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A Characterization of Absolute Continuity by means of Mellin Integral Operators

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Abstract. In the case of classical convolution operators, an important characterization of absolute continuity is given in terms of convergence in variation. In this paper we will study this problem for Mellin integral operators, proving analogous characterizations in the frame of the classical BV-spaces, both in the one-dimensional and in the multidimensional setting.

Keywords. Mellin integral operators, absolute continuity, multidimensional variation, convergence in variation

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1. Introduction

Properties of convergence in variation for classical convolution integral operators play a fundamental role in order to establish characterizations of absolute continuity. Indeed it can be proved (see [12]) that $f \in AC(\mathbb{R})$ if and only if $\lim_{w\to+\infty} V[T_w f - f] = 0$, where $T_w f$ are classical convolution integral operators with regular (AC) kernels, V denotes the (Jordan) variation functional and $AC(\mathbb{R})$ is the space of the absolutely continuous functions on $\mathbb{R}$, i.e., the functions which are absolutely continuous on every interval $[a,b] \subset \mathbb{R}$ and of bounded variation on $\mathbb{R}$. This result has been also extended to the multidimensional case ([12]) using the Tonelli variation (see [28–30] for the definition). The problem has been studied also in the setting of the Musielak-Orlicz φ-variation defined for $g:[a,b]\to\mathbb{R}$ as $V^{\varphi}_{[a,b]}[g] := \sup_D \sum_{i=1}^{n} \varphi(|g(s_i) - g(s_{i-1})|)$, where the supremum is taken over all the divisions $D = \{s_0 = a, s_1, \ldots, s_n = b\}$ of $[a,b]$ and φ is a φ-function, both in the one-dimensional (see [27]) and in the multidimensional case (see [1, 4]). Similar results in the case of nonlinear

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convolution integral operators can be found in [2, 3, 15] and, in other settings, in [5, 13, 14, 17, 22, 31].

In case of Mellin convolution integral operators, the convergence in variation has been studied in [6–9] without giving a characterization in terms of absolute continuity. This is the problem that we face in this paper for the classical variation both in the one and in the multidimensional setting. The importance of Mellin integral operators is well-known not only in approximation theory (see, e.g., [18,26]), but also because of the several applications for example in optical physics and engineering (see, e.g., [16,21]). Indeed they can be successfully used in problems of signal reconstruction where the samples are not uniformly spaced, as in the classical Shannon Sampling Theorem, but exponentially spaced, e.g., in situations in which information accumulates near time $t = 0$ (see, e.g., [19]).

Even if Mellin integral operators, defined as

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}) f(\mathtt{st}) \langle \mathtt{t} \rangle^{-1} d\mathtt{t}, \quad w > 0,\ \mathtt{s} \in \mathbb{R}^N_+, \tag{I}$$

where $\mathtt{st} := (s_1 t_1, \ldots, s_N t_N)$ and $\langle \mathtt{t} \rangle := \prod_{i=1}^N t_i$, for $\mathtt{s} = (s_1, \ldots, s_N)$, $\mathtt{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N_+$, can be regarded as a kind of convolution integral operators with respect to a different measure (the logarithmic Haar measure $\mu(A) := \int_A \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}$, where A is a Borel subset of $\mathbb{R}^N_+$, $N \geq 1$), the group operation (multiplication) becomes the problem of the characterization of absolutely continuous functions not suitable to be studied through a "direct" approach. In particular it is not possible, for example, to prove directly via the definition of absolute continuity that the above operators (I) are AC.

In this paper we will show that it is possible to overcome the problem using a notion of absolute continuity (log-absolute continuity), equivalent to the classical one, which takes into account of the logarithmic measure μ. Indeed the absolute continuity of the Mellin integral operators, in case of AC-kernels (Propositions 3.7 and 4.4), together with the fact that the set of the absolutely continuous functions is a closed subspace of the set of the BV-functions, allows us to obtain the suitable characterizations for AC-functions (Theorems 3.8 and 4.6).

The paper is organized as follows: after a preliminary section (Section 2) where we introduce the Mellin integral operators under consideration and we present some examples, in Section 3 we give the characterization for AC-functions in case of the classical Jordan variation, while in Section 4 we face the problem in the multidimensional case using a concept of variation on $\mathbb{R}^N_+$ in the sense of Tonelli introduced in [9].

2. A class of Mellin operators

In this paper we will give results for a class of Mellin integral operators of the form (I) for

$$f \in \widetilde{L}^1(\mathbb{R}^N_+) := \left\{ f : \mathbb{R}^N_+ \to \mathbb{R} \quad \text{s.t.} \int_{\mathbb{R}^N_+} |f(\mathtt{t})| \langle \mathtt{t} \rangle^{-1} \, d\mathtt{t} < +\infty \right\},$$

where $\mathtt{st} := (s_1 t_1, \ldots, s_N t_N)$, for $\mathtt{s} = (s_1, \ldots, s_N), \mathtt{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N_+$. Since we work with Mellin type operators, the most suitable setting is to consider $\mathbb{R}^N_+$ as a group with the multiplicative operation and equipped with the logarithmic Haar measure $\mu(A) := \int_A \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}$, where $\langle \mathtt{t} \rangle := \prod_{i=1}^N t_i$. For the same reason, it is natural to consider functions belonging to the space $\widetilde{L}^1(\mathbb{R}^N_+)$, instead of the usual Lebesgue space $L^1(\mathbb{R}^N_+)$.

Here $\{K_w\}_{w>0}$ is a family of kernel functions which satisfy the following assumptions:

$\mathbf{K_w.1}$) $K_w : \mathbb{R}^N_+ \to \mathbb{R}$ is a measurable essentially bounded function such that $K_w \in \widetilde{L}^1(\mathbb{R}^N_+)$, $\|K_w\|_{\widetilde{L}^1} \leq A$ for an absolute constant $A > 0$ and $\int_{\mathbb{R}^N_+} K_w(\mathtt{t}) \langle \mathtt{t} \rangle^{-1} d\mathtt{t} = 1$, for every $w > 0$,

$\mathbf{K_w.2}$) for every fixed $0 < \delta < 1$, $\int_{|\mathbf{1}-\mathtt{t}|>\delta} |K_w(\mathtt{t})| \langle \mathtt{t} \rangle^{-1} d\mathtt{t} \to 0$, as $w \to +\infty$, where $\mathbf{1} := (1, \ldots, 1)$ is the unit vector of $\mathbb{R}^N_+$,

i.e., $\{K_w\}_{w>0}$ is a *bounded approximate identity* (see, e.g., [20]).

In the following we will say that $\{K_w\}_{w>0} \subset \mathcal{K}_w$ if $K_w.1$) and $K_w.2$) are fulfilled.

We point out that, with the above assumptions, $(T_w f)(\mathtt{s})$ is well defined for every $\mathtt{s} \in \mathbb{R}^N_+$ and $w > 0$, for $f \in \widetilde{L}^1(\mathbb{R}^N_+)$. Indeed, it is sufficient to notice that

$$|(T_w f)(\mathtt{s})| \leq \int_{\mathbb{R}^N_+} |K_w(\mathtt{t})| |f(\mathtt{st})| \langle \mathtt{t} \rangle^{-1} \, d\mathtt{t} \leq \|K_w\|_{L^\infty} \|f\|_{\widetilde{L}^1} < +\infty,$$

for every $\mathtt{s} \in \mathbb{R}^N_+$, $w > 0$. Note that, in the particular case $N = 1$, it is no more necessary to assume that $f \in \widetilde{L}^1(\mathbb{R}^N_+)$ in order to guarantee that the operators are well defined (see Proposition 3.4).

It is not difficult to find examples of kernel functions which fulfill all the previous assumptions. Among them there are, for example, the moment-type kernels, defined as

$$M_w(\mathtt{t}) := w^N \langle \mathtt{t} \rangle^w \chi_{]0,1[^N}(\mathtt{t}), \quad \mathtt{t} \in \mathbb{R}^N_+, \; w > 0.$$

First of all, it is easy to see that they fulfill assumption $K_w.1$). Moreover, for every fixed $\delta \in]0, 1[$, $|\mathbf{1} - \mathtt{t}| > \delta$ implies that $|1 - t_j| > \frac{\delta}{\sqrt{N}}$, for some $j = 1, \ldots, N$;

hence

$$\left\{\mathtt{t}\in]0,1[^N:\ |\mathbf{1}-\mathtt{t}|>\delta\right\} \subset \bigcup_{j=1}^{N} \left\{\mathtt{t}\in\mathbb{R}^N_+:\ 0<t_j<1-\frac{\delta}{\sqrt{N}},\ 0<t_i<1,\ \forall i\neq j\right\}$$

and therefore

$$\begin{aligned} I_w := & \int_{|\mathbf{1}-\mathtt{t}|>\delta} |M_w(\mathtt{t})|\langle \mathtt{t}\rangle^{-1}\, d\mathtt{t} \\ \leq & \sum_{j=1}^{N} \left\{ \left(\prod_{i\neq j} \int_0^1 w t_i^{w-1}\, dt_i \right) \int_0^{1-\frac{\delta}{\sqrt{N}}} w t_j^{w-1}\, dt_j \right\} \\ = & N\left(1-\frac{\delta}{\sqrt{N}}\right)^w \to 0, \end{aligned}$$

as $w\to+\infty$.

Other examples of kernel functions which satisfy our assumptions are the Mellin-Gauss-Weierstrass kernels, defined as

$$G_w(\mathtt{t}) := \frac{w^N}{\pi^{\frac{N}{2}}} e^{-w^2|\log \mathtt{t}|^2}, \quad \mathtt{t}\in\mathbb{R}^N_+,\ w>0,$$

where $\log \mathtt{t} := (\log t_1, \ldots, \log t_N)$, or the Mellin-Picard kernels, defined as

$$P_w(\mathtt{t}) := \frac{w^N}{2\pi^{\frac{N}{2}}} \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} e^{-w|\log \mathtt{t}|}, \quad \mathtt{t}\in\mathbb{R}^N_+,\ w>0,$$

where Γ is the Gamma-Euler function. These last families are examples of kernel functions of Fejér type, namely of the form

$$K_w(\mathtt{t}) = w^N K(\mathtt{t}^w), \quad \mathtt{t}\in\mathbb{R}^N_+,\ w>0,$$

where $K\in\widetilde{L}^1(\mathbb{R}^N_+)$ is such that $\int_{\mathbb{R}^N_+} K(\mathtt{t})\langle\mathtt{t}\rangle^{-1}\, d\mathtt{t} = 1$. For Fejér type kernels it can be proved (see [9]) that all the previous assumptions are implied by the classical condition that their absolute moments of order α ($\alpha>0$), defined in this setting as

$$m(K,\alpha) := \int_{\mathbb{R}^N_+} |\log \mathtt{t}|^\alpha |K(\mathtt{t})|\langle\mathtt{t}\rangle^{-1}\, d\mathtt{t},$$

are finite. Therefore, since the absolute moments of order α of both the families $\{G_w\}_{w>0}$ and $\{P_w\}_{w>0}$ are finite (see [9]), they satisfy the above assumptions.

3. The one-dimensional case

Before treating the problem in the multidimensional frame, we study the one-dimensional case since the multidimensional variation requires a different approach and, moreover, it lays on the one-dimensional setting.

3.1. Definitions. In this section we study the problem in the one-dimensional case, namely in the frame of $BV(\mathbb{R}_+)$, the space of functions with bounded (Jordan) variation on $\mathbb{R}_+$. In particular we will give a characterization of the absolute continuity for the class of Mellin integral operators (I) in the one-dimensional case, i.e.,

$$(T_w f)(s) = \int_{\mathbb{R}_+} K_w(t) f(st) t^{-1}\, dt, \quad w > 0,\ s > 0,$$

for $f \in \widetilde{L}^1(\mathbb{R}_+)$ and where the family of kernels $\{K_w\}_{w>0}$ is a bounded approximate identity.

We first recall some well-known definitions about the BV-spaces.

Definition 3.1. The (Jordan) *variation* of a function $f : [a,b] \to \mathbb{R}$ on $[a,b] \subset \mathbb{R}_+$ is defined as

$$V_{[a,b]}[f] := \sup_D \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all the possible divisions $D = \{a = x_0, x_1, \dots, x_m = b\}$ of the interval $[a,b]$ and f is said to be *of bounded variation* on $[a,b]$ if $V_{[a,b]}[f] < \infty$. If $f : \mathbb{R}_+ \to \mathbb{R}$, $V[f] := \sup_{[a,b]\subset\mathbb{R}_+} V_{[a,b]}[f]$ is the variation of f over the whole space $\mathbb{R}_+$ and by

$$BV(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{R} :\ V[f] < +\infty\}$$

we denote the *space of functions of bounded variation on* $\mathbb{R}_+$.

Definition 3.2. A function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every collection of non-overlapping intervals $[\alpha^\nu, \beta^\nu]$, $\nu = 1, \dots, n$ in $[a,b]$ such that $\sum_{\nu=1}^{n} |\beta^\nu - \alpha^\nu| < \delta$, then

$$\sum_{\nu=1}^{n} |f(\beta^\nu) - f(\alpha^\nu)| < \varepsilon.$$

By $AC(\mathbb{R}_+)$ (space of *absolutely continuous* functions) we will denote the space of functions $f : \mathbb{R}_+ \to \mathbb{R}$ which are of bounded variation on $\mathbb{R}_+$ and absolutely continuous on every $[a,b] \subset \mathbb{R}_+$.

Since we work with Mellin integral operators, in order to reach our results, we have to deal with the logarithmic Haar measure. Therefore, for the sake of simplicity, we use the following notion of absolute continuity (*logarithmic* or *log-absolute continuity*), compatible with the multiplicative structure of $\mathbb{R}_+^N$, which is equivalent to the classical one (see Proposition 3.5).

Definition 3.3. We say that $f : \mathbb{R}_+ \to \mathbb{R}$ is *log-absolutely continuous* on $[a,b] \subset \mathbb{R}_+$ ($f \in AC_{\log}([a,b])$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every collection of nonoverlapping intervals $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^n$ in $[a,b]$ such that

$$\sum_{\nu=1}^{n} |\log(\beta^\nu) - \log(\alpha^\nu)| < \delta,$$

then

$$\sum_{\nu=1}^{n} |f(\beta^\nu) - f(\alpha^\nu)| < \varepsilon.$$

By $AC_{\log}(\mathbb{R}_+)$ (space of *log-absolutely continuous* functions) we will denote the space of functions which are of bounded variation on $\mathbb{R}_+$ and log-absolutely continuous on every $[a,b] \subset \mathbb{R}_+$.

3.2. Results. We first prove that the operators (I) map $BV(\mathbb{R}_+)$ into itself: this obviously also implies that, if $f \in BV(\mathbb{R}_+)$, then $(T_w f)(s) < +\infty$, for every $s, w > 0$.

Proposition 3.4. *If* $f \in BV(\mathbb{R}_+)$ *and* $\{K_w\}_{w>0}$ *satisfy assumption* $\mathrm{K_w}.1)$, *then* $V[T_w f] \le AV[f]$, *for every* $w > 0$, *where* A *is the constant in* $\mathrm{K_w}.1)$, *i.e.,* $\{T_w f\}_{w>0}$ *are equibounded in variation.*

Proof. If $f \in BV(\mathbb{R}_+)$ and $\{s_0 = a, s_1, \ldots, s_n = b\}$ is a partition of $[a,b] \subset \mathbb{R}_+$, then

$$\begin{aligned}\sum_{i=1}^{n} |(T_w f)(s_i) - (T_w f)(s_{i-1})| &= \sum_{i=1}^{n} \left| \int_{\mathbb{R}_+} K_w(t) f(s_i t) \frac{dt}{t} - \int_{\mathbb{R}_+} K_w(t) f(s_{i-1} t) \frac{dt}{t} \right| \\ &\le \int_{\mathbb{R}_+} |K_w(t)| \sum_{i=1}^{n} |f(s_i t) - f(s_{i-1} t)| \frac{dt}{t} \\ &\le \int_{\mathbb{R}_+} |K_w(t)| V_{[a,b]}[f(t\cdot)] \frac{dt}{t} \\ &\le \int_{\mathbb{R}_+} |K_w(t)| V[f] \frac{dt}{t},\end{aligned}$$

and hence, passing to the supremum over all the partitions of $[a,b]$, and by $\mathrm{K_w}.1)$, $V_{[a,b]}[T_w f] \le AV[f]$. Therefore, by the arbitrariness of $[a,b] \subset \mathbb{R}_+$,

$$V[T_w f] \le AV[f]. \tag{1}$$

This finishes the proof. □

We now prove that the concept of log-absolute continuity is equivalent to the classical absolute continuity.

Proposition 3.5. *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is absolutely continuous ($f \in AC(\mathbb{R}_+)$) if and only if f is log-absolutely continuous ($f \in AC_{log}(\mathbb{R}_+)$).*

Proof. We have to prove that f is absolutely continuous on $[a, b] \subset \mathbb{R}_+$ if and only if f is log-absolutely continuous on $[a, b]$. Let us fix $\varepsilon > 0$. If f is absolutely continuous on $[a, b]$, it is easy to see that, if $\eta : [c, d] \to [a, b]$ is absolutely continuous and strictly increasing, then $(f \circ \eta)$ is absolutely continuous on $[c, d]$. Hence, if we take $\eta(x) := e^x$ and a collection $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^n$ of nonoverlapping intervals in $[a, b]$ such that $\sum_{\nu=1}^n |\log(\beta^\nu) - \log(\alpha^\nu)| < \delta$, where δ is the constant of the absolute continuity of $(f \circ \eta)$ on $[\log a, \log b]$, then $\{[a^\nu := \log(\alpha^\nu), b^\nu := \log(\beta^\nu)]\}_{\nu=1}^n$ is a collection of nonoverlapping intervals in $[\log a, \log b]$ and therefore

$$\sum_{\nu=1}^n |f(\beta^\nu) - f(\alpha^\nu)| = \sum_{\nu=1}^n |(f \circ \eta)(a^\nu) - (f \circ \eta)(b^\nu)| < \varepsilon,$$

that is, f is log-absolutely continuous on $[a, b]$.

For the converse, let $\widetilde{\delta} > 0$ be the constant of the log-absolute continuity of f on $[a, b]$ and δ the constant of the absolute continuity of the function $\log x$ on $[a, b]$ in correspondence to $\bar{\varepsilon} := \widetilde{\delta}$. Then, if $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^n$ is a collection of nonoverlapping intervals in $[a, b]$ such that $\sum_{\nu=1}^n (\beta^\nu - \alpha^\nu) < \delta$, then $\sum_{\nu=1}^n \big(\log(\beta^\nu) - \log(\alpha^\nu)\big) < \widetilde{\delta}$ and hence, by the log-absolute continuity of f,

$$\sum_{\nu=1}^n |f(\beta^\nu) - f(\alpha^\nu)| < \varepsilon. \qquad \square$$

We can now prove that Mellin integral operators, as the classical convolution operators, preserve absolute continuity both of the function and of the kernel functions.

Proposition 3.6. *If $f \in AC(\mathbb{R}_+)$ and $\{K_w\}_{w>0}$ satisfies assumption* $\mathrm{K_w}.1$)*, then $T_w f \in AC(\mathbb{R}_+)$.*

Proof. Let us fix $\varepsilon > 0$ and $[a, b] \subset \mathbb{R}_+$. By Proposition 3.5, we know that f is log-absolutely continuous on $[a, b]$: let us consider a collection $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^m$ of nonoverlapping intervals in $[a, b]$ such that $\sum_{\nu=1}^m (\log(\beta^\nu) - \log(\alpha^\nu)) < \delta$, where δ is the number of the logarithmic absolute continuity of f on $[a, b]$ in correspondence to $\frac{\varepsilon}{A}$ and A is the constant of assumption $\mathrm{K_w}.1$). Then there holds

$$\sum_{\nu=1}^m |(T_w f)(\beta^\nu) - (T_w f)(\alpha^\nu)| \ \leq \ \int_{\mathbb{R}_+} |K_w(t)| \sum_{\nu=1}^m |f(\beta^\nu t) - f(\alpha^\nu t)| \frac{dt}{t}.$$

Since $\sum_{\nu=1}^{m}(\log(\beta^\nu t) - \log(\alpha^\nu t)) < \delta$, by the logarithmic absolute continuity of f on $[a,b]$, $\sum_{\nu=1}^{m} |f(\beta^\nu t) - f(\alpha^\nu t)| < \frac{\varepsilon}{A}$, and hence

$$\sum_{\nu=1}^{m} |(T_w f)(\beta^\nu) - (T_w f)(\alpha^\nu)| < \frac{\varepsilon}{A} \int_{\mathbb{R}_+} |K_w(t)| \frac{dt}{t} \leq \varepsilon.$$

This proves that $T_w f$ is log-absolutely continuous and so, again by Proposition 3.5, $T_w f$ is absolutely continuous on $[a,b]$. Hence the result is proved taking into account that, by Proposition 3.4, $T_w f \in BV(\mathbb{R}_+)$, since $f \in BV(\mathbb{R}_+)$. □

Proposition 3.7. *If $f \in \widetilde{L}^1(\mathbb{R}_+) \cap BV(\mathbb{R}_+)$ and $K_w \in AC(\mathbb{R}_+)$, then $T_w f \in AC(\mathbb{R}_+)$.*

Proof. We will first prove that $T_w f$ is log-absolutely continuous on every interval $[a,b] \subset \mathbb{R}_+$, since this implies, by Proposition 3.5, that $T_w f$ is absolutely continuous on $[a,b]$. Let us fix $\varepsilon > 0$, $[a,b] \subset \mathbb{R}_+$, and a collection $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^{n}$ of nonoverlapping intervals in $[a,b]$ such that $\sum_{\nu=1}^{n}(\log(\beta^\nu) - \log(\alpha^\nu)) < \delta$, where δ is the number of the logarithmic absolute continuity of K_w on $[a,b]$ in correspondence to $\frac{\varepsilon}{\|f\|_{\widetilde{L}^1}}$ (without any loss of generality we assume that $\|f\|_{\widetilde{L}^1} \neq 0$, since the other case is trivial). Taking into account that

$$(T_w f)(s) = \int_{\mathbb{R}_+} K_w(t) f(st) \frac{dt}{t} = \int_{\mathbb{R}_+} K_w\left(\frac{t}{s}\right) f(t) \frac{dt}{t},$$

there holds

$$\sum_{\nu=1}^{n} |(T_w f)(\beta^\nu) - (T_w f)(\alpha^\nu)| \leq \int_{\mathbb{R}_+} \sum_{\nu=1}^{n} \left| K_w\left(\frac{t}{\beta^\nu}\right) - K_w\left(\frac{t}{\alpha^\nu}\right) \right| |f(t)| \frac{dt}{t}. \quad (2)$$

Now, since $\sum_{\nu=1}^{n} \left| \log\left(\frac{t}{\beta^\nu}\right) - \log\left(\frac{t}{\alpha^\nu}\right) \right| < \delta$, by the logarithmic absolute continuity of K_w on $[a,b]$,

$$\sum_{\nu=1}^{n} \left| K_w\left(\frac{t}{\beta^\nu}\right) - K_w\left(\frac{t}{\alpha^\nu}\right) \right| < \frac{\varepsilon}{\|f\|_{\widetilde{L}^1}},$$

and hence $\sum_{\nu=1}^{n} |(T_w f)(\beta^\nu) - (T_w f)(\alpha^\nu)| \leq \varepsilon$. Finally, by (1), $T_w f \in BV(\mathbb{R}_+)$, since $f \in BV(\mathbb{R}_+)$, and therefore $T_w f \in AC(\mathbb{R}_+)$. □

The previous results show that Mellin integral operators have some important properties of the classical convolution integral operators, i.e., they preserve absolute continuity. By means of this property we are now able to prove that the absolute continuity of the function is a necessary and sufficient condition for the convergence in variation of Mellin integral operators. In other words, as it happens for the convolution operators (see, e.g., [12]), the absolute continuity is a characterization of the convergence in variation.

Theorem 3.8. *Let* $\{K_w\}_{w>0} \subset \mathcal{K}_w \cap AC(\mathbb{R}_+)$ *and* $f \in \widetilde{L}^1(\mathbb{R}_+) \cap BV(\mathbb{R}_+)$. *Then* $f \in AC(\mathbb{R}_+)$ *if and only if* $\lim_{w\to+\infty} V[T_w f - f] = 0$.

Proof. By Proposition 3.7, the Mellin integral operators $\{T_w f\}_{w>0}$ turn out to be AC on $\mathbb{R}_+$ and hence, since $AC(\mathbb{R}_+)$ is a closed subspace of $BV(\mathbb{R}_+)$ with respect to the variation functional, the convergence in variation of $T_w f$ to f implies the absolute continuity of f. The converse implication is a particular case (for $N = 1$) of [9, Theorem 2]. □

4. The multidimensional case

In this section we will show that the results of Section 3 can be extended to the multidimensional setting, using the concepts of multidimensional variation and absolute continuity in the sense of Tonelli introduced in [9]. We now recall the definitions.

4.1. Definitions. Let $f : \mathbb{R}^N_+ \to \mathbb{R}$ and $\mathtt{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N_+$, $N \in \mathbb{N}$. We put $\mathtt{x}'_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}_+$, so that $\mathtt{x} = (\mathtt{x}'_j, x_j)$ and $f(\mathtt{x}) = f(\mathtt{x}'_j, x_j)$. If $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N_+$, $[\mathtt{a}'_j, \mathtt{b}'_j]$ denotes the $(N-1)$-dimensional interval obtained deleting by I the j-th coordinate, so that

$$I = [\mathtt{a}'_j, \mathtt{b}'_j] \times [a_j, b_j].$$

In order to compute the variation of a function $f \in \widetilde{L}^1(\mathbb{R}^N_+)$, we first consider the euclidean norm of $(\Phi_1(f, I), \ldots, \Phi_N(f, I))$, namely

$$\Phi(f, I) := \left\{ \sum_{j=1}^{N} [\Phi_j(f, I)]^2 \right\}^{\frac{1}{2}},$$

where

$$\Phi_j(f, I) := \int_{\mathtt{a}'_j}^{\mathtt{b}'_j} V_{[a_j, b_j]}[f(\mathtt{x}'_j, \cdot)] \frac{d\mathtt{x}'_j}{\langle \mathtt{x}'_j \rangle},$$

and $\langle \mathtt{x}'_j \rangle$ denotes the product $\prod_{i=1, i\neq j}^N x_i$. We put $\Phi(f, I) = +\infty$ if $\Phi_j(f, I) = +\infty$ for some $j = 1, \ldots, N$.

The multidimensional variation of f on an interval $I \subset \mathbb{R}^N_+$ is defined as

$$V_I[f] := \sup \sum_{i=1}^{m} \Phi(f, J_i),$$

where the supremum is taken over all the finite families of N-dimensional intervals $\{J_1, \ldots, J_m\}$ which form partitions of I.

The variation of f over $\mathbb{R}_+^N$ is defined as

$$V[f] := \sup_{I \subset \mathbb{R}_+^N} V_I[f],$$

where the supremum is taken over all the intervals $I \subset \mathbb{R}_+^N$.

Definition 4.1. By $BV(\mathbb{R}_+^N) = \{f \in \widetilde{L}^1(\mathbb{R}_+^N) : \ V[f] < +\infty\}$ we denote the *space of functions of bounded variation* on $\mathbb{R}_+^N$.

Definition 4.2. A function $f : \mathbb{R}_+^N \to \mathbb{R}$ is *absolutely continuous* on $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$ if for every $j = 1, 2, \dots, N$, the j-th sections of f, $f(\mathtt{x}'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$, are (uniformly) absolutely continuous for almost every $\mathtt{x}'_j \in \mathbb{R}_+^{N-1}$ (see [11, 23]). This means that the following property holds:

for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{\nu=1}^{n} |f(\mathtt{x}'_j, \beta^\nu) - f(\mathtt{x}'_j, \alpha^\nu)| < \varepsilon,$$

for a.e. $\mathtt{x}'_j \in \mathbb{R}_+^{N-1}$ and for all finite collections of non-overlapping intervals $[\alpha^\nu, \beta^\nu] \subset [a_j, b_j]$, $\nu = 1, \dots, n$, for which $\sum_{\nu=1}^n (\beta^\nu - \alpha^\nu) < \delta$.

Now, by $AC(\mathbb{R}_+^N)$ (space of *absolutely continuous functions* on $\mathbb{R}_+^N$) we denote the space of functions $f : \mathbb{R}_+^N \to \mathbb{R}$ which are of bounded variation and absolutely continuous on every $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$.

As in Section 3, we now introduce the multidimensional version of the log-absolute continuity.

Definition 4.3. We say that $f : \mathbb{R}_+^N \to \mathbb{R}$ is *log-absolutely continuous* on $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$ if for every $j = 1, 2, \dots, N$, the j-th sections of f, $f(\mathtt{x}'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$, are (uniformly) log-absolutely continuous for almost every $\mathtt{x}'_j \in \mathbb{R}_+^{N-1}$.

4.2. Results. First of all let us point out that, as an immediate consequence of the one-dimensional case, the absolute continuity is equivalent to the log-absolute continuity: indeed it is sufficient to apply Proposition 3.5 to the sections of the function f. This fact will be important in order to prove the next results.

Proposition 4.4. *If $f \in BV(\mathbb{R}_+^N)$ and $K_w \in AC(\mathbb{R}_+^N)$, then $T_w f \in AC(\mathbb{R}_+^N)$.*

Proof. First of all, by [9, Proposition 1], $T_w f \in BV(\mathbb{R}_+^N)$, since $f \in BV(\mathbb{R}_+^N)$. In order to prove that $T_w f$ is absolutely continuous on every $I = \prod_{i=1}^N [a_i, b_i] \subset R_+^N$, let us fix $\varepsilon > 0$, and a collection $\{[\alpha^\nu, \beta^\nu]\}_{\nu=1}^n$ of nonoverlapping intervals in $[a_i, b_i]$ such that $\sum_{\nu=1}^n (\log(\beta^\nu) - \log(\alpha^\nu)) < \delta$, where δ is the number of the log-absolute continuity of $K_w(\mathtt{x}'_i, \cdot)$ in correspondence to $\frac{\varepsilon}{\|f\|_{\widetilde{L}^1}}$ (again without any loss of generality we assume that $\|f\|_{\widetilde{L}^1} \neq 0$, since the other case is trivial), a.e. $\mathtt{x}'_i \in \mathbb{R}^{N-1}_+$. Similarly to (2) of Proposition 3.7 we can write

$$\sum_{\nu=1}^{n} |(T_w f)(\mathtt{x}'_i, \beta^\nu) - (T_w f)(\mathtt{x}'_i, \alpha^\nu)| \leq \int_{\mathbb{R}^N_+} |f(\mathtt{t})| \sum_{\nu=1}^{n} \left| K_w\left(\frac{\mathtt{t}'_i}{\mathtt{x}'_i}, \frac{t_i}{\beta^\nu}\right) - K_w\left(\frac{\mathtt{t}'_i}{\mathtt{x}'_i}, \frac{t_i}{\alpha^\nu}\right)\right| \frac{d\mathtt{t}}{\langle \mathtt{t} \rangle}.$$

Since $\sum_{\nu=1}^{n} \left| \log\left(\frac{t_i}{\beta^\nu}\right) - \log\left(\frac{t_i}{\alpha^\nu}\right)\right| < \delta$, by the logarithmic absolute continuity of K_w on I,

$$\sum_{\nu=1}^{n} \left| K_w\left(\frac{\mathtt{t}'_i}{\mathtt{x}'_i}, \frac{t_i}{\beta^\nu}\right) - K_w\left(\frac{\mathtt{t}'_i}{\mathtt{x}'_i}, \frac{t_i}{\alpha^\nu}\right)\right| < \frac{\varepsilon}{\|f\|_{\widetilde{L}^1}},$$

which implies that $\sum_{\nu=1}^{n} |(T_w f)(\mathtt{x}'_i, \beta^\nu) - (T_w f)(\mathtt{x}'_i, \alpha^\nu)| < \varepsilon$, a.e. $\mathtt{x}'_i \in \mathbb{R}^{N-1}_+$, namely, $(T_w f)(\mathtt{x}'_i, \cdot)$ is log-absolutely continuous on I, and hence, by Proposition 3.5, absolutely continuous. □

In an analogous way, the proof of Proposition 3.6 can be extended to the multidimensional case; hence we have the following:

Proposition 4.5. *If $f \in AC(\mathbb{R}^N_+)$ and $\{K_w\}_{w>0}$ satisfies assumption* $\mathrm{K_w}$.1)*, then $T_w f \in AC(\mathbb{R}^N_+)$.*

Exactly as in the one-dimensional case, Proposition 4.4 guarantees that, also in $\mathbb{R}^N_+$, the absolute continuity of the function is equivalent to the convergence in variation of the Mellin integral operators.

Theorem 4.6. *Let $\{K_w\}_{w>0} \subset \mathcal{K}_w \cap AC(\mathbb{R}^N_+)$ and $f \in BV(\mathbb{R}^N_+)$. Then $f \in AC(\mathbb{R}^N_+)$ if and only if $\lim_{w\to+\infty} V[T_w f - f] = 0$.*

Proof. The proof is analogous to the one-dimensional case, taking into account of [9, Theorem 2] for the sufficient part and, for the necessary one, of Proposition 4.4 and of the fact that $AC(\mathbb{R}^N_+)$ is a closed subspace of $BV(\mathbb{R}^N_+)$ (see [10, Proposition 5.3]). □

Remark 4.7. We point out that the assumption that $f \in \widetilde{L}^1(\mathbb{R}^N_+)$ is needed in order to prove Proposition 4.4 (and also Proposition 3.7 in the one-dimensional case) and therefore to furnish the characterization of $AC(\mathbb{R}^N_+)$. In general, in order to obtain just the convergence in variation for the operators (I) or to define the multidimensional variation, alternatively, one could denote by

$$\mathcal{D} := \{f : \mathbb{R}^N_+ \to \mathbb{R} : \ f \text{ is measurable and } (T_w f)(\mathtt{s}) < +\infty, \ \forall \mathtt{s} \in \mathbb{R}^N_+, \ w > 0\}$$

the domain of $\{T_w\}_{w>0}$, and define the space BV as

$$BV(\mathbb{R}^N_+) := \{f \in \mathcal{D} : \ V[f] < +\infty\}.$$

We notice that, in this case, it is not necessary to assume that the kernels are essentially bounded. We finally point out that the domain $\mathcal{D}$ is not trivial: as an example, it contains all the essentially bounded functions. Nevertheless, we remark that the space of the functions with bounded multidimensional variation is in general defined as a subspace of L^1, also taking into account of the distributional variation (see, e.g., [24, 25]).

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