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Multiplicity of Nontrivial Solutions of a Class of Fractional p-Laplacian Problem

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Abstract. In this paper, we deal with existence of nontrivial solutions to the fractional p-Laplacian problem of the type

$$\begin{cases} (-\triangle)_p^\alpha u = \frac{1}{r}\frac{\partial F(x,u)}{\partial u} + \lambda a(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $a \in C(\Omega)$, $p \geq 2$, $\alpha \in (0,1)$ such that $p\alpha < n$, $1 < q < p < r < \frac{np}{n-\alpha p}$, and $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Using the decomposition of the Nehari manifold, we prove that the non-local elliptic problem has at least two nontrivial solutions.

Keywords. Nontrivial solutions, sign-changing weight function, Nehari manifold
Mathematics Subject Classification (2010). Primary 35J35, secondary 35J50, 35J60

1. Introduction

In this paper, we are concerned with the multiplicity of nontrivial solutions for the following problem

$$\textbf{(P)} \quad \begin{cases} (-\triangle)_p^\alpha u = \frac{1}{r}\frac{\partial F(x,u)}{\partial u} + \lambda a(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $a \in C(\Omega)$, $\lambda > 0$, $p \geq 2$, such that $n > p\alpha$ and $1 < q < p < r < p_\alpha^*$, $p_\alpha^* = \frac{np}{n-\alpha p}$. The function $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree r , that is, $F(x,tu) = t^r F(x,u,v)(t > 0)$ holds for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$.

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Throughout this paper the sign changing weight function a satisfies the following condition

$$\textbf{(A)} \quad a \in C(\Omega) \text{ with } \|a\|_\infty = 1 \text{ and } a^\pm := \max(\pm a, 0) \not\equiv 0,$$

and the fractional p-Laplacian operator may be defined for $p \in (1, \infty)$ as

$$(-\triangle)_p^\alpha u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+p\alpha}} dy, \quad x \in \mathbb{R}^n.$$

Recently, a lot of attention is given to the study of fractional and non-local operators of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc. Dirichlet boundary value problem in case of fractional Laplacian using variational methods is recently studied in [3,5,7,9,10]. Also existence and multiplicity results for nonlocal operators with convex-concave type nonlinearity is shown in [12]. Moreover multiplicity results with sign-changing weight functions using Nehari manifold and fibering map analysis is also studied in many papers (see [1, 2, 4, 10]).

In this paper, we propose a very simple variational method to prove the existence of at least two nontrivial solutions of problem (P). In fact, we use the decomposition of the Nehari manifold as λ vary to prove our main result. Before stating our main result, we need the following assumptions:

(H$_1$) $F : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function such that

$$F(x, tu) = t^r F(x, u) (t > 0) \quad \text{for all } x \in \overline{\Omega},\ u \in \mathbb{R}.$$

(H$_2$) $F(x, 0) = \frac{\partial F}{\partial u}(x, 0) = 0$.

(H$_3$) $F^\pm(x, u) = \max(\pm F(x, u), 0) \neq 0$ for all $u \neq 0$.

We remark that assumption (H$_1$) leads to the so-called Euler identity

$$u \frac{\partial F}{\partial u}(x, u) = r F(x, u)$$

and

$$|F(x, u)| \leq K|u|^r \quad \text{for some constant } K > 0. \tag{1}$$

Our main result is the following

Theorem 1.1. *Under the assumptions* (A) *and* (H$_1$)-(H$_3$)*, there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$*, problem* (P) *has at least two nontrivial solutions.*

This paper is organized as follows. In Section 2, we give some notations and preliminaries. Proofs of Theorem 1.1 is given in Section 3.

2. Preliminaries

In this preliminary section, for the reader's convenience, we collect some basic results that will be used in the forthcoming sections. in the following, For all $1 \leq r \leq \infty$ denote by $\|.\|_r$ the norm of $L^r(\Omega)$. The Gagliardo seminorm is defined for all measurable function $u : \mathbb{R}^n \to \mathbb{R}$ by

$$|u|_{\alpha,p} := \left(\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\alpha}} dxdy \right)^{\frac{1}{p}}.$$

We define the fractional Sobolev space

$$W^{\alpha,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) : u \text{ measurable }, |u|_{\alpha,p} < \infty\}$$

endowed with the norm

$$||u||_{\alpha,p} := \left(||u||_p^p + |u|_{\alpha,p}^p\right)^{\frac{1}{p}}.$$

For a detailed account on the properties of $W^{\alpha,p}(\mathbb{R}^n)$ we refer the reader to [6].

We shall work in the closed linear subspace

$$E := \{u \in W^{\alpha,p}(\mathbb{R}^n) : u(x) = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

which can be equivalenty renormed by setting $||\cdot|| = |\cdot|_{\alpha,p}$, note that these type of spaces were introduced in [9]. It is readily seen that $(E, ||\cdot||)$ is a uniformly convex Banach space and that the embedding $E \hookrightarrow L^r(\Omega)$ is continuous for all $1 \leq r \leq p_\alpha^*$, and compact for all $1 \leq r < p_\alpha^*$. The dual space of $(E, ||\cdot||)$ is denoted by $(E^*, ||\cdot||_*)$, and $\prec \cdot, \cdot \succ$ denotes the usual duality between E and E^*.

Definition 2.1. We say that $u \in E$ is a weak solution of (P) if for every $v \in E$ we have

$$\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+p\alpha}} dxdy$$
$$= \frac{1}{r} \int_\Omega \frac{\partial F}{\partial u}(x, u(x))v(x)dx + \lambda \int_\Omega a(x)|u(x)|^{q-2}u(x)v(x)dx.$$

The Euler functional $J_\lambda \colon E \to \mathbb{R}$ associated to the problem (P) is defined as

$$J_\lambda(u) = \frac{1}{p}||u||^p - \frac{1}{r}\int_\Omega F(x, u)dx - \frac{\lambda}{q}\int_\Omega a(x)|u(x)|^q dx.$$

Then J_λ is Fréchet differentiable and, for all $u \in E$, we have

$$\prec J_\lambda'(u), u \succ = ||u||^p - \int_\Omega F(x, u)dx - \lambda \int_\Omega a(x)|u(x)|^q dx,$$

which shows that the weak solutions of (P) are critical points of the functional J_λ. It is easy to see that the energy functional J_λ is not bounded below on the space E, but is bounded below on an appropriate subset of E and a minimizer on subsets of this set gives raise to solutions of (P). In order to obtain the existence result, we introduce the Nehari manifold

$$\mathcal{N}_\lambda := \{u \in E : \prec J'_\lambda(u), u \succ = 0\}.$$

Then, $u \in \mathcal{N}_\lambda$ if and only if

$$||u||^p - \int_\Omega F(x,u)dx - \lambda \int_\Omega a(x)|u(x)|^q dx = 0. \tag{2}$$

We note that $\mathcal{N}_\lambda$ contains every non zero solution of (P).

Lemma 2.2. *J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$, then we have

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{r}\right)||u||^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega a(x)|u(x)|^q dx \geq c_1||u||^p - c_2||u||^q.$$

Hence, J_λ is bounded below and coercive on $\mathcal{N}_\lambda$. □

Now as we know that the Nehari manifold is closely related to the behavior of the functions $\Phi_u : [0,\infty) \to \mathbb{R}$ defined as

$$\Phi_u(t) = J_\lambda(tu).$$

Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [7].

For $u \in E$, we have

$$\Phi_u(t) = \frac{t^p}{p}||u||^p - \frac{t^r}{r}\int_\Omega F(x,u)dx - \lambda\frac{t^q}{q}\int_\Omega a(x)|u(x)|^q dx,$$
$$\Phi'_u(t) = t^{p-1}||u||^p - t^{r-1}\int_\Omega F(x,u)dx - \lambda t^{q-1}\int_\Omega a(x)|u(x)|^q dx,$$
$$\Phi''_u(t) = (p-1)t^{p-2}||u||^p - (r-1)t^{r-2}\int_\Omega F(x,u)dx - \lambda(q-1)t^{q-2}\int_\Omega a(x)|u(x)|^q dx.$$

Then, it is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(1) = 0$. Thus it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection. For this we set

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \Phi_u''(1) > 0\} = \{tu \in E : \Phi_u'(t) = 0, \Phi_u''(t) > 0\},\\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \Phi_u''(1) < 0\} = \{tu \in E : \Phi_u'(t) = 0, \Phi_u''(t) < 0\},\\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \Phi_u''(1) = 0\} = \{tu \in E : \Phi_u'(t) = 0, \Phi_u''(t) = 0\}.\end{aligned}$$

Before studying the behavior of Nehari manifold using fibering maps, we introduce some notations

$$\mathcal{F}^+ = \left\{u \in E : \int_\Omega F(x,u)dx > 0\right\}, \qquad \mathcal{F}^- = \left\{u \in E : \int_\Omega F(x,u)dx < 0\right\},$$
$$\mathcal{A}^+ = \left\{u \in E : \int_\Omega a(x)|u(x)|^q dx > 0\right\}, \quad \mathcal{A}^- = \left\{u \in E : \int_\Omega a(x)|u(x)|^q dx < 0\right\}.$$

Now we study the fiber map Φ_u according to the sign of $\int_\Omega a(x)|u(x)|^q dx$ and $\int_\Omega F(x,u)dx$.

Case 1. $u \in \mathcal{F}^- \cap \mathcal{A}^-$.
In this case $\Phi_u(0) = 0$ and $\Phi_u'(t) > 0$, $\forall t > 0$ which implies that Φ_u is strictly increasing and hence no critical point.

Case 2. $u \in \mathcal{F}^+ \cap \mathcal{A}^-$.
In this case, firstly we define $m_u : [0,\infty) \to \mathbb{R}$ by

$$m_u(t) = t^{p-q}||u||^p - t^{r-q}\int_\Omega F(x,u)dx.$$

Clearly, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if t is a solution of

$$m_u(t) = \lambda \int_\Omega a(x)|u(x)|^q dx.$$

As we have $m_u(t) \to -\infty$ as $t \to \infty$ and

$$m_u'(t) = (p-q)t^{p-q-1}||u||^p - (r-q)t^{r-q-1}\int_\Omega F(x,u)dx.$$

Therefore, $m_u'(t) > 0$ as $t \to 0$. Since $u \in \mathcal{A}^-$, there exists T such that $m_u(t) = \lambda\int_\Omega a(x)|u(x)|^q dx$. Thus, for $0 < t < T$, $\Phi_u'(t) = t^{q-1}\big(m_u(t) - \lambda\int_\Omega a(x)|u(x)|^q dx\big) > 0$ and for $t > T, \Phi_u'(t) < 0$. Hence, $\Phi_u(t)$ is increasing on $(0,T)$, decreasing on (T,∞). Since $\Phi_u(t) > 0$ for t close to 0 and $\Phi_u(t) \to -\infty$ as $t \to \infty$, we get Φ_u has exactly one critical point t_1, which is a global maximum point. Hence $t_1 u \in \mathcal{N}_\lambda^-$.

Case 3. $u \in \mathcal{F}^- \cap \mathcal{A}^+$.
In this case, $m_u(0) = 0$, $m_u'(t) > 0, \forall t > 0$, which implies that m_u is strictly increasing and since $u \in \mathcal{A}^+$, there exists a unique $t_1 > 0$ such that $m_u(t_1) = \lambda \int_\Omega a(x)|u(x)|^q dx$. This implies that Φ_u is decreasing on $(0, t_1)$, increasing on (t_1, ∞) and $\Phi_u'(t_1) = 0$. Thus, Φ_u has exactly one critical point t_1, corresponding to global minimum point. Hence $t_1 u \in \mathcal{N}_\lambda^+$.

Case 4. $u \in \mathcal{F}^+ \cap \mathcal{A}^+$.
In this case, we claim that there exists $\mu_0 > 0$ such that for $\lambda \in (0, \mu_0)$, Φ_u has exactly two critical points t_1 and t_2. Moreover, t_1 is a local minimum point and t_2 is a local maximum point. Thus $t_1 u \in \mathcal{N}_\lambda^+$ and $t_2 u \in \mathcal{N}_\lambda^-$. We prove this claim in the following Lemma:

Lemma 2.3. *There exists $\mu_0 > 0$ such that for $\lambda \in (0, \mu_0)$, Φ_u take positive value for all non-zero $u \in E$. Moreover, if $u \in \mathcal{F}^+ \cap \mathcal{A}^+$, then Φ_u has exactly two critical points.*

Proof. Let $u \in E$, define

$$M_u(t) = \frac{t^p}{p}||u||^p - \frac{t^r}{r}\int_\Omega F(x,u)dx.$$

Then

$$M_u'(t) = t^{p-1}||u||^p - t^{r-1}\int_\Omega F(x,u)dx$$

and M_u attains its maximum value at $T = \left(\frac{||u||^p}{\int_\Omega F(x,u)dx}\right)^{\frac{1}{r-p}}$. Moreover,

$$M_u(T) = \left(\frac{1}{p} - \frac{1}{r}\right)\left(\frac{||u||^r}{\int_\Omega F(x,u)dx}\right)^{\frac{p}{r-p}}$$

and

$$M_u''(T) = (p-r)\frac{||u||^{\frac{p(r-2)}{r-p}}}{\left(\int_\Omega F(x,u)dx\right)^{\frac{p-2}{r-p}}} < 0.$$

For $1 \le \nu < p_\alpha^*$ we denoted by S_ν be the Sobolev constant of embedding $E \hookrightarrow L^\nu(\Omega)$, then, by (1) we have

$$M_u(T) \ge \frac{r-p}{rp(KS_r^r)^{\frac{p}{r-p}}} = \delta, \tag{3}$$

which is independent of u. We now show that there exists $\mu_0 > 0$ such that $\Phi_u(T) > 0$. Using condition (A) and the Soblev imbedding, we get

$$\frac{T^q}{q}\int_\Omega a(x)|u(x)|^q dx \le \frac{S_q^q}{q}||u||^q T^q = \frac{S_q^q}{q}||u||^q\left(\frac{||u||^p}{\int_\Omega F(x,u)dx}\right)^{\frac{q}{r-p}} = \frac{S_q^q}{q}M_u(T)^{\frac{q}{p}}.$$

Thus

$$\Phi_u(T) = M_u(T) - \lambda \frac{T^q}{q} \int_\Omega a(x)|u(x)|^q dx \le M_u(T) - \lambda c M_u(T)^{\frac{q}{p}} = \delta^{\frac{q}{p}} \left(\delta^{\frac{p-q}{p}} - \lambda c \right),$$

where δ is the constant given in (3). Let

$$\mu_0 = \frac{q \delta^{\frac{p-q}{p}}}{S_q^q}.$$

Then, the choice of such μ_0 completes the proof. □

Corollary 2.4. *If $\lambda < \mu_0$, then there exists $\delta_1 > 0$ such that $J_\lambda(u) > \delta_1$ for all $u \in \mathcal{N}_\lambda^-$.*

Proof. Let $u \in \mathcal{N}_\lambda^-$, then Φ_u has a positive global maximum at $T = 1$ and $\int_\Omega a(x)|u(x)|^q dx > 0$. Thus, if $\lambda < \mu_0$, then we have

$$J_\lambda(u) = \Phi_u(1) \;=\; \Phi_u(T) \ge \delta^{\frac{q}{p}} \left(\delta^{\frac{p-q}{p}} - \lambda c \right) > 0,$$

where δ is the same as in Lemma 2.3, and so the result follows immediately. □

Lemma 2.5. *There exists μ_1 such that if $0 < \lambda < \mu_1$, then $\mathcal{N}_\lambda^0 = \varnothing$.*

Proof. Let

$$\mu_1 = \frac{r-p}{S_q^q (r-q)} \left(\frac{p-q}{K S_r^r (r-q)} \right)^{\frac{p-q}{r-p}},$$

where K is given by (1).

Suppose otherwise, that $0 < \lambda < \mu_1$ such that $\mathcal{N}_\lambda^0 \neq \varnothing$. Then, for $u \in \mathcal{N}_\lambda^0$, we have

$$0 = \Phi_u''(1) = (p-1)||u||^p - (r-1) \int_\Omega F(x,u) dx - \lambda (q-1) \int_\Omega a(x)|u(x)|^q dx.$$

So, it follows from (2) that $(r-p)||u||^p = \lambda(r-q) \int_\Omega a(x)|u|^q dx \le \lambda(r-q) S_q^q ||u||^q$, and so

$$||u|| \le \left(\lambda S_q^q \frac{r-q}{r-p} \right)^{\frac{1}{p-q}}. \tag{4}$$

On the other hand, by (1) we get $(p-q)||u||^p = \lambda(r-q) \int_\Omega F(x,u) dx \le K(r-q) S_r^r ||u||^r$, then

$$||u|| \ge \left(\frac{p-q}{K S_r^r (r-q)} \right)^{\frac{1}{r-p}}. \tag{5}$$

Combining (4) and (5) we obtain $\lambda \ge \mu_1$, which is a contradiction. □

Here and always, we define λ_0 as

$$\lambda_0 = \min(\mu_0, \mu_1). \tag{6}$$

We remark that if $0 < \lambda < \lambda_0$, then all the above Lemmas hold true.

Lemma 2.6. *Let u be a local minimizer for J_λ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$, then u is a critical point of J_λ.*

Proof. Since u is a minimizer for J_λ under the constraint $I_\lambda(u) :=\prec J'_\lambda(u), u \succ = 0$, by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $J'_\lambda(u) = \mu I'_\lambda(u)$. Thus

$$\prec J'_\lambda(u), u \succ = \mu \prec I'_\lambda(u), u \succ = \mu \Phi''_u(1) = 0,$$

but $u \notin \mathcal{N}_\lambda^0$ and so $\Phi''_u(1) \neq 0$. Hence $\mu = 0$. This completes the proof. □

3. Proof of our result

Throughout this section, we assume that the parameter λ satisfies $0 < \lambda < \lambda_0$, where λ_0 is the constant given by (6). That leads us consequently to the following results on the existence of minimizers in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$.

Lemma 3.1. *If $0 < \lambda < \lambda_0$, then J_λ achieves its minimum on $\mathcal{N}_\lambda^+$.*

Proof. Since J_λ is bounded below on $\mathcal{N}_\lambda$ and so on $\mathcal{N}_\lambda^+$, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u).$$

As J_λ is coercive on $\mathcal{N}_\lambda$, $\{u_k\}$ is a bounded sequence in E. Therefore, for all $1 \leq \nu < p_s^*$ we have

$$\begin{cases} u_k \rightharpoonup u_\lambda & \text{weakly} \quad \text{in } E \\ u_k \to u_\lambda & \text{strongly} \quad \text{in } L^\nu(\mathbb{R}^n). \end{cases}$$

If we choose $u \in E$ such that $\int_\Omega a(x)|u(x)|^q dx > 0$, then there exists $t_1 > 0$ such that $t_1 u \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1 u) < 0$, Hence, $\inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0$.

On the other hand, since $\{u_k\} \subset \mathcal{N}_\lambda$ we have

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{r}\right) ||u_k||^p - \lambda \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega a(x)|u_k(x)|^q dx,$$

and so $\lambda(\frac{1}{q} - \frac{1}{r}) \int_\Omega a(x)|u_k(x)|^q dx = (\frac{1}{p} - \frac{1}{r})||u_k||^p - J_\lambda(u_k)$. Letting k tends to infinity, we get

$$\int_\Omega a(x)|u_\lambda(x)|^q dx > 0. \tag{7}$$

Next we claim that $u_k \to u_\lambda$. Suppose this is not true, then

$$||u_\lambda||^p < \liminf_{k\to\infty} ||u_k||^p.$$

Since $\Phi'_{u_\lambda}(t_1) = 0$, it follows that $\Phi'_{u_k}(t_1) > 0$ for sufficiently large k. So, we must have $t_1 > 1$ but $t_1 u_\lambda \in \mathcal{N}_\lambda^+$ and so

$$J_\lambda(t_1 u_\lambda) < J_\lambda(u_\lambda) \leq \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),$$

which is a contradiction. Since $\mathcal{N}_\lambda^0 = \varnothing$, then $u_\lambda \in \mathcal{N}_\lambda^+$. Finally, u_λ is a minimizer for J_λ on $\mathcal{N}_\lambda^+$. □

Lemma 3.2. *If* $0 < \lambda < \lambda_0$*, then* J_λ *achieves its minimum on* $\mathcal{N}_\lambda^-$.

Proof. Let $u \in \mathcal{N}_\lambda^-$, then from Corollary 2.4, there exists $\delta_1 > 0$ such that $J_\lambda(u) \geq \delta_1$. So, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^-$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) > 0. \tag{8}$$

On the other hand, since J_λ is coercive, $\{u_k\}$ is a bounded sequence in E. Therefore, for all $1 \leq \nu < p_s^*$ we have

$$\begin{cases} u_k \rightharpoonup v_\lambda & \text{weakly} \quad \text{in } E \\ u_k \to v_\lambda & \text{strongly} \quad \text{in } L^\nu(\mathbb{R}^n). \end{cases}$$

Since $u \in \mathcal{N}_\lambda$, then we have

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{q}\right) ||u_k||^p + \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega F(x, u_k)dx. \tag{9}$$

Letting k goes to infinity, it follows from (8) and (9) that

$$\int_\Omega F(x, v_\lambda)dx > 0. \tag{10}$$

Hence, $v_\lambda \in \mathcal{F}^+$ and so Φ_{v_λ} has a global maximum at some point T and consequently, $Tv_\lambda \in \mathcal{N}_\lambda^-$. on the other hand, $u_k \in \mathcal{N}_\lambda^-$ implies that 1 is a global maximum point for Φ_{u_k}, i.e.

$$J_\lambda(tu_k) = \Phi_{u_k}(t) \leq \Phi_{u_k}(1) = J_\lambda(u_k). \tag{11}$$

Next we claim that $u_k \to u_\lambda$. Suppose this is not true, then

$$||u_\lambda||^p < \liminf_{k\to\infty} ||u_k||^p,$$

and from (11) it follows that

$$\begin{aligned} J_\lambda(Tv_\lambda) &= \frac{T^p}{p}||v_\lambda||^p - \frac{T^r}{r}\int_\Omega F(x,v_\lambda)dx - \lambda\frac{T^q}{q}\int_\Omega a(x)|v_\lambda|^q dx \\ &< \inf_{k\to\infty}\left(\frac{T^p}{p}||u_k||^p - \frac{T^r}{r}\int_\Omega F(x,u_k)dx - \lambda\frac{T^q}{q}\int_\Omega a(x)|u_k|^q dx\right) \\ &\le \lim_{k\to\infty} J_\lambda(Tu_k) \\ &\le \lim_{k\to\infty} J_\lambda(u_k) \\ &= \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u), \end{aligned}$$

which is a contradiction. Hence, $u_k \to v_\lambda$. Since $\mathcal{N}_\lambda^0 = \varnothing$, then $v_\lambda \in \mathcal{N}_\lambda^-$. □

Proof of Theorem 1.1. By Lemma 3.1 and Lemma 3.2, Problem (P) has two weak solutions $u_\lambda \in \mathcal{N}_\lambda^+$ and $v_\lambda \in \mathcal{N}_\lambda^+$. On the other hand, from (7) and (10), these solutions are nontrivial. Since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \varnothing$, then u_λ and v_λ are distinct. □

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