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Existence of a Positive Solution to Kirchhoff Problems Involving the Fractional Laplacian

Bin Ge and Chao Zhang

Abstract. The goal of this paper is to establish the existence of a positive solution to the following fractional Kirchhoff-type problem

$$\left(1 + \lambda \int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^2 + V(x)u^2\right) dx\right) \left[(-\Delta)^\alpha u + V(x)u\right] = f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $\lambda \geq 0$ is a parameter, $\alpha \in (0,1)$, $(-\Delta)^\alpha$ stands for the fractional Laplacian, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$. Using a variational method combined with suitable truncation techniques, we obtain the existence of at least one positive solution without compactness conditions.

Keywords. Fractional-Laplacian, Variational method, Cut-off function, Pohozaev type identity

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1. Introduction

In this paper, we are concerned with the existence of positive solutions for a class of fractional Kirchhoff-type problem

$$\left(1+\lambda \int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{\alpha}{2}} u(x)\right|^2 + V(x)u^2\right) dx\right) \left[(-\Delta)^\alpha u + V(x)u\right] = f(u) \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $N \geq 2$, $\lambda \geq 0$ is a parameter, $\alpha \in (0,1)$, $(-\Delta)^\alpha$ stands for the fractional Laplacian, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$. The fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ of a function $\phi \in \mathcal{S}$ is defined by

$$\mathcal{F}\left(\left((-\Delta)^\alpha\right) \phi\right)(\xi) = |\xi|^{2\alpha} \mathcal{F}(\phi)(\xi), \quad \forall \alpha \in (0,1),$$

B. Ge: Department of Applied Mathematics, Harbin Engineering University, Harbin 150001, P. R. China; gebin79@gmail.com
C. Zhang: Department of Mathematics, Harbin Institute of Technology, Harbin 150001, P. R. China; czhangmath@hit.edu.cn

where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞-functions in $\mathbb{R}^N$, $\mathcal{F}$ is the Fourier transform, i.e.,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} \phi(x) dx.$$

If ϕ is smooth enough, it can also be computed by the following singular integral:

$$(-\Delta)^\alpha \phi(x) = c_{N,\alpha} \, \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2\alpha}} dy.$$

Here P.V. is the principal value and $c_{N,\alpha}$ is a normalization constant.

The fractional Kirchhoff equation was first studied by Fiscella and Valdinoci [10]. If $\lambda = 0$, then problem (P) becomes problem (P_1) as follows:

$$(-\Delta)^\alpha u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{P_1}$$

which has been extensively studied in the past few years by many authors. We just mention the earlier works by Autuori and Pucci [1], Chang [2], Chang and Wang [3], Cheng [5], Dipierro, Palatucci and Valdinoci [6], Evéquoz and Fall [8], Felmer, Quaas and Tan [9], Secchi [14], Shen and Gao [17], Shang and Zhang [16] and the references therein. Here we just mention some results related to our problems.

- For the case $V \equiv 1$, the authors in [9] studied the existence of positive solutions of (P_1) when f has subcritical growth and satisfies the Ambrosetti-Rabinowitz condition.
- For the case $V \equiv 0$, Chang and Wang in [3] obtained the existence of a positive ground state under the general Berestycki-Lions type assumptions.
- Moreover, for the general potential V that is allowed to vary, ground states were found by imposing a coercivity assumption on V, i.e.,

  $$\lim_{|x| \to +\infty} V(x) = +\infty.$$

  We refer the readers to [5, 14] for details. Recently, when the potential V satisfies the following conditions $\mathbf{H}(\mathbf{V})(1)$ and $\mathbf{H}(\mathbf{V})(3)$, Chang in [2] proved the existence of ground state solutions under the assumption that $f(u)$ is asymptotically linear with respect to u.

Inspired by the above-mentioned papers, we are concerned with the existence of positive solutions of (P). The novelties in this paper are mainly two parts. First, we just assume that the nonlinear term $f(u)$ is superlinear with respect to u at infinity instead of the asymptotically linear condition or Ambrosetti-Rabinowitz condition, which is completely different from those appeared in the literatures. To compensate the lack of compactness, we employ the Pohozaev Identity and a cut-off functional to obtain the boundedness of Palais-Smale sequence. Second, it is worth mentioning that in the present paper we consider the existence of non-radial positive solutions of (P). We have to prove a new compact embedding theorem by virtue of some assumptions imposed on the potential V.

Throughout this paper, we assume the following conditions which are considerably weaker than the ones in the previous works:

$\mathbf{H}(\mathbf{V})$: (1) $V \in C(\mathbb{R}^N, \mathbb{R}^N)$, $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$;

(2) $\langle \nabla V(x), x\rangle \in L^{\frac{N}{2\alpha}}(\mathbb{R}^N)$ and $|\langle \nabla V(x), x\rangle|_{L^{\frac{N}{2\alpha}}(\mathbb{R}^N)} < 2\alpha S_\alpha$, where S_α is the best Sobolev constant of the embedding $\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N)$, i.e., $S_\alpha = \inf_{u\in\dot{H}^\alpha(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{|u|^2_{2^*_\alpha}}$, (see [4]);

(3) There exists $r > 0$ such that for any $b > 0$

$$\lim_{|y|\to\infty} \mu(\{x \in \mathbb{R}^N : V(x) \leq b\} \cap B_r(y)) = 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^N$.

$\mathbf{H}(\mathbf{f})$: (1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $C^{1,\gamma}$ for some $\gamma > \max\{0, 1 - 2\alpha\}$;

(2) $|f(u)| \leq C(|u| + |u|^{p-1})$ for all $u \in \mathbb{R}_+ = [0, +\infty)$ and $p \in (2, 2^*_\alpha)$, where $2^*_\alpha = \frac{2N}{N-2\alpha}$ for $N \geq 2$;

(3) $\lim_{u\to 0} \frac{f(u)}{u} = 0$;

(4) $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$.

Next, we state our main result.

Theorem 1.1. *Assume that* $\mathbf{H}(\mathbf{V})$ *and* $\mathbf{H}(\mathbf{f})$ *hold. Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda \in [0, \lambda_0)$*, problem* (P) *has at least one positive solution.*

The rest of this paper is organized as follows. In Section 2, we state and prove some preliminary results that will be used later. We will finish the proof of our main result (Theorem 1.1) in Section 3.

2. Preliminary

In this section we recall some results on Sobolev spaces of fractional order. A very complete introduction to fractional Sobolev spaces can be found in [7].

Consider the fractional order Sobolev space

$$H^\alpha(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \big(|\xi|^{2\alpha}\hat{u}^2 + \hat{u}^2\big) d\xi < +\infty\Big\},$$

where $\hat{u} \doteq \mathcal{F}(u)$. The norm is defined by

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\xi|^{2\alpha}\hat{u}^2 + \hat{u}^2) d\xi\right)^{\frac{1}{2}}.$$

The homogeneous fractional Sobolev space $D^{\alpha,2}(\mathbb{R}^N)$, also denoted by $\dot{H}^\alpha(\mathbb{R}^N)$, is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\dot{H}^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\xi|^{2\alpha}\hat{u}^2 d\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 dx\right)^{\frac{1}{2}}.$$

In this paper we consider its subspace:

$$E = \left\{u \in \dot{H}^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\right\}$$

with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 dx + \int_{\mathbb{R}^N} V(x)u^2 dx\right)^{\frac{1}{2}}.$$

Lemma 2.1. ([9, Lemma 2.1]) *$H^\alpha(\mathbb{R}^N)$ continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_\alpha^*]$, and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2_\alpha^*)$.*

Using the above lemma, we can obtain the following result.

Theorem 2.2. *Assume that* $\mathbf{H}(\mathbf{V})(1)$ *and* $\mathbf{H}(\mathbf{V})(3)$ *hold. Then*

(i) *we have a compact embedding $E \hookrightarrow L^2(\mathbb{R}^N)$;*

(ii) *for any $p \in (2, 2_\alpha^*)$, we have a compact embedding $E \hookrightarrow L^p(\mathbb{R}^N)$.*

Proof. (i) Assume $u_n \rightharpoonup 0$ in E, and $\|u_n\|_E \le c_1$. We need to show $u_n \to 0$ in $L^2(\mathbb{R}^N)$. By Lemma 2.1, we have $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. It suffices to show that for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R^c} |u_n|^2 dx \le \varepsilon \quad \text{forall } n = 1, 2, \ldots,$$

where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$, $B_R^c = \{x \in \mathbb{R}^N : |x| \ge R\}$.

Firstly, choose $\{y_i\} \subset \mathbb{R}^N$ such that $\mathbb{R}^N \subset \bigcup_{i=1}^\infty B_r(y_i)$ and each $x \in \mathbb{R}^N$ is covered by at most 2^N such balls. So

$$\begin{aligned}\int_{B_R^c} |u_n|^2 dx &\le \sum_{|y_i| \ge R-r} \int_{B_r(y_i)} |u_n|^2 dx \\ &= \sum_{|y_i| \ge R-r} \left[\int_{B_r(y_i)\cap\{x\in\mathbb{R}^N: V(x)>b\}} |u_n|^2 dx + \int_{A_b(y_i)} |u_n|^2 dx\right],\end{aligned}$$

where $A_b(y_i) = B_r(y_i) \cap \{x \in \mathbb{R}^N : V(x) \leq b\}$.

On the one hand,

$$\int_{B_r(y_i)\cap\{x\in\mathbb{R}^N:V(x)>b\}} |u_n|^2 dx \leq \frac{1}{b}\int_{B_r(y_i)} V(x)|u_n|^2 dx,$$

where $B_r(y) = \{x \in \mathbb{R}^N : |x - y| < r\}$.

On the other hand, since the Sobolev embedding $E \hookrightarrow H^\alpha(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N)$ is continuous, there exists a constant $c_2 > 0$ such that

$$|u_n|_{2^*_\alpha} \leq c_2\|u_n\|_E \leq c_1c_2.$$

Applying Hölder's inequality, we get

$$\begin{aligned}\int_{A_b(y_i)} |u_n|^2 dx &\leq \left(\int_{A_b(y_i)} |u_n|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \left(\int_{A_b(y_i)} 1^{\frac{N}{2\alpha}} dx\right)^{\frac{2\alpha}{N}}\\ &\leq \left(\int_{A_b(y_i)} |u_n|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \sup_{|y_i|\geq R-r} \big[\mu(A_b(y_i))\big]^{\frac{2\alpha}{N}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{B_R^c} |u_n|^2 dx\\ &\leq \sum_{|y_i|\geq R-r}\left[\frac{1}{b}\int_{B_r(y_i)} V(x)|u_n|^2 dx + \left(\int_{A_b(y_i)} |u_n|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \sup_{|y_i|\geq R-r} \big[\mu(A_b(y_i))\big]^{\frac{2\alpha}{N}}\right]\\ &\leq \frac{2^N}{b}\int_{B_{R-2r}^c} V(x)|u_n|^2 dx + \left(2^N\int_{B_{R-2r}^c} |u_n|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \sup_{|y_i|\geq R-r} \big[\mu(A_b(y_i))\big]^{\frac{2\alpha}{N}}\\ &\leq \frac{2^N}{b}\int_{\mathbb{R}^N} V(x)|u_n|^2 dx + 2^{N-2\alpha}\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \sup_{|y|\geq R-r} \big[\mu(A_b(y))\big]^{\frac{2\alpha}{N}}\\ &\leq \frac{2^N}{b}\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 + V(x)|u_n|^2\right) dx + 2^{N-2\alpha}|u_n|^2_{2^*_\alpha} \sup_{|y|\geq R-r} \big[\mu(A_b(y))\big]^{\frac{2\alpha}{N}}\\ &\leq \frac{2^N}{b}\|u_n\|_E^2 + 2^{N-2\alpha}c_2^2\|u_n\|_E^2 \sup_{|y|\geq R-r} \big[\mu(A_b(y))\big]^{\frac{2\alpha}{N}}\\ &\leq \frac{2^N}{b}c_1^2 + 2^{N-2\alpha}c_2^2c_1^2 \sup_{|y|\geq R-r} \big[\mu(A_b(y))\big]^{\frac{2\alpha}{N}}.\end{aligned}$$

Now, choose $b > 0$ such that

$$\frac{2^N}{b}c_1^2 < \frac{\varepsilon}{2}. \tag{1}$$

For such a fixed $b > 0$, since $\sup_{|y|\geq R-r}\left[\mu(A_b(y))\right]^{\frac{2\alpha}{N}} \to 0$, as $R \to +\infty$, there exists $R > 0$ large enough such that

$$\sup_{|y|\geq R-r}\left[\mu(A_b(y))\right]^{\frac{2\alpha}{N}} \leq \frac{\varepsilon}{2^{N-2\alpha+1}c_2^2c_1^2}. \tag{2}$$

It follows from (1) and (2) that

$$\int_{B_R^c} |u_n|^2 dx \leq \varepsilon,$$

from which conclusion (i) of the lemma follows.

(ii) To prove the lemma for general exponent $p \in (2, 2_\alpha^*)$, we use an interpolation argument. Let $u_n \rightharpoonup 0$ in E, we have just proved that $u_n \to 0$ in $L^2(\mathbb{R}^N)$.

Moreover, since the embedding $E \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^N)$ is continuous and $\{u_n\}$ is bounded in E, we also have $\sup_n \int_{\mathbb{R}^N} |u_n|^{2_\alpha^*} dx < +\infty$.

Since $2 < p < 2_\alpha^*$, there exists $\lambda \in (0,1)$ such that $\frac{1}{p} = \frac{\lambda}{2} + \frac{1-\lambda}{2_\alpha^*}$. Then we have

$$s = \frac{2}{p\lambda} > 1 \quad \text{and} \quad t = \frac{2_\alpha^*}{p(1-\lambda)} > 1.$$

Using Hölder's inequality, we deduce that

$$\int_{\mathbb{R}^N} |u_n|^p dx = \int_{\mathbb{R}^N} |u_n|^{\frac{2}{s}} |u_n|^{\frac{2_\alpha^*}{t}} dx \leq \left||u_n|^{\frac{2}{s}}\right|_s \left||u_n|^{\frac{2_\alpha^*}{t}}\right|_t = |u_n|_2^{\frac{2}{s}} |u_n|_{2_\alpha^*}^{\frac{2_\alpha^*}{t}} \to 0,$$

as $n \to +\infty$. This implies $u_n \to 0$ in $L^p(\mathbb{R}^N)$, and the proof of conclusion (ii) is completed. □

Next, we state the following version of Pohozaev identity, which will be used to obtain the boundedness of $\|u_n\|_E$. Similar results can be found in [3, 7, 13, 15]. Its proof is a mixture of many ingredients that are scattered through the literature. We refer the readers to [15, Proposition 4.1] (see also [3, Proposition 4.1] for the case that $V(x) \equiv 0$) for the details.

Lemma 2.3. *Let $N \geq 2$. Assume that* $\mathbf{H}(\mathbf{f})(1)$ *and* $\mathbf{H}(\mathbf{f})(2)$ *hold. If $u \in E$ is a weak solution* (P)*, then the following Pohozaev type identity holds:*

$$\frac{N-2\alpha}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}}u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u(x)|^2 dx$$
$$= N\int_{\mathbb{R}^N} F(u(x))dx.$$

Remark 2.4. We would like to mention that the regularity condition $\mathbf{H}(\mathbf{f})(1)$ is necessary to prove the Pohozaev identity. The smoothness of f will only be used here.

In order to discuss the problem (P), we need to define a functional in E:

$$\varphi_\lambda(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4}\|u\|_E^4 - \int_{\mathbb{R}^N} F(u(x))dx, \quad \forall u \in E.$$

Then we have from $\mathbf{H}(\mathbf{f})(1)$ and $\mathbf{H}(\mathbf{f})(2)$ that φ is well defined on E and is of C^1 for all $\lambda \geq 0$, and

$$\begin{aligned}&\langle \varphi_\lambda'(u), v\rangle \\ &= (1+\lambda\|u\|_E^2)\int_{\mathbb{R}^N}\Big((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v + V(x)uv\Big)dx - \int_{\mathbb{R}^N} f(u)v dx, \quad \forall u, v \in E.\end{aligned}$$

It is standard to verify that the weak solutions of (P) correspond to the critical points of the functional φ.

To overcome the difficulty of finding bounded Palais-Smale (PS)-sequences for the associated functional φ_λ, following [12], we use a cut-off function $\eta \in C^\infty(\mathbb{R}_+)$ satisfying

$$\begin{cases} \eta(t) = 1, & t \in [0,1], \\ 0 \leq \eta(t) \leq 1, & t \in (1,2), \\ \eta(t) = 0, & t \in [2,+\infty), \\ |\eta|_\infty \leq 2, \end{cases} \tag{3}$$

and study the following modified functional $\varphi_\lambda^M : E \to \mathbb{R}$ defined by

$$\varphi_\lambda^M(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4}h_M(u)\|u\|_E^4 - \int_{\mathbb{R}^N} F(u(x))dx, \quad \forall u \in E,$$

where for every $M > 0$, $h_M(u) = \eta\Big(\frac{\|u\|_E^2}{M^2}\Big)$. With this penalization, for $M > 0$ sufficiently large and for λ sufficiently small, we are able to find a critical point u of φ_λ^M such that $\|u\|_E \leq M$ and so u is also a critical point of φ_λ.

Next we recall a monotonicity method due to Struwe [18] and Jeanjean [11], which will be used in our proof. The version here is from [11].

Theorem 2.5. *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1-functionals on X*

$$\varphi_\mu(u) = A(u) - \mu B(u), \quad \mu \in I,$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $\varphi_\mu(0) = 0$.

For any $\mu \in I$ we set $\mathcal{T}_\mu = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \varphi_\mu(\gamma(1)) < 0\}$. If for every $\mu \in I$, the set $\mathcal{T}_\mu$ is nonempty and

$$c_\mu = \inf_{\gamma\in\mathcal{T}_\mu}\max_{t\in[0,1]}\varphi_\mu(\gamma(t)) > 0,$$

then for almost every $\mu \in I$ there is a sequence $\{u_n\} \subset X$ such that

(1) $\{u_n\}$ *is bounded;*
(2) $\varphi_\mu(u_n) \to c_\mu$ *as* $n \to \infty$*;*
(3) $\varphi'_\mu(u_n) \to 0$ *as* $n \to \infty$*, in the dual space* X^{-1} *of* X*.*

In our case, $X = E$,

$$A(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4}h_M(u)\|u\|_E^4 \quad \text{and} \quad B(u) = \int_{\mathbb{R}^N} F(u)dx.$$

So the perturbed functional which we study is

$$\varphi_{\lambda,\mu}^M(u) = \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4}h_M(u)\|u\|_E^4 - \mu\int_{\mathbb{R}^N} F(u(x))dx, \quad \forall u \in E,$$

and

$$\begin{aligned}&\langle(\varphi_{\lambda,\mu}^M)'(u), v\rangle \\ &= \left(1+\lambda h_M(u)\|u\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u\|_E^2}{M^2}\right)\|u\|_E^4\right)\int_{\mathbb{R}^N}\left((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v+V(x)uv\right)dx \\ &\quad - \mu\int_{\mathbb{R}^N} f(u)vdx, \quad \forall u, v\in E.\end{aligned}$$

3. Some lemmas and proof of main result

In this section, to overcome the lack of compactness, we need to consider the functional $\varphi_{\lambda,\mu}^M$ in the functions space E. We shall prove that $\varphi_{\lambda,\mu}^M$ satisfies the conditions of Theorem 2.2 in the next several lemmas.

Lemma 3.1. *Let* $\mathcal{T}_\mu$ *be the set of paths defined in Theorem* 2.2*. Then* $\mathcal{T}_\mu \neq \emptyset$ *for any* $\mu \in I = [\delta, 1]$*, where* $\delta \in (0,1)$ *is a positive constant.*

Proof. We choose $\xi \in C_0^\infty(\mathbb{R}^N)$ with $\|\xi\|_E = 1$ and $\sup\xi \subset B_r(0)$ for some $r > 0$. From $\mathbf{H(f)}(4)$, we know that for any $c_3 > 0$ with $c_3\delta\int_{B_r(0)}\xi^2dx > \frac{1}{2}$, there exists $c_4 > 0$ such that

$$F(u) \geq c_3|u|^2 - c_4, \quad u \in \mathbb{R}_+. \tag{4}$$

Then for $t^2 > 2M^2$ we have

$$\begin{aligned}\varphi_{\lambda,\mu}^M(t\xi) &= \frac{1}{2}\|t\xi\|_E^2 + \frac{\lambda}{4}h_M(t\xi)\|t\xi\|_E^4 - \mu\int_{\mathbb{R}^N} F(t\xi)dx \\ &= \frac{t^2}{2} + \frac{\lambda}{4}\eta\Big(\frac{t^2\|\xi\|_E^2}{M^2}\Big)t^4 - \mu\int_{\mathbb{R}^N} F(t\eta)dx \\ &\leq \frac{t^2}{2} - \delta c_3t^2\int_{B_r(0)}\eta^2dx + c_4|B_r(0)| \\ &= t^2\left(\frac{1}{2} - \delta c_3\int_{B_r(0)}\eta^2dx\right) + c_4|B_r(0)|.\end{aligned}$$

Therefore, we can choose $t > 0$ large such that $\varphi^M_{\lambda,\mu}(t\xi) < 0$. The proof is completed. □

Lemma 3.2. *Let c_μ be the set of paths defined in Theorem 2.2. Then there exists a constant $\bar{c} > 0$ such that $c_\mu \geq \bar{c}$ for $\mu \in [\delta, 1]$.*

Proof. By $\mathbf{H}(\mathbf{f})(2)$ and $\mathbf{H}(\mathbf{f})(3)$, for any $\varepsilon \in (0, \frac{1}{2})$, there exists $c_\varepsilon > 0$ such that

$$|F(u)| \leq \frac{\varepsilon V_0}{2}|u|^2 + c_\varepsilon |u|^p, \quad u \in \mathbb{R}_+. \tag{5}$$

Furthermore, by Lemma 2.1, we have $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous. Then there exists $c_5 > 0$ such that $|u|_p \leq c_5 \|u\|_E$. Hence, for any $u \in E$ and $\mu \in [\delta, 1]$, using (5) it follows that

$$\begin{aligned}
\varphi^M_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|_E^2 + \frac{\lambda}{4} h_M(t\xi)\|u\|_E^4 - \mu \int_{\mathbb{R}^N} F(u)dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} \big(\frac{\varepsilon V_0}{2}|u|^2 + c_\varepsilon |u|^p\big)dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - c_\varepsilon \int_{\mathbb{R}^N} |u|^p dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + V(x)|u|^2\big)dx - c_\varepsilon \int_{\mathbb{R}^N} |u|^p dx \\
&\geq \frac{1}{4}\|u\|_E^2 - c_\varepsilon |u|_p^p \\
&\geq \frac{1}{4}\|u\|_E^2 - c_\varepsilon c_5^p \|u\|_E^p,
\end{aligned}$$

which implies that there exists $\rho > 0$ such that $\varphi^M_{\lambda,\mu}(u) > 0$ for every $u \in E$ and $\|u\|_E \in (0, \rho]$. In particular, for $\|u\|_E = \rho$, we have $\varphi^M_{\lambda,\mu}(u) \geq \bar{c} > 0$. Fix $\mu \in [\delta, 1]$ and $\gamma \in \mathcal{T}_\mu$. By the definition of $\mathcal{T}_\mu$ we can see that $\|\gamma(1)\|_E > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\|_E = \rho$. Thus, for any $\mu \in [\delta, 1]$,

$$c_\mu \geq \inf_{\gamma \in \mathcal{T}_\mu} \varphi^M_{\lambda,\mu}(\gamma(t_\gamma)) \geq \bar{c} > 0.$$

Therefore, we complete the proof. □

Next we prove that the functional $\varphi^M_{\lambda,\mu}$ can achieve the critical value at c_μ for any $\mu \in [\delta, 1]$.

Lemma 3.3. *For any $\mu \in [\delta, 1]$ and $\lambda M^2 < \frac{1}{8}$, each bounded (PS)-sequence of the functional $\varphi^M_{\lambda,\mu}$ admits a convergent subsequence.*

Proof. Let $\mu \in [\delta, 1]$. Suppose that $\{u_n\} \subset E$ is a (PS)-sequence for $\varphi^M_{\lambda,\mu}$, that is, $\{u_n\}$ and $\varphi^M_{\lambda,\mu}(u_n)$ are bounded, $(\varphi^M_{\lambda,\mu})'(u_n) \to 0$ in E', where E' is the dual space of E. Then there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. Thus, Theorem 2.1 implies that

$$u_n \to u \quad \text{in } L^p(\mathbb{R}^N) \quad \text{and} \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^N.$$

By virtue of hypothesis $\mathbf{H(f)}(2)$ and $\mathbf{H(f)}(3)$, for any $\varepsilon \in (0, \min\{\frac{1}{2}, \frac{V_0}{2N}\})$, there exists $c_\varepsilon > 0$ such that

$$|f(u)| \le \varepsilon|u| + c_\varepsilon|u|^{p-1}, \quad u \in \mathbb{R}_+. \tag{6}$$

So it follows from (6) that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx\right| &\le \int_{\mathbb{R}^N} |f(u_n)||u_n-u|dx \\ &\le \int_{\mathbb{R}^N} (\varepsilon|u_n| + c_\varepsilon|u_n|^{p-1})|u_n-u|dx \\ &\le \varepsilon|u_n|_2|u_n-u|_2 + c_\varepsilon\big||u_n|^{p-1}\big|_{p'}|u_n-u|_p \\ &\le \varepsilon|u_n|_2|u_n-u|_2 + c_\varepsilon|u_n|_p^{p-1}|u_n-u|_p \\ &\le \varepsilon c_5\|u_n\|_E|u_n-u|_2 + c_\varepsilon c_5^{p-1}\|u_n\|_E^{p-1}|u_n-u|_p,\end{aligned}$$

which implies that $\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}&\langle(\varphi^M_{\lambda,\mu})'(u_n), u_n-u\rangle \\ &= \left(1+\lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4\right) \\ &\quad\times \int_{\mathbb{R}^N}\Big((-\Delta)^{\frac{\alpha}{2}}u_n(-\Delta)^{\frac{\alpha}{2}}(u_n-u) + V(x)u_n(u_n-u)\Big)dx \\ &\quad- \lambda\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx \\ &= \left(1+\lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4\right) \\ &\quad\times \int_{\mathbb{R}^N}\Big((-\Delta)^{\frac{\alpha}{2}}u_n(-\Delta)^{\frac{\alpha}{2}}(u_n-u) + V(x)u_n(u_n-u)\Big)dx + o(1)\end{aligned}$$

and

$$\begin{aligned}&\left(1+\lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4\right) \\ &\times \int_{\mathbb{R}^N}\Big((-\Delta)^{\frac{\alpha}{2}}u_n(-\Delta)^{\frac{\alpha}{2}}(u_n-u) + V(x)u_n(u_n-u)\Big)dx \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Note that $\left|\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4\right| \leq 8M^4$ and $\lambda M^2 < \frac{1}{8}$. Therefore we conclude that $\|u_n\|_E^2 \to \|u\|_E^2$. This together with $u_n \rightharpoonup u$ in E shows that $u_n \to u$ in E. The proof is completed. □

Now we are in the position to show that the modified functional $\varphi_{\lambda,\mu}^M$ has a nontrivial critical point.

Lemma 3.4. *Assume $\lambda M^2 < \frac{1}{8}$. Then for almost every $\mu \in [\delta, 1]$, there exists $u_\mu \in E\backslash\{0\}$ such that $(\varphi_{\lambda,\mu}^M)'_\lambda(u_\mu) = 0$ and $\varphi_{\lambda,\mu}^M(u_\mu) = c_\mu$.*

Proof. By virtue of Theorem 2.2, for almost every $\mu \in [\delta, 1]$, there exists a bounded sequence $\{u_\mu^n\} \subset E$ such that

$$\varphi_{\lambda,\mu}^M(u_\mu^n) \to c_\mu \quad \text{and} \quad (\varphi_{\lambda,\mu}^M)'(u_\mu^n) \to 0 \quad \text{as } n \to \infty.$$

According to Lemma 3.3, we may assume that there exists $u_\mu \in E$ such that $u_\mu^n \to u_\mu$ in E. Then it follows that $\varphi_{\lambda,\mu}^M(u_\mu) = c_\mu$ and $(\varphi_{\lambda,\mu}^M)'(u_\mu) = 0$ and $u_\mu \neq 0$ from Lemma 3.2. □

From Lemma 3.4 we know that there exist a sequence $\mu_n \in [\delta, 1]$ with $\mu_n \to 1^-$ and an associated sequence $\{u_n\} \subset E$ such that

$$\varphi_{\lambda,\mu_n}^M(u_n) = c_{\mu_n} \quad \text{and} \quad (\varphi_{\lambda,\mu_n}^M)'(u_n) = 0. \tag{7}$$

Next, we will show that the sequence $\{u_n\}$ is bounded, which is a key ingredient in this paper.

Lemma 3.5. *Let u_n be a critical point of $\varphi_{\lambda,\mu_n}^M$ at the level c_{μ_n} as defined in (7). Then for $M > 0$ sufficiently large, there exists $\lambda_0 = \lambda_0(M)$ with $\lambda_0 M^2 < \frac{1}{8}$ such that for any $\lambda \in [0, \lambda_0)$, we have $\|u_n\|_E \leq M$ for all n.*

Proof. We argue by contradiction. Firstly, if we set

$$g(u_n) = \frac{f(u_n)}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4},$$

then from Lemma 2.2 and (7), we infer that u_n satisfies the following Pohozaev identity

$$\begin{aligned} &\frac{N-2\alpha}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx + \frac{N}{2}\int_{\mathbb{R}^N}V(x)|u_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\langle\nabla V(x), x\rangle|u_n|^2dx \\ &= \mu_n N\int_{\mathbb{R}^N}G(u_n)dx, \end{aligned} \tag{8}$$

where $G(t) = \int_0^t g(s)ds$.

Recall that $\varphi^M_{\lambda,\mu_n}(u_n) = c_{\mu_n}$. So we have

$$\frac{N}{2}\|u_n\|_E^2 + \frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^4 - \mu_n N \int_{\mathbb{R}^N} F(u_n)dx = c_{\mu_n} N. \tag{9}$$

Then from (8), (9) and hypothesis $\mathbf{H}(\mathbf{f})(2)$, it follows that

$$\begin{aligned}
&\alpha \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \\
=&\frac{N}{2}\|u_n\|_E^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_n|^2 dx - \mu_n N \int_{\mathbb{R}^N} G(u_n)dx \\
=&\frac{N}{2}\|u_n\|_E^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_n|^2 dx \\
&- \frac{\mu_n N \int_{\mathbb{R}^N} F(u_n)dx}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4} \\
=&\frac{N}{2}\|u_n\|_E^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_n|^2 dx \\
&+ \frac{C_{\mu_n} N - \frac{N}{2}\|u_n\|_E^2 - \frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^4}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4} \\
=&\frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_n|^2 dx \\
&+ \frac{C_{\mu_n} N + \frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^2 + \frac{\lambda N}{4M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^6}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4} \\
\leq&\frac{1}{2}|\langle \nabla V(x), x\rangle|_{L^{\frac{N}{2\alpha}}(\mathbb{R}^N)} |u_n|_{2^*_\alpha}^2 \\
&+ \frac{C_{\mu_n} N + \frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^2 + \frac{\lambda N}{4M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^6}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4} \\
\leq&\frac{1}{2S_\alpha}|\langle \nabla V(x), x\rangle|_{L^{\frac{N}{2\alpha}}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \\
&+ \frac{C_{\mu_n} N + \frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^2 + \frac{\lambda N}{4M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^6}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4}.
\end{aligned}$$

Set $M_0 = \alpha - \frac{1}{2S_\alpha}|\langle \nabla V(x), x\rangle|_{L^{\frac{N}{2\alpha}}(\mathbb{R}^N)}$. Then

$$M_0 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \leq \frac{C_{\mu_n} N + \frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^2 + \frac{\lambda N}{4M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^6}{1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4}. \tag{10}$$

We estimate the right hand side of (10). By the min-max definition of the mountain pass level c_{λ_n}, Lemma 3.1 and (4), we have

$$
\begin{aligned}
c_{\mu_n} &\le \max_t \varphi^M_{\lambda,\mu_n}(t\eta) \\
&\le \max_t \left\{ \frac{t^2}{2} - \mu_n \int_{\mathbb{R}^N} F(t\eta)dx \right\} + \frac{\lambda}{4} \max_t \eta\Big(\frac{t^2}{M^2}\Big) t^4 \\
&\le \max_t \left\{ t^2 \left(\frac{1}{2} - \delta c_3 \int_{B_r(0)} \eta^2 dx \right) + c_4 |B_r(0)| \right\} + \lambda M^4 \\
&\le c_4|B_r(0)| + \lambda M^4.
\end{aligned}
\tag{11}
$$

On the other hand, from (3), we infer that

$$
\begin{aligned}
\frac{\lambda N}{4} h_M(u_n)\|u_n\|_E^4 &\le \lambda N M^4, \\
\frac{\lambda N}{4M^2} \eta'\Big(\frac{\|u_n\|_E^2}{M^2}\Big)\|u_n\|_E^6 &\le 4\lambda N M^4.
\end{aligned}
\tag{12}
$$

Using (11) and (12) in (10), we obtain

$$
M_0 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 dx \le \frac{C_{\mu_n} N + \frac{\lambda N}{2} M^2 + 4\lambda M^4}{1 - 4\lambda M^2}.
$$

Note that $E \subset \dot{H}^\alpha(\mathbb{R}^N)$ and the embedding $\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N)$ is continuous. Then we have

$$
|u|^2_{2^*_\alpha} \le \frac{1}{S_\alpha} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx, \quad \forall u \in E,
$$

where S_α is the best Sobolev constant of the embedding $\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^N)$, i.e., $S_\alpha = \inf_{u \in \dot{H}^\alpha(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{|u|^2_{2^*_\alpha}}$, (see [4]).

Recall from (7)

$$
\left(1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2} \eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4\right)\|u_n\|_E^2 = N\mu_n \int_{\mathbb{R}^N} f(u_n) u_n dx. \tag{13}
$$

Using (6) in (13), we obtain

$$\begin{aligned}
\frac{1}{2}\|u_n\|_E^2 \leq & \left(1 + \lambda h_M(u_n)\|u_n\|_E^2 + \frac{\lambda}{2M^2}\eta'\left(\frac{\|u_n\|_E^2}{M^2}\right)\|u_n\|_E^4\right)\|u_n\|_E^2 \\
\leq & N\int_{\mathbb{R}^N}\left(\varepsilon|u_n|^2 + c_\varepsilon|u_n|^p\right)dx \\
\leq & N\int_{\mathbb{R}^N}\left(\varepsilon\frac{V(x)}{V_0}|u_n|^2 + c_\varepsilon|u_n|^{2_\alpha^*}\right)dx \\
\leq & \frac{N\varepsilon}{V_0}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{\alpha}{2}}u_n|^2 + V(x)|u_n|^2\right)dx + c_\varepsilon N\int_{\mathbb{R}^N}|u_n|^{2_\alpha^*}dx \\
\leq & \frac{N\varepsilon}{V_0}\|u_n\|_E^2 + c_\varepsilon N\left[\frac{1}{S_\alpha}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right]^{\frac{N}{N-2\alpha}},
\end{aligned}$$

which implies that

$$\begin{aligned}
\left(\frac{1}{2} - \frac{N\varepsilon}{V_0}\right)\|u_n\|_E^2 \leq & c_\varepsilon N\left[\frac{1}{S_\alpha}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2dx\right]^{\frac{N}{N-2\alpha}} \\
\leq & c_\varepsilon N\left[\frac{1}{S_\alpha M_0}\frac{C_{\mu_n}N + \frac{\lambda N}{2}M^2 + 4\lambda M^4}{1 - 4\lambda M^2}\right]^{\frac{N}{N-2\alpha}} \\
\leq & c_\varepsilon N\left[\frac{2}{S_\alpha M_0}\left(c_4|B_r(0)| + \lambda M^4 + \frac{\lambda N}{2}M^2 + 4\lambda M^4\right)\right]^{\frac{N}{N-2\alpha}}.
\end{aligned}$$

We suppose by contradiction that there exists no subsequence of $\{u_n\}_{n=1}^\infty$ which is uniformly bounded by M. Then we can assume that $\|u_n\|_E > M$, $n \in N$. This means that

$$\begin{aligned}
M < & \|u_n\|_E \\
\leq & c_\varepsilon N\left[\frac{2}{S_\alpha M_0}\left(c_4|B_r(0)| + \lambda M^4 + \frac{\lambda N}{2}M^2 + 4\lambda M^4\right)\right]^{\frac{N}{N-2\alpha}}\frac{2V_0}{(V_0 - 2N\varepsilon)} \\
\leq & c_\varepsilon N\left[\frac{2}{S_\alpha M_0}\left(c_4|B_r(0)| + \lambda M^4 + \frac{\lambda N}{2}M^4 + 4\lambda M^4\right)\right]^{\frac{N}{N-2\alpha}}\frac{2V_0}{(V_0 - 2N\varepsilon)},
\end{aligned}$$

which is not true for M large and $\lambda M^4 < \frac{1}{8}$. So by setting $\lambda_0 < \frac{1}{8M^4}$, we obtain the conclusion. □

Finally, we are ready to prove our main theorem.

Proof of Theorem 1.1. Let M, λ_0 be defined as in Lemma 3.5. Let u_n be a critical point for $\varphi_{\lambda,\mu_n}^M$ at the level c_{μ_n}. Then from Lemma 3.5 we may assume that $\|u_n\|_E \leq c_5$.

Note that $\mu_n \to 1$, we can show that $\{u_n\}$ is a (PS)-sequence of φ_λ. Indeed, the boundedness of $\{u_n\}$ implies that $\{\varphi_\lambda(u_n)\}$ is bounded. Also

$$\langle \varphi'_\lambda(u_n), v\rangle = \langle (\varphi^M_{\lambda,\mu_n})'(u_n), v\rangle + (\mu_n - 1)\int_{\mathbb{R}^N} f(u_n)v dx, \quad \forall v \in E.$$

Hence, $\varphi'_\lambda(u_n) \to 0$, and consequently $\{u_n\}$ is a bounded (PS)-sequence of φ_λ. By Lemma 3.3, $\{u_n\}$ has a convergent subsequence, hence without loss of generality we may assume that $u_n \to u$. Therefore, $\varphi'_\lambda(u) = 0$. By virtue of Lemma 3.2, we have

$$\varphi_\lambda(u) = \lim_{n\to\infty} \varphi_\lambda(u_n) = \lim_{n\to\infty} \varphi^M_{\lambda,\lambda_n}(u_n) \geq \overline{c} > 0,$$

and u is a positive solution by the condition $\mathbf{H}(\mathbf{f})(1)$. The proof is completed. □

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