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Uniform Exponential Stability of Discrete Semigroup and Space of Asymptotically Almost Periodic Sequences

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Abstract. We prove that the discrete semigroup $\mathbb{T} = \{\mathcal{T}(n) : n \in \mathbb{Z}_+\}$ is uniformly exponentially stable if and only if for each $z(n) \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ the solution of the Cauchy problem

$$\begin{cases} y_{n+1} = \mathcal{T}(1)y_n + z(n+1), \\ y(0) = 0 \end{cases}$$

belongs to $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$. Where $\mathcal{T}(1)$ is the algebraic generator of $\mathbb{T}$, $\mathbb{Z}_+$ is the set of all non-negative integers and $\mathcal{X}$ is a complex Banach space. Our proof uses the approach of discrete evolution semigroups.

Keywords. Exponential stability, discrete semigroups, periodic sequences, almost periodic sequences

Mathematics Subject Classification (2010). 35D35, 47D03, 47D06

1. Introduction

Let A be a bounded linear operator acting on a complex Banach space $\mathcal{X}$. A well known theorem of M. G. Krein [10, 13] says that the system $\dot{x}(t) = Ax(t)$ is uniformly exponentially stable if and only if for each $\mu \in \mathbb{R}$ and each $y_0 \in \mathcal{X}$ the solution of the Cauchy problem

$$\begin{cases} \dot{y}(t) = Ay(t) + e^{i\mu t}y_0, \\ y(0) = 0 \end{cases}$$

is bounded. The proof of this classic result can be found in [1]. This result can also be extended for strongly continuous bounded semigroups, see [17].

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Under a slightly different assumption the result on stability remains valid for any strongly continuous semigroups acting on complex Hilbert spaces, see for example [18, 19] and references therein. See also, [11], for counter-examples. In [7, 21] the same result were extended for square size matrices in both continuous and discrete cases.

Recently Zada et al. [22] proved that the system $x_{n+1} = \mathcal{T}(1)x_n$ is uniformly exponentially stable if and only if for each q-periodic bounded sequence $z(n)$ with $z(0) = 0$ the solution of the Cauchy problem

$$\begin{cases} y_{n+1} = \mathcal{T}(1)y_n + z(n+1), \\ y(0) = 0 \end{cases}$$

is bounded.

In this article we extended the result of last quoted paper to the space of asymptotically almost periodic sequences denoted by $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$. Similar results in the continuous case can be found in [3, 5, 6, 15] and in the discrete case can be seen in [4, 12, 20, 23]. For the periodic and almost periodic sequences we recommend [2, 9].

2. Notations and Preliminaries

Let $\mathcal{X}$ be a real or complex Banach space and $\mathcal{B}(\mathcal{X})$ the Banach algebra of all linear and bounded operators acting on $\mathcal{X}$.

We denote by $\|\cdot\|$ the norms of operators and vectors. Denote by $\mathbb{R}_+$ the set of real numbers, by $\mathbb{Z}$ the set of all integers and by $\mathbb{Z}_+$ the set of non-negative integers. Let us consider the following spaces

- $\mathbb{B}(\mathbb{Z}, \mathcal{X})$ is the spaces of all $\mathcal{X}$-valued bounded sequences with the supremum norm.
- $\mathbb{C}_0(\mathbb{Z}, \mathcal{X})$ is the sub space of $\mathbb{B}(\mathbb{Z}, \mathcal{X})$ consisting of all $\mathcal{X}$-valued sequences $z(n)$ such that $\lim_{|n|\to\infty} |z(n)| = 0$.
- $\mathbb{P}^q(\mathbb{Z}, \mathcal{X})$ is the space of q-periodic $\mathcal{X}$-valued (with $q \geq 2$) sequences $z(n)$.
- $\mathbb{P}_0^q(\mathbb{Z}, \mathcal{X}) \subseteq \mathbb{P}^q(\mathbb{Z}, \mathcal{X})$, with $z(0) = 0$.
- $\mathbb{AP}(\mathbb{Z}, \mathcal{X})$ is the space of all $\mathcal{X}$-valued almost periodic sequences.

Throughout this paper, $\mathcal{A} \in \mathcal{B}(\mathcal{X})$, $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$, and $r(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$ denotes the spectral radius of $\mathcal{A}$. It is well known that $r(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|^{\frac{1}{n}}$. The resolvent set of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) := \mathbb{C}\backslash\sigma(\mathcal{A})$, i.e., the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{A} - \lambda I$ is an invertible operator in $\mathcal{B}(\mathcal{X})$.

We give some results in the framework of general Banach spaces and spaces of sequences as defined above.

The operator $\mathcal{A}$ is said to be power bounded if there exists a positive constant M such that $\|\mathcal{A}^n\| \leq M$ for all $n \in \mathbb{Z}_+$.

We recall that the discrete semigroup is a family $\mathbb{T} = \{\mathcal{T}(n) : n \in \mathbb{Z}_+\}$ of bounded linear operators acting on X which satisfies the following conditions

(1) $\mathcal{T}(0) = I$, the identity operator on X,

(2) $\mathcal{T}(n+m) = \mathcal{T}(n)\mathcal{T}(m)$ for all $n, m \in \mathbb{Z}_+$.

Let $\mathcal{T}(1)$ denote the algebraic generator of the semigroup $\mathbb{T}$. Then it is clear that $\mathcal{T}(n) = \mathcal{T}^n(1)$ for all $n \in \mathbb{Z}_+$. The growth bound of $\mathbb{T}$ is denoted by $\omega_0(\mathbb{T})$ and is defined as

$$\omega_0(\mathbb{T}) := \inf\{\omega \in \mathbb{R} : \exists\, M_\omega > 0 \text{ such that } \|\mathcal{T}(n)\| \leq M_\omega e^{\omega n} \text{ for all } n \in \mathbb{Z}_+\}.$$

The family $\mathbb{T}$ is uniformly exponentially stable if $\omega_0(\mathbb{T})$ is negative, or equivalently, if there exist two positive constants M and ω such that $\|\mathcal{T}(n)\| \leq Me^{-\omega n}$ for all $n \in \mathbb{Z}_+$.

We recall the following lemma without proof from [1], so that the paper will be self contained.

Lemma 2.1. *Let $A \in \mathcal{L}(X)$. If*

$$\sup_{n\in\mathbb{Z}_+} \Big\| \sum_{k=0}^{n} e^{i\mu k} A^k \Big\| = M_\mu < \infty, \quad \textit{for all } \mu \in \mathbb{R},$$

holds, then $r(A) < 1$.

Given $\mathcal{T}(1) \in \mathcal{B}(\mathcal{X})$ is the algebraic generator of $\mathbb{T}$. Consider the discrete system

$$x_{n+1} = \mathcal{T}(1)x_n, \quad n \in \mathbb{Z}_+. \tag{$\mathcal{T}(1)$}$$

The solution of the above system is $x_n = \mathcal{T}(n)x_0$, i.e., there exists a bijection between the discrete semigroups $\mathbb{T} = \{\mathcal{T}(n) : n \in \mathbb{Z}_+\}$ and the problem $x_{n+1} = \mathcal{T}(1)x_n$.

Also consider the following discrete Cauchy problem:

$$\begin{cases} y_{n+1} = \mathcal{T}(1)y_n + z(n+1), & n \in \mathbb{Z}_+, \\ y_0 = 0, \end{cases} \tag{$\mathcal{T}(1), 0$}$$

The solution of the above Cauchy problem is given by

$$y_n = \sum_{k=0}^{n} \mathcal{T}(n-k)z(k).$$

In [22], the exponential stability of the system $(\mathcal{T}(1))$ in terms of the boundedness of the solution of the Cauchy problem $(\mathcal{T}(1), 0)$ on space $\mathbb{P}^q(\mathbb{Z}_+, \mathcal{X})$, where $q \geq 2$ is a fixed integer, are given as follows.

Theorem 2.2 ([22, Corollary 3.3])**.** *The system $x_{n+1} = \mathcal{T}(1)x_n$ is uniformly exponentially stable if and only if for each q-periodic bounded sequence $z(n)$ with $z(0) = 0$ the unique solution of the Cauchy problem $(\mathcal{T}(1), 0)$ is bounded.*

3. Evolution semigroups and uniform exponential stability of $\mathbb{T}$ on space $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$

Let $\mathbb{T} = \{\mathcal{T}(n);\ n \in \mathbb{Z}_+\}$ is an exponentially bounded discrete semigroup of bounded linear operators on Banach space $\mathcal{X}$. For every $n \in \mathbb{Z}_+$ and each $F \in \mathbb{C}_0(\mathbb{Z}, \mathcal{X})$ the sequence $\big(\mathcal{K}(n)F\big)$, given by

$$s \to (\mathcal{K}(n)F)(s) := \mathcal{T}(n)F(s-n) : \mathbb{Z} \to \mathcal{X}$$

belongs to $\mathbb{C}_0(\mathbb{Z}, \mathcal{X})$ and the family $\mathbb{K} = \{\mathcal{K}(n) : n \in \mathbb{Z}\}$ is a discrete semi group on $\mathbb{C}_0(\mathbb{Z}, \mathcal{X})$, for similar results in continuous case we recommend [14].

If $\mathbb{T} = \{\mathcal{T}(n);\ n \in \mathbb{Z}_+\}$ is a q-periodic discrete semigroup, $n \in \mathbb{Z}_+$ and $G \in \mathbb{AP}(\mathbb{Z}_+, \mathcal{X})$ then the sequence $\big(\mathcal{S}(n)G\big)$ given by

$$s \to (\mathcal{S}(n)G)(s) := \mathcal{T}(n)G(s-n) : \mathbb{Z} \to \mathcal{X},$$

belongs to $\mathbb{AP}(\mathbb{Z}, \mathcal{X})$ and the one parameter discrete family $\mathbb{S} = \{\mathcal{S}(n) : n \in \mathbb{Z}\}$ is discrete semigroup on $\mathbb{AP}(\mathbb{Z}, \mathcal{X})$, for continuous case we recommend [16]. $\mathbb{K}$ and $\mathbb{S}$ are called evolution semigroups on $\mathbb{C}_0(\mathbb{Z}, \mathcal{X})$ and $\mathbb{AP}(\mathbb{Z}, \mathcal{X})$, respectively.

Let us consider the spaces $\mathbb{C}_0(\mathbb{Z}_+, \mathcal{X})$ and $\mathbb{AP}(\mathbb{Z}_+, \mathcal{X})$ consisting of sequences $f, g : \mathbb{Z}_+ \to \mathcal{X}$ for which there exists $F \in \mathbb{C}_0(\mathbb{Z}, \mathcal{X})$ such that $F(n) = f(n)$ for all $n \in \mathbb{Z}_+$ and there exists $G \in \mathbb{AP}(\mathbb{Z}, \mathcal{X})$ such that $G(n) = g(n)$ for all $n \in \mathbb{Z}_+$. Also consider the subspaces of above spaces as:

- $\mathbb{C}_{00}(\mathbb{Z}_+, \mathcal{X}) \subseteq \mathbb{C}_0(\mathbb{Z}_+, \mathcal{X})$, consisting of all $\mathcal{X}$-valued sequences f such that $f(0) = 0$.
- $\mathbb{AP}_0(\mathbb{Z}_+, \mathcal{X}) \subseteq \mathbb{AP}(\mathbb{Z}_+, \mathcal{X})$, consisting of all $\mathcal{X}$-valued sequences g such that $g(0) = 0$.
- $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X}) = \mathbb{C}_{00}(\mathbb{Z}_+, \mathcal{X}) \oplus \mathbb{AP}_0(\mathbb{Z}_+, \mathcal{X})$, i.e., $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ consisting of all $\mathcal{X}$-valued sequences h such that there exists $f \in \mathbb{C}_{00}(\mathbb{Z}_+, \mathcal{X})$ and $g \in \mathbb{AP}_0(\mathbb{Z}_+, \mathcal{X})$ with the condition that $h = f + g$. Clearly $h(0) = 0$.

For each $h \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ and every $n \in \mathbb{Z}_+$ consider the sequence $\mathcal{V}(n)h$ given by

$$\big(\mathcal{V}(n)h\big)(m) = \begin{cases} \mathcal{T}(n)h(m-n), & \text{if } m \geq n, \\ 0, & \text{if } 0 \leq m < n. \end{cases} \tag{1}$$

The semigroup $\mathbb{V} = \{\mathcal{V}(n) : n \in \mathbb{Z}_+\}$ is called the evolution semigroup associated to $\mathbb{T} = \{\mathcal{T}(n) : n \in \mathbb{Z}_+\}$ on the space $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$.

Lemma 3.1. *The discrete semigroup $\mathbb{V} := \{\mathcal{V}(n) : n \in \mathbb{Z}_+\}$ leave the space $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ invariant.*

Proof. Let $h = f + g$ with $f \in \mathbb{C}_{00}(\mathbb{Z}_+, \mathcal{X})$ and $g \in \mathbb{AP}_0(\mathbb{Z}_+, \mathcal{X})$ such that $h(0) = 0$ and let $F \in \mathbb{C}_0(\mathbb{Z}, \mathcal{X})$ and $G \in \mathbb{AP}(\mathbb{Z}, \mathcal{X})$ such that $F(m) = f(m)$ and $G(m) = g(m)$ for all $m \in \mathbb{Z}_+$. It can be seen that for all $n \in \mathbb{Z}_+$ we have

$$\mathcal{V}(n)h = (1_{\{0,1,2\dots\}}\mathcal{S}(n)G) + \big(1_{\{n,n+1,\dots\}}\mathcal{K}(n)F - 1_{\{0,1,\dots,n-1\}}\mathcal{S}(n)G\big).$$

As $\{\mathcal{S}(n)\}_{n\in\mathbb{Z}}$ is the evolution semigroup on $\mathbb{AP}(\mathbb{Z},\mathcal{X})$ and $1_{\mathcal{A}}$ is the characteristic function on the set $\mathcal{A}$. If we put $g_1 := 1_{\{0,1,2\ldots\}}\mathcal{S}(n)G$ and $f_1 := 1_{\{n,n+1,\ldots\}}\mathcal{K}(n)F - 1_{\{0,1,2\ldots n\}}\mathcal{S}(n)G$ then $g_1 \in \mathbb{AAP}_0(\mathbb{Z}_+,\mathcal{X})$, $f_1 \in \mathbb{C}_{00}(\mathbb{Z}_+,\mathcal{X})$ and $(f_1+g_1)(0) = 0$, thus $\mathbb{V}$ acts on $\mathbb{AAP}_0(\mathbb{Z}_+,\mathcal{X})$. □

Before going to next lemma we mention here the following remark.

Remark 3.2. Let $A := \mathcal{V}(1) - I$, where $\mathcal{V}(1)$ is called the algebraic generator of the discrete evolution semigroup $\mathbb{V}$. Then For discrete semigroups, the Taylor formula of order one is

$$\mathcal{V}(n)f - f = \sum_{k=0}^{n-1} \mathcal{V}(k)\mathrm{A}f, \quad n \in \mathbb{Z}_+ \quad \text{with } n \geq 1, \tag{2}$$

for all $f \in \mathcal{X}$.

With the help of the above remark, in the following lemma the discrete version of [15, Lemma 1.1] and [5, Lemma 3.6] is obtain.

Lemma 3.3. *Let $\mathbb{T} = \{\mathcal{T}(n);\ n \in \mathbb{Z}_+\}$ be a q-periodic discrete semigroup of bounded linear operators on $\mathcal{X}$, $\mathbb{V} = \{\mathcal{V}(n) : n \in \mathbb{Z}_+\}$ is the evolution semigroup associated to $\mathbb{T}$ on the space $\mathbb{AAP}_0(\mathbb{Z}_+,\mathcal{X})$, given in* (1)*, and $A = \mathcal{V}(1) - I$. Let $u, f \in \mathbb{AAP}_0(\mathbb{Z}_+,\mathcal{X})$. Then the following two statements are equivalent.*

(1) $\mathrm{A}u(n) = -f(n)$.

(2) $u(n) = \sum_{k=0}^{n} \mathcal{T}(n-k)f(k)$ *for all* $n \in \mathbb{Z}_+$.

Proof. (1) ⇒ (2): Using the Taylor formula (2), we have

$$\mathcal{V}(n)u - u = \sum_{m=0}^{n-1} \mathcal{V}(m)\mathrm{A}u = -\sum_{m=0}^{n-1} \mathcal{V}(m)f.$$

Hence, for every $n \in \mathbb{Z}_+$, $u(n) = (\mathcal{V}(n)u)(n) + \sum_{m=0}^{n-1}(\mathcal{V}(m)f)(n) = \mathcal{T}(n)u(0) + \sum_{m=0}^{n-1} \mathcal{T}(m)f(n-m) = \sum_{k=0}^{n} \mathcal{T}(n-k)f(k)$.

(2) ⇒ (1): For the converse implication as $A = \mathcal{V}(1) - I$, thus

$$\begin{aligned}
\mathrm{A}u(n) &= (\mathcal{V}(1) - \mathrm{I})u(n) \\
&= \mathcal{V}(1)u(n) - u(n) \\
&= \mathcal{T}(1)\sum_{k=0}^{n-1} \mathcal{T}(n-1-k)f(k) - u(n) \\
&= \sum_{k=0}^{n-1} \mathcal{T}(n-k)f(k) - \sum_{k=0}^{n} \mathcal{T}(n-k)f(k) \\
&= -f(n)
\end{aligned}$$

The proof is complete. □

The following theorem gives a characterization of the uniform exponential stability for the discrete semigroup $\mathbb{T} = \{\mathcal{T}(n) : n \in \mathbb{Z}_+\}$ on the space $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$, the space of all $\mathcal{X}$-valued asymptotically almost periodic sequences.

Theorem 3.4. *Let* $\mathbb{T}$, $\mathbb{V}$ *and* A *be as in Lemma* 3.3. *The following statements are equivalent:*

(1) $\mathbb{T}$ *is uniformly exponentially stable.*
(2) *The evolution semigroup* $\mathbb{V}$ *associated to* $\mathbb{T}$ *on* $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ *is uniformly exponentially stable.*
(3) A *is an invertible operator.*
(4) *For every* $f \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ *the sequence* $\sum_{k=0}^{n} \mathcal{T}(n-k)f(k)$ *belongs to* $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$.
(5) *For every* $f \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ *the sequence* $\sum_{k=0}^{n} \mathcal{T}(n-k)f(k)$ *is bounded on* $\mathbb{Z}_+$.
(6) *For any* $f \in \mathbb{P}_0^q(\mathbb{Z}_+, \mathcal{X})$ *the sequence* $t \mapsto \sum_{k=0}^{n} \mathcal{T}(n-k)f(k)$ *is bounded on* $\mathbb{Z}_+$.

Proof. (1) $\Rightarrow$ (2): Let N and ν be two positive constants such that $||T(n)|| \leq Ne^{-\nu n}$ for all $n \in \mathbb{Z}_+$. Let $f \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ then for any $n \in \mathbb{Z}_+$, we have

$$\|T(n)f\|_{\mathbb{AAP}_0} = \sup_{m \geq n} \|T(n)f(m-n)\| \leq Ne^{-\nu n}\|f(m-n)\|_{\mathbb{AAP}_0} = Ne^{-\nu n}\|f\|_{\mathbb{AAP}_0}.$$

Thus the evolution semigroup $\mathbb{V}$ associated to $\mathbb{T}$ on $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ is uniformly exponentially stable.

(2) $\Rightarrow$ (3): It is well known that the evolution semigroup $\mathbb{V}$ is uniformly exponentially stable if and only if $r(\mathcal{V}(1)) < 1$. Then the assumption assure us that $1 \in \rho(\mathcal{V}(1))$ and so $\mathcal{V}(1) - I$ is invertible, i.e., A is invertible.

(3) $\Rightarrow$ (4): As $\mathcal{V}(1) - I$ is invertible thus for every $f \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ there exists a unique $u \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ such that $[\mathcal{V}(1) - I]u = -f$. Thus by Lemma 3.3 we get that $u(n) = \sum_{k=0}^{n} \mathcal{T}(n-k)f(k)$ and by Lemma 3.1 $u(n)$ belongs to $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$, i.e., $\sum_{k=0}^{n} \mathcal{T}(n-k)f(k) \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$.

(4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1): This is a direct consequence of Theorem 2.2.

The proof is complete. □

Remark 3.5. A similar result in the continuous case was stated in [5, Theorem 3.7]. There the authors did not closed the chain with (5) $\Rightarrow$ (1). But in our case with the help of Theorem 2.2, we close the chain, i.e., (6) $\Rightarrow$ (1), as a consequence we can state the following corollary.

Corollary 3.6. *The system* $\zeta_{n+1} = \mathcal{T}(1)\zeta(n)$ *is uniformly exponentially stable if and only if for every* $z(n) \in \mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$ *the unique solution of the Cauchy problem*

$$\begin{cases} \zeta_{n+1} = \mathcal{T}(1)\zeta_n + z(n+1), & n \in \mathbb{Z}_+, \\ \zeta_0 = 0, \end{cases}$$

belongs to $\mathbb{AAP}_0(\mathbb{Z}_+, \mathcal{X})$.

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