33}$$

Writing the inequality (26) for $\boldsymbol{w}_1(t)$ with $\boldsymbol{v} = \boldsymbol{w}_2(t)$, then for $\boldsymbol{w}_2(t)$ with $\boldsymbol{v} = \boldsymbol{w}_1(t)$ and adding the resulting inequalities, we get

$$\begin{aligned}
&\langle A(\boldsymbol{w}_1(t)) - A(\boldsymbol{w}_2(t)), \boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\rangle_{V^*\times V}\\
&\leqslant \int_{\Gamma_C} (\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t))\big(\varphi(\gamma\boldsymbol{w}_2(t)) - \varphi(\gamma\boldsymbol{w}_1(t))\big)\, d\Gamma\\
&\quad + \int_{\Gamma_C} \big(j^0\big(\gamma\boldsymbol{w}_1(t); \gamma\boldsymbol{w}_2(t) - \gamma\boldsymbol{w}_1(t)\big) + j^0\big(\gamma\boldsymbol{w}_2(t); \gamma\boldsymbol{w}_1(t) - \gamma\boldsymbol{w}_2(t)\big)\big) d\Gamma.
\end{aligned}$$

Using hypotheses $H(\mathcal{A})$(d), $H(p_\nu)$, $H(k_\nu)$, $H(k_\tau)$, $H(j_\tau)$(d), we obtain

$$\begin{aligned}
m_{\mathcal{A}}\|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V^2 &\leqslant \|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_{L^2(\Gamma_C)} \overline{k}_\nu L_\nu \|\gamma(\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t))\|_{L^2(\Gamma_C;\mathbb{R}^d)}\\
&\quad + \overline{k}_\tau c_{2\tau}\|\gamma(\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t))\|^2_{L^2(\Gamma_C;\mathbb{R}^d)},
\end{aligned}$$

thus

$$(m_{\mathcal{A}} - \overline{k}_\tau c_{2\tau}\|\gamma\|^2)\|\boldsymbol{w}_1(t) - \boldsymbol{w}_2(t)\|_V \;\leqslant\; \overline{k}_\nu L_\nu \|\gamma\| \|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_{L^2(\Gamma_C)}. \tag{34}$$

Next, using the hypothesis $H(p_\nu)$, we find

$$\begin{aligned}
&\|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_{L^2(\Gamma_C)}\\
&= \|(\mathcal{R}\boldsymbol{\mu}_1)(t) - (\mathcal{R}\boldsymbol{\mu}_2)(t)\|_{L^2(\Gamma_C)}\\
&= \Big\|p_\nu\Big(\int_0^t \mu_{1\nu}(s)\, ds + u_{0\nu} - g\Big) - p_\nu\Big(\int_0^t \mu_{2\nu}(s)\, ds + u_{0\nu} - g\Big)\Big\|_{L^2(\Gamma_C)}\\
&\leqslant L_\nu \Big\|\int_0^t |\mu_{1\nu}(s) - \mu_{2\nu}(s)|\, ds\Big\|_{L^2(\Gamma_C)}\\
&\leqslant L_\nu\|\gamma\| \int_0^t \|\boldsymbol{\mu}_1(s) - \boldsymbol{\mu}_2(s)\|_V\, ds.
\end{aligned} \tag{35}$$

It follows from (33), (34), (35) and hypothesis H_1(a), that

$$\|(\Lambda_\eta \boldsymbol{\mu}_1)(t) - (\Lambda_\eta \boldsymbol{\mu}_2)(t)\|_V \leqslant c \int_0^t \|\boldsymbol{\mu}_1(s) - \boldsymbol{\mu}_2(s)\|_V \, ds \quad \text{for all } t \in [0,T].$$

Theorem 2.9 asserts that Λ_η has a unique fixed point $\boldsymbol{\mu}^* \in C([0,T];V)$. Thus, we have $\boldsymbol{z}_{\mu^*}(t) = (\mathcal{R}\boldsymbol{\mu}^*)(t)$, $\boldsymbol{w}_{\eta\mu^*}(t) = \boldsymbol{\mu}^*(t)$ for all $t \in [0,T]$. Writing inequality (26) with $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ we conclude that $\boldsymbol{\mu}^*$ is a solution of Problem P_η^3. By the uniqueness of the fixed point of Λ_η we see that $\boldsymbol{w}_\eta \equiv \boldsymbol{\mu}^*$ is the unique solution of Problem P_η^3. This completes the proof of the lemma. □

Let $\boldsymbol{w}_\eta \in C([0,T];V)$ be the unique solution to Problem P_η^3. Defining

$$\begin{aligned} \boldsymbol{\sigma}_\eta(t) &= \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_\eta(t))) + \boldsymbol{\eta}(t) \quad \text{for all } t \in [0,T], \\ \boldsymbol{u}_\eta(t) &= \int_0^t \boldsymbol{w}_\eta(s)\, ds + \boldsymbol{u}_0 \qquad \text{for all } t \in [0,T], \end{aligned}$$

we find that $(\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta)$ is the unique solution to Problem P_η^1. This completes the proof of Proposition 5.2. □

We turn to the problem of damage with given damage source function.

Proposition 5.5. *Under the hypotheses of Theorem* 5.1*, for every* $\theta \in C([0,T];L^2(\Omega))$ *and* $\beta_0 \in \mathscr{K}$*, Problem* P_θ *admits a unique solution* β_θ*.*

Proof. The assertion follows from standard results for parabolic variational inequalities (see, e.g., Barbu [4, p. 124]). □

Now we are ready for the proof of Theorem 5.1.

Proof of Theorem 5.1. Using Propositions 5.2 and 5.5, the operator

$$\Lambda \colon C([0,T];\mathscr{H} \times L^2(\Omega)) \longrightarrow C([0,T];\mathscr{H} \times L^2(\Omega))$$

given by

$$\Lambda(\boldsymbol{\eta},\theta) = \big(\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta),\beta_\theta),\ \phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta),\beta_\theta)\big) \quad \text{for all } (\boldsymbol{\eta},\theta) \in C([0,T];\mathscr{H} \times L^2(\Omega))$$

is well defined. Here, $\boldsymbol{u}_\eta$ is the unique solution of Problem P_η^1 (Proposition 5.2) and β_θ is the unique solution to Problem P_θ (Proposition 5.5). We show that operator Λ has a unique fixed point $(\boldsymbol{\eta}^*,\beta^*) \in C([0,T];\mathscr{H} \times L^2(\Omega))$. To this end let $(\boldsymbol{\eta}_1,\beta_1),(\boldsymbol{\eta}_2,\beta_2) \in C([0,T];\mathscr{H} \times L^2(\Omega))$. We denote $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$, $\boldsymbol{w}_i = \dot{\boldsymbol{u}}_{\eta_i}$, $\beta_i = \beta_{\theta_i}$ for $i = 1,2$. Using hypotheses $H(\mathcal{G})$(b) and $H(\phi)$(b) we deduce that for all $t \in [0,T]$,

$$\begin{aligned}
&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathscr{H}\times L^2(\Omega)} \\
&= \|\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)),\beta_1(t)) - \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)),\beta_2(t))\|_{\mathscr{H}} \\
&\quad + \|\phi(t,\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)),\beta_1(t)) - \phi(t,\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)),\beta_2(t))\|_{L^2(\Omega)} \\
&\leqslant (L_{\mathcal{G}} + L_\phi)\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(t))\|_{\mathscr{H}} + \|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}\big) \\
&= (L_{\mathcal{G}} + L_\phi)\big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}\big). \qquad (36)
\end{aligned}$$

Since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$ (cf. (22)), using (25), we find

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leqslant \int_0^t \|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V\, ds \quad \text{for all } t \in [0,T]. \qquad (37)$$

For $s \in [0,t]$, writing the inequality (21) for $\boldsymbol{w}_1(s)$ with $\boldsymbol{v} = \boldsymbol{w}_2(s)$, then for $\boldsymbol{w}_2(s)$ with $\boldsymbol{v} = \boldsymbol{w}_1(s)$ and adding the resulting inequalities, we get

$$\begin{aligned}
&\langle \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_1(s))) - \mathcal{A}(\boldsymbol{\varepsilon}(\boldsymbol{w}_2(s))),\ \boldsymbol{\varepsilon}(\boldsymbol{w}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{w}_2(s))\rangle_{\mathscr{H}} \\
&\leqslant \int_{\Gamma_C} k_\nu\big(p_\nu(u_{2\nu}(s)-g) - p_\nu(u_{1\nu}(s)-g)\big)(w_{1\nu}(s) - w_{2\nu}(s))\, d\Gamma \\
&\quad + \int_{\Gamma_C} k_\tau\big(j_\tau^0(\boldsymbol{w}_{1\tau}(s); \boldsymbol{w}_{2\tau}(s) - \boldsymbol{w}_{1\tau}(s)) + j_\tau^0(\boldsymbol{w}_{2\tau}(s); \boldsymbol{w}_{1\tau}(s) - \boldsymbol{w}_{2\tau}(s))\big)\, d\Gamma \\
&\quad + \langle \boldsymbol{\eta}_2(s) - \boldsymbol{\eta}_1(s),\ \boldsymbol{\varepsilon}(\boldsymbol{w}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{w}_2(s))\rangle_{\mathscr{H}}.
\end{aligned}$$

Using assumptions $H(\mathcal{A})$(d), $H(p_\nu)$, $H(k_\nu)$, $H(k_\tau)$, $H(j_\tau)$(d), we obtain

$$\begin{aligned}
&(m_{\mathcal{A}} - \overline{k}_\tau c_{2\tau}\|\gamma\|^2)\|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V \\
&\leqslant L_\nu \overline{k}_\nu \|\gamma\|^2 \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V + \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathscr{H}},
\end{aligned}$$

then hypothesis H_1(a) implies

$$\|\boldsymbol{w}_1(s) - \boldsymbol{w}_2(s)\|_V \leqslant c\big(\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V + \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathscr{H}}\big) \qquad (38)$$

for all $s \in [0,t]$. From (37), (38) and the Gronwall inequality, we obtain

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leqslant c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathscr{H}}\, ds \quad \text{for all } t \in [0,T]. \qquad (39)$$

Next, writing inequality (23) for $\beta_1(s)$ with $\zeta = \beta_2(s)$, then for $\beta_2(s)$ with $\zeta = \beta_1(s)$ and adding the resulting inequalities, we obtain

$$\begin{aligned}
&\langle \dot\beta_1(s) - \dot\beta_2(s), \beta_1(s) - \beta_2(s)\rangle_{L^2(\Omega)} + \kappa\langle \nabla\beta_1(s) - \nabla\beta_2(s), \nabla\beta_1(s) - \nabla\beta_2(s)\rangle_{L^2(\Omega;\mathbb{R}^d)} \\
&\leqslant \langle \theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s)\rangle_{L^2(\Omega)} \quad \text{for all } s \in [0,T].
\end{aligned}$$

Integrating this inequality over $(0,t)$ for $t \in (0,T)$ and using integration by parts, we get

$$\frac{1}{2}\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}-\frac{1}{2}\|\beta_1(0)-\beta_2(0)\|^2_{L^2(\Omega)}+\kappa\int_0^t\|\nabla\beta_1(s)-\nabla\beta_2(s)\|^2_{L^2(\Omega;\mathbb{R}^d)}\,ds$$
$$\leqslant \int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}\,ds \quad \text{for all } t\in[0,T].$$

Since $\beta_1(0)=\beta_2(0)=\beta_0$ (cf. (24)), by using the Young inequality, we find

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)} \leqslant\ c\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t \|\beta_1(s)-\beta_2(s)\|^2_{L^2(\Omega)}\,ds,$$

for all $t\in(0,T)$. Using the Gronwall inequality yields

$$\|\beta_1(t)-\beta_2(t)\|^2_{L^2(\Omega)}\ \leqslant\ c\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds \quad \text{for all } t\in[0,T]. \tag{40}$$

Applying (39) and (40) in (36), we obtain

$$\begin{aligned}
&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathscr{H}\times L^2(\Omega)}\\
&\leqslant\ c\int_0^t\big(\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|^2_{\mathscr{H}}+\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\big)ds\\
&\leqslant\ c\int_0^t\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|^2_{\mathscr{H}\times L^2(\Omega)}\,ds.
\end{aligned}$$

It follows from Theorem 2.9 that Λ has a unique fixed point $(\boldsymbol{\eta}^*,\theta^*)$.

We now establish the existence of a solution to Problem $\mathrm{P_V}$. Let $(\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*})$ be the solution of Problem P^1_η for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ (Proposition 5.2) and let β_{θ^*} be the solution of Problem P_θ for $\theta=\theta^*$ (Proposition 5.5). From the definition of Λ we have that

$$\boldsymbol{\eta}^*\ =\ \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}),\beta_{\theta^*})\quad\text{and}\quad \theta^*\ =\ \phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}),\beta_{\theta^*}),$$

therefore, $(\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\beta_{\theta^*})$ is a solution of Problem $\mathrm{P_V}$.

Finally, we show that $(\boldsymbol{u}_{\eta^*},\boldsymbol{\sigma}_{\eta^*},\beta_{\theta^*})$ is the unique solution of Problem $\mathrm{P_V}$. To that end, let $(\boldsymbol{u},\boldsymbol{\sigma},\beta)$ be any solution of Problem $\mathrm{P_V}$. Let $\boldsymbol{\eta}\in C^1([0,T];V)$ and $\theta\in C([0,T];L^2(\Omega))$ denote the functions

$$\boldsymbol{\eta}\ =\ \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta)\quad\text{and}\quad\theta\ =\ \phi(\cdot,\boldsymbol{\varepsilon}(\boldsymbol{u}),\beta). \tag{41}$$

From (16), (17) and (20) it is clear that $(\boldsymbol{u},\boldsymbol{\sigma})$ is a solution to Problem P^1_η. But by Proposition 5.2 we know that Problem P^1_η has a unique solution $(\boldsymbol{u}_\eta,\boldsymbol{\sigma}_\eta)$, thus $\boldsymbol{u}=\boldsymbol{u}_\eta$ and $\boldsymbol{\sigma}=\boldsymbol{\sigma}_\eta$. Similarly, from (18), (19) and (20) it is clear that β

is a solution to Problem P_θ. However, by Proposition 5.5 Problem P_θ has a unique solution β_θ, thus $\beta = \beta_\theta$. Using (41), we see that $\Lambda(\boldsymbol{\eta}, \theta) = (\boldsymbol{\eta}, \theta)$ and by the uniqueness of the fixed point of the operator Λ we deduce that $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and $\theta = \theta^*$. Thus the solution of Problem P_V is unique.

Finally proceeding as in the proof of [23, Theorem 7.5, pp. 210–211], using assumptions $H(\mathcal{A})$, $H(\mathcal{G})$, relation (16) and regularity of $\boldsymbol{u}$ and β, we can show that

$$\operatorname{Div} \boldsymbol{\sigma}(t) = -\boldsymbol{f}_0(t) \quad \text{in } \Omega, \text{ for all } t \in [0, T].$$

So, it follows from $H_0(\mathrm{a})$ that $\operatorname{Div} \boldsymbol{\sigma} \in C([0,T]; \mathscr{H})$ and than we can easily obtain that $\boldsymbol{\sigma} \in C([0,T]; \mathscr{H}_1)$.

The proof of our main result, Theorem 5.1, is now complete. $\square$

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