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# Positive Solutions for Nonlinear Nonhomogeneous Robin Problems

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**Abstract.** We consider a nonlinear, nonhomogeneous Robin problem with a Carathéodory reaction which satisfies certain general growth conditions near $0^+$ and near $+\infty$. We show the existence and regularity of positive solutions, the existence of a smallest positive solution and under an additional condition on the reaction, we show the uniqueness of the positive solutions. We then show that our setting incorporates certain parametric Robin equations of interest such as nonlinear equidiffusive logistic equations.

**Keywords.** Positive solution, nonlinear regularity, nonlinear maximum principle, logistic equations, smallest positive solution

**Mathematics Subject Classification (2010).** Primary 35J20, secondary 35J60

## 1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial\Omega$. In this paper we study the following nonlinear Robin problem:

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

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In this problem $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a strictly monotone, continuous function satisfying certain other regularity and growth conditions which are listed in hypotheses H($a$).

These conditions are general enough and include as special cases various differential operators of interest, such as the $p$-Laplacian. In the boundary condition $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative $a(|\nabla u|)\frac{\partial u}{\partial n}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$. The reaction $f(z,\zeta)$ is a Carathéodory function (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous), which exhibits general growth conditions near $+\infty$ and near $0^+$, related to the spectrum of the negative $p$-Laplacian with Robin boundary condition (denoted by $-\Delta_p^R$).

These growth conditions incorporate in our setting various parametric problems of interest, such as $p$-logistic equations with equidiffusive reaction. Recently, a particular class of such parametric equations driven by the Robin $p$-Laplacian, were studied by Papageorgiou-Rădulescu [22], who for all large values of the parameter produced results providing precise sign information for all the solutions. Here we focus on positive solutions and derive conditions for the existence and uniqueness of such solutions. For other problems with nonhomogeneous operators generalizing $p$-Laplacian we refer to Gasiński-Papageorgiou [10, 14] (problems with Dirichlet boundary condition) and Gasiński [8, 9] (problems with periodic boundary condition).

We should also mention the recent works of Autuori-Pucci-Varga [2], Colasuonno-Pucci-Varga [4], Filippucci-Pucci-Rădulescu [7] and Perera-Pucci-Varga [23]. In [4] the authors consider parametric equations (eigenvalue problems) on a bounded domain and employ Dirichlet or Robin boundary condition. The differential operator is nonhomogeneous and $z$-dependent. They establish existence and multiplicity of nontrivial solutions (not necessarily positive) valid for certain values of the parameter $\lambda$. Their approach uses an abstract result of Ricceri.

In [2, 7, 23] the domain is unbounded (in [7] it as an exterior domain) and they use Robin boundary conditions (nonhomogeneous in [23]). We mention that in [7, 23] the differential operator is a weighted $p$-Laplacian plus a potential term and so it is coercive. In all three papers the authors establish existence of nontrivial solutions. Positive solutions are obtained in [7] for large values of the parameter (see [7, Theorem 1.1]). Therefore their result is in a sense complementary to Propositions 4.6 and 4.7 of this paper (of course as it was mentioned, in [7] the set $\Omega$ is unbounded and exterior domain). We point out that here the critical parameter $\lambda^*$ is precisely identified.

In general we can say that in [2,4,7], the differential operator is more general at the expense of adding a potential term in order to have coercivity (see [7]) and the weak solutions they obtain are less regular and without sign information (except for [7]).

## 2. Preliminaries

In the analysis of problem (1), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will use the ordered Banach space $C^1(\overline{\Omega})$. The order cone of this space is given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has nonempty interior, given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

On $\partial\Omega$ we consider the $(N$–$1)$-dimensional surface (Hausdorff) measure $\sigma(\cdot)$. Then we can define the Lebesgue spaces $L^s(\partial\Omega)$ with $1 \leqslant s \leqslant +\infty$. We know that there exists a unique continuous linear map $\gamma_0 \colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$ such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \quad \forall u \in C^1(\overline{\Omega}).$$

This map is known as the trace map. We have that $\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$ and $\ker\gamma_0 = W_0^{1,p}(\Omega)$. In what follows, for the sake of notational simplicity, we drop the use of $\gamma_0$ in order to denote the restriction of a Sobolev function on $\partial\Omega$. All such restrictions are understood in the sense of traces.

In what follows by $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$, defined by

$$\|u\| = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

To distinguish, by $|\cdot|$ we denote the norm of $\mathbb{R}^N$. Given $\zeta \in \mathbb{R}$, we define $\zeta^\pm = \max\{\pm\zeta, 0\}$ and then for $u \in W^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Let $\Delta_p$ denote the $p$-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall u \in W^{1,p}(\Omega).$$

We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $1 < p < +\infty$. Here $\frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}\frac{\partial u}{\partial n}$. We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of $-\Delta_p^R$, if problem (2) admits a nontrivial solution. Throughout this work our hypotheses on the boundary weight function $\beta$ are the following

H($\beta$): $\beta \in C^{0,\tau}(\partial\Omega)$ with $0<\tau<1$, $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Eigenvalue problem (2) was studied by Lê [17] (with $\beta(z)=\beta>0$) and by Papageorgiou-Rădulescu [22]. We know that there exists a smallest eigenvalue $\widehat{\lambda}_1(p,\beta)>0$ which is simple, isolated and admits the following variational characterization

$$\widehat{\lambda}_1(p,\beta)=\inf\left\{\frac{\|\nabla u\|_p^p+\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma}{\|u\|_p^p}\ :\ u\in W^{1,p}(\Omega),\ u\neq 0\right\}. \tag{3}$$

The infimum in (3) is realized on the one-dimensional eigenspace corresponding to the eigenvalue $\widehat{\lambda}_1(p,\beta)>0$. It is clear from (3) that the elements of this eigenspace do not change sign. In what follows by $\widehat{u}_1(p,\beta)\in W^{1,p}(\Omega)$ we denote the positive, $L^p$-normalized (that is, $\|\widehat{u}_1(p,\beta)\|_p=1$) eigenfunction corresponding to $\widehat{\lambda}_1(p,\beta)>0$. From nonlinear regularity theory and the nonlinear maximum principle (see Lieberman [19] and Pucci-Serrin [24]), we have $\widehat{u}_1(p,\beta)\in \operatorname{int} C_+$. For the higher parts of the spectrum of $-\Delta_p^R$, we refer to Lê [17] and Papageorgiou-Rădulescu [22].

As a consequence of the above properties, we have the following result.

**Lemma 2.1.** *If $\vartheta\in L^\infty(\Omega)_+$, $\vartheta(z)\leqslant\widehat{\lambda}_1(p,\beta)$ for almost all $z\in\Omega$, $\vartheta\not\equiv\widehat{\lambda}_1(p,\beta)$, then there exists $\xi_0>0$ such that*

$$\sigma_0(u)=\|\nabla u\|_p^p+\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma-\int_\Omega\vartheta(z)|u|^p\,dz\geqslant\xi_0\|u\|^p\quad\forall u\in W^{1,p}(\Omega).$$

*Proof.* It is clear from (3) and the hypothesis on $\vartheta$, that $\sigma_0\geqslant 0$. Arguing by contradiction, suppose that the result of the lemma is not true. Then exploiting the $p$-homogeneity of $\sigma$, we can find a sequence $\{u_n\}_{n\geqslant 1}\subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\|=1\quad\forall n\geqslant 1,\quad\text{and}\quad\sigma_0(u_n)\ \longrightarrow\ 0^+. \tag{4}$$

By passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned}&u_n\longrightarrow u\quad\text{weakly in } W^{1,p}(\Omega),\\&u_n\longrightarrow u\quad\text{in } L^p(\Omega)\text{ and } L^p(\partial\Omega)\end{aligned}$$

(by the Sobolev embedding theorem). Then $\sigma_0(u)\leqslant 0$, hence

$$\|\nabla u\|_p^p+\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma\leqslant\int_\Omega\vartheta(z)|u|^p\,dz\leqslant\widehat{\lambda}_1(p,\beta)\|u\|_p^p, \tag{5}$$

so $u=\xi\widehat{u}_1(p,\beta)$ for some $\xi\in\mathbb{R}$ (see (3)).

If $\xi=0$, then $u=0$ and so $u_n\longrightarrow 0$ in $W^{1,p}(\Omega)$, contradicting (4). If $\xi\neq 0$, then from (5) and since $\widehat{u}(p,\beta)\in\operatorname{int} C_+$, we have

$$\|\nabla u\|_p^p+\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma<\widehat{\lambda}_1(p,\beta)\|\widehat{u}_1\|_p^p=\widehat{\lambda}_1(p,\beta),$$

which contradicts (3). This proves the lemma. $\square$

Now, we will introduce our conditions on the map $a$ involved in the definition of the differential operator. So, let $\kappa \in C^1(0,\infty)$, with $\kappa(t) > 0$ for all $t > 0$ and

$$0 < \widehat{c} \leqslant \frac{t\kappa'(t)}{\kappa(t)} \leqslant c_0 \quad \forall t > 0 \tag{6}$$

and

$$c_1 t^{p-1} \leqslant \kappa(t) \leqslant c_2(1 + t^{p-1}) \quad \forall t > 0 \tag{7}$$

for some $\widehat{c}, c_0, c_1, c_2 > 0$.

Using $\kappa$ we can introduce our hypotheses on the map $a$ which are the following.

<u>H($a$):</u> $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,+\infty)$, the function $t \longmapsto ta_0(t)$ is strictly increasing on $(0,+\infty)$, $ta_0(t) \longrightarrow 0^+$ as $t \searrow 0$ and

$$\lim_{t\searrow 0} \frac{ta_0'(t)}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$, such that

$$|\nabla a(y)| \leqslant c_3 \frac{\kappa(|y|)}{|y|} \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(iii) we have

$$\big(\nabla a(y)\xi,\ \xi\big)_{\mathbb{R}^N} \geqslant \frac{\kappa(|y|)}{|y|}|\xi|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}^N;$$

(iv) if

$$G_0(t) = \int_0^t s a_0(s)\, ds \quad \forall t > 0,$$

then there exists $q \in (1,p]$ such that the function $t \longmapsto G_0(t^{\frac{1}{q}})$ is convex on $(0,+\infty)$ and

$$\lim_{t\searrow 0} \frac{qG_0(t)}{t^q} = \widetilde{c} > 0.$$

**Remark 2.2.** These conditions on the map $a$ are designed to fit the regularity results of Lieberman [20, p. 320] and the nonlinear maximum principle of Pucci-Serrin [24, pp. 111, 120]. We have adopted exactly the conditions imposed on $a(\cdot)$ by Lieberman [20] and Pucci-Serrin [24] in order to facilitate our calculations and be consistent with the above two references. Of course we could have set $a(y) = \frac{k(|y|)y}{|y|}$ for $y \in \mathbb{R}^N$ and proceed with this map.

It is clear that the primitive $G_0$ is strictly convex and strictly increasing. We set

$$G(y) = G_0(|y|) \quad \forall y \in \mathbb{R}^N$$

and we have that $G$ is convex and $G(0)=0$. Moreover

$$\nabla G(y)=G_0'(|y|)\frac{y}{|y|}=a_0(|y|)y=a(y)\quad \forall y\in\mathbb{R}^N\setminus\{0\},\quad \nabla G(0)=0.$$

Therefore $G$ is the primitive of $a$. Since $G$ is convex and $G(0)=0$, we have

$$G(y)\leqslant (a(y),y)_{\mathbb{R}^N}\quad \forall y\in\mathbb{R}^N. \tag{8}$$

Using hypotheses H($a$) above and (6), (7), we obtain the following lemma summarizing the property of $a$.

**Lemma 2.3.** *If hypotheses* H($a$)(i)–(iii) *hold, then*

(a) *the function* $y\longmapsto a(y)$ *is continuous and strictly monotone, hence maximal monotone too;*
(b) *there exists* $c_4>0$ *such that* $|a(y)|\leqslant c_4(1+|y|^{p-1})$ *for all* $y\in\mathbb{R}^N$*;*
(c) $(a(y),y)_{\mathbb{R}^N}\geqslant \frac{c_1}{p-1}|y|^p$ *for all* $y\in\mathbb{R}^N$.

This lemma together with (8), lead to the following growth estimate for the primitive $G$.

**Corollary 2.4.** *If hypotheses* H($a$)(i)–(iii) *hold, then there exists* $c_5>0$ *such that*

$$\frac{c_1}{p(p-1)}|y|^p\leqslant G(y)\leqslant c_5(1+|y|^p)\quad \forall y\in\mathbb{R}^N.$$

**Example 2.5.** The following maps satisfy hypotheses H($a$):

(a) $a(y)=|y|^{p-2}y$ with $1<p<+\infty$; This map corresponds to the $p$-Laplace differential operator $\Delta_p$;
(b) $a(y)=|y|^{p-2}y+|y|^{q-2}y$ with $1<q<p<+\infty$. This map corresponds to the $(p,q)$-Laplacian

$$\Delta_p u+\Delta_q u\quad \forall u\in W^{1,p}(\Omega).$$

Such operators arise in many physical applications (see Gasiński-O'Regan-Papageorgiou [16] and Gasiński-Papageorgiou [15] and references therein).
(c) $a(y)=(1+|y|^2)^{\frac{p-2}{p}}y$ with $1<p<+\infty$. This map corresponds to the generalized $p$-mean curvature differential operator

$$\operatorname{div}\big((1+|\nabla u|^2)^{\frac{p-2}{p}}\nabla u\big)\quad \forall u\in W^{1,p}(\Omega).$$

(d) $a(y)=|y|^{p-2}y+\frac{|y|^{p-2}y}{1+|y|^p}$ with $1<p<+\infty$.

We introduce the operator $A\colon W^{1,p}(\Omega)\longrightarrow W^{1,p}(\Omega)^*$ defined by

$$\langle A(u),y\rangle=\int_\Omega (a(\nabla u),\nabla y)_{\mathbb{R}^N}dz\quad \forall u,y\in W^{1,p}(\Omega). \tag{9}$$

Here and in the sequel, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}(\Omega)^*,W^{1,p}(\Omega))$. From Gasiński-Papageorgiou [12], we have

**Proposition 2.6.** *If $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ is the nonlinear map defined by* (9)*, then $A$ is bounded* (*i.e., maps bounded sets into bounded ones*)*, continuous, monotone* (*hence maximal monotone too*) *and of type $(S)_+$* (*i.e., if $u_n \longrightarrow u$ weakly in $W^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$*)*.*

## 3. Existence and uniqueness of positive solutions

In this section we examine the existence and uniqueness of positive solutions for problem (1). We introduce the following hypotheses on the reaction $f$:

H($f$): $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exist a function $a \in L^\infty(\Omega)_+$ and $r \in (p, p^*)$, where

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } \ p < N, \\ +\infty & \text{if } N \leqslant p \end{cases}$$

such that

$$|f(z,\zeta)| \leqslant a(z)(1 + \zeta^{r-1}) \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \geqslant 0;$$

(ii) there exists a function $\vartheta \in L^\infty(\Omega)_+$ such that

$$\vartheta(z) \leqslant \widehat{\lambda}_1(p, \widehat{\beta}) \quad \text{for almost all } z \in \Omega$$

and the inequality is strict on a set of positive measure, with $\widehat{\beta} = \frac{p-1}{c_1}\beta$, and

$$\limsup_{\zeta\to+\infty} \frac{f(z,\zeta)}{\zeta^{p-1}} \leqslant \vartheta(z) \quad \text{for almost all } z \in \Omega;$$

(iii) there exists a function $\eta \in L^\infty(\Omega)_+$ such that

$$\eta(z) \geqslant \widetilde{c}\widehat{\lambda}_1(q, \widetilde{\beta}) \quad \text{for almost all } z \in \Omega$$

and the inequality is strict on a set of positive measure, with $\widetilde{\beta} = \frac{\beta}{\widetilde{c}}$ and $\widetilde{c} > 0$ is as in hypothesis H($a$)($iv$) and

$$\liminf_{\zeta\to0^+} \frac{f(z,\zeta)}{\zeta^{q-1}} \geqslant \eta(z) \quad \text{uniformly for almost all } z \in \Omega.$$

**Remark 3.1.** Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality that $f(z,\zeta) = 0$ for almost all $z \in \Omega$, all $\zeta \leqslant 0$. Note that, if $a(y) = |y|^{p-2}y$ with $1 < p < +\infty$ (i.e., the differential operator is the $p$-Laplacian), then $c_1 = p-1$, $q = p$ and $\widetilde{c} = 1$ and so hypotheses H$(f)$(i)–(iii) imply that, as $\zeta$ moves from $0^+$ to $+\infty$, the quotient $\frac{f(z,\zeta)}{\zeta^{p-1}}$ crosses at least the principal eigenvalue $\widehat{\lambda}_1(p,\beta) > 0$. Since $q \leqslant p$, given $\varrho > 0$, we can find $\xi_\varrho > 0$ such that

$$f(z,\zeta) + \xi_\varrho \zeta^{p-1} \geqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [0,\varrho]. \tag{10}$$

In H$(f)$(ii) we can take $\vartheta(z) = \max\left\{\limsup_{\zeta\to+\infty} \frac{f(z,\zeta)}{\zeta^{p-1}}, 0\right\}$.

We introduce the following truncation-perturbation of the reaction $f$:

$$\widehat{f}(z,\zeta) = \begin{cases} 0 & \text{if } \ \zeta \leqslant 0, \\ f(z,\zeta) + \zeta^{p-1} & \text{if } \ 0 < \zeta. \end{cases} \tag{11}$$

This is a Carathéodory function. Let

$$F(z,\zeta) = \int_0^\zeta f(z,s)\,ds \quad \text{and} \quad \widehat{F}(z,\zeta) = \int_0^\zeta \widehat{f}(z,s)\,ds.$$

**Proposition 3.2.** *If hypotheses* H$(a)$, H$(\beta)$ *and* H$(f)$ *hold, then problem* (1) *has at least one positive solution* $u_0 \in \operatorname{int} C_+$.

*Proof.* We consider the functional $\widehat{\varphi}\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \int_\Omega G(\nabla u)\,dz + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\,d\sigma - \int_\Omega \widehat{F}(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

Evidently $\widehat{\varphi} \in C^1(W^{1,p}(\Omega))$. By virtue of hypotheses H$(f)$(i),(ii), given $\varepsilon > 0$, we can find $c_6 = c_6(\varepsilon) > 0$ such that

$$F(z,\zeta) \leqslant \frac{1}{p}(\vartheta(z)+\varepsilon)(\zeta^+)^p + c_6 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{12}$$

Using Corollary 2.4 and (12), we have

$$\begin{aligned}
\widehat{\varphi}(u) &\geqslant \frac{c_1}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p\,d\sigma \\
&\quad - \frac{1}{p}\int_\Omega (\vartheta(z)+\varepsilon)(u^+)^p\,dz - c_6|\Omega|_N \\
&\geqslant \frac{c_1}{p(p-1)}\Big(\|\nabla u^+\|_p^p + \int_{\partial\Omega}\widehat{\beta}(z)(u^+)^p\,d\sigma - \int_\Omega \vartheta(z)(u^+)^p\,dz\Big) \\
&\quad - \frac{\varepsilon}{p}\|u^+\|^p + \frac{c_1}{p(p-1)}\|\nabla u^-\|_p^p + \frac{1}{p}\|u^-\|_p^p - c_6|\Omega|_N \\
&\geqslant \frac{c_7-\varepsilon}{p}\|u^+\|^p + \frac{c_7}{p}\|u^-\|^p - c_6|\Omega|_N
\end{aligned}$$

for some $c_7 > 0$ (see Lemma 2.1).

Choosing $\varepsilon \in (0, c_7)$, we infer that $\widehat{\varphi}$ is coercive. Also, using the Sobolev embedding theorem and the fact that the trace map $\gamma_0 \colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$ is compact, we infer that $\widehat{\varphi}$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem (see Buttazzo-Giaquinta-Hildebrandt [3, p. 4] and Tikhomirov [25, p. 31]), we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}(u_0) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}(u). \tag{13}$$

Hypotheses H$(a)$(iv) and H$(f)$(iii) imply that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$G(y) \leqslant \frac{\widetilde{c} + \varepsilon}{q}|y|^q \qquad \forall |y| \leqslant \delta \tag{14}$$

$$F(z,\zeta) \geqslant \frac{1}{q}(\eta(z) - \varepsilon)\zeta^q \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [0,\delta]. \tag{15}$$

Let $\widehat{\xi} \in (0,1)$ be small such that

$$\widehat{\xi}\widehat{u}_1(q,\widetilde{\beta})(z),\ \widehat{\xi}\left|\nabla \widehat{u}_1(q,\widetilde{\beta})(z)\right| \in [0,\delta] \quad \forall z \in \overline{\Omega} \tag{16}$$

(recall that $\widehat{u}_1(q,\widetilde{\beta}) \in \operatorname{int} C_+$). Then

$$\begin{aligned}
\widehat{\varphi}(\widehat{\xi}\widehat{u}_1(q,\widetilde{\beta})) &\leqslant \frac{(\widetilde{c}+\varepsilon)\widehat{\xi}^q}{q}\|\nabla \widehat{u}_1(q,\widetilde{\beta})\|_q^q + \frac{\widehat{\xi}^q}{p}\int_{\partial\Omega} \beta(z)\widehat{u}_1(q,\widetilde{\beta})^q \, d\sigma \\
&\quad - \frac{\widehat{\xi}^q}{q}\int_\Omega (\eta(z) - \varepsilon)\widehat{u}_1(q,\widetilde{\beta})^q \, dz \\
&\leqslant \frac{\widehat{\xi}^q}{q}\Big(\int_\Omega (\widetilde{c}\widehat{\lambda}_1(q,\widetilde{\beta}) - \eta(z))\widehat{u}_1(q,\widetilde{\beta})^q dz + \varepsilon(\widehat{\lambda}_1(q,\widetilde{\beta}) + 1)\Big)
\end{aligned} \tag{17}$$

(see (14)–(16) and recall that $\|\widehat{u}_1(q,\widetilde{\beta})\|_q = 1$). Note that

$$\int_\Omega \big(\eta(z) - \widetilde{c}\widehat{\lambda}_1(q,\widetilde{\beta})\big)\widehat{u}_1(q,\widetilde{\beta})^q \, dz = \xi^* > 0.$$

From (17), we have

$$\widehat{\varphi}(\widehat{\xi}\widehat{u}_1(q,\widetilde{\beta})) \leqslant \frac{\widehat{\xi}^q}{q}(-\xi^* + \varepsilon\mu) \quad \text{with } \mu = \widehat{\lambda}_1(q,\widetilde{\beta}) + 1 > 0.$$

Choosing $\varepsilon \in (0, \frac{\xi^*}{\mu})$, we have $\widehat{\varphi}(\widehat{\xi}\widehat{u}_1(q,\widetilde{\beta})) < 0$, so $\widehat{\varphi}(u_0) < 0 = \widehat{\varphi}(0)$ (see (13)), hence $u_0 \neq 0$.

From (13), we have $\widehat{\varphi}'(u_0) = 0$, so

$$\begin{aligned} &\langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h\,dz + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1} h\,d\sigma \\ &= \int_\Omega \widehat{f}(z, u_0)h\,dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{18}$$

In what follows, by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair of spaces $(W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*, W_0^{1,p}(\Omega))$ (where $\frac{1}{p'} + \frac{1}{p} = 1$). From the representation theorem for the elements of $W^{-1,p'}(\Omega)$ (see e.g., Gasiński-Papageorgiou [11, p. 212]), we have

$$\operatorname{div} a(\nabla u_0) \in W^{-1,p'}(\Omega).$$

Performing integration by parts, we obtain

$$\langle A(u_0), v\rangle = \langle -\operatorname{div} a(\nabla u_0), v\rangle_0 \quad \forall v \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

Using this equality in (18), we have

$$\langle -\operatorname{div} a(\nabla u_0), v\rangle + \int_\Omega |u_0|^{p-2}u_0 v\,dz = \int_\Omega \widehat{f}(z, u_0)v\,dz \quad \forall v \in W_0^{1,p}(\Omega),$$

so

$$-\operatorname{div} a(\nabla u_0(z)) + |u_0(z)|^{p-2}u_0(z) = \widehat{f}(z, u_0(z)) \quad \text{for almost all } z \in \Omega. \tag{19}$$

So, we can apply the nonlinear Green's identity (see e.g., Gasiński-Papageorgiou [11, p. 210]) and have

$$\langle A(u_0), h\rangle + \int_\Omega \operatorname{div} a(\nabla u_0)h\,dz = \left\langle \frac{\partial u_0}{\partial n_a}, h\right\rangle_{\partial\Omega} \quad \forall h \in W^{1,p}(\Omega), \tag{20}$$

where by $\langle\cdot,\cdot\rangle_{\partial\Omega}$ we denote the duality brackets for the pair of spaces

$$(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega)).$$

Returning to (18) and using (20), we obtain

$$\begin{aligned} &-\int_\Omega \operatorname{div} a(\nabla u_0)h\,dz + \left\langle \frac{\partial u}{\partial n_a}, h\right\rangle_{\partial\Omega} + \int_\Omega |u_0|^{p-2}u_0 h\ dz + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1}h\,d\sigma \\ &= \int_\Omega \widehat{f}(z, u_0)h\,dz \quad \forall h \in W^{1,p}(\Omega), \end{aligned}$$

so

$$\left\langle \frac{\partial u}{\partial n_a}, h\right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1}h\,d\sigma = 0 \quad \forall h \in W^{1,p}(\Omega) \tag{21}$$

(see (19)). Since $\gamma_0(W^{1,p}(\Omega)) = W^{\frac{1}{p'},p}(\partial\Omega)$, from (21) it follows that

$$\frac{\partial u_0}{\partial n_a} + \beta(z)(u_0^+)^{p-1} = 0 \quad \text{on } \partial\Omega. \tag{22}$$

In (18) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Using Lemma 2.3, we have

$$\frac{c_1}{p-1}\|\nabla u_0^-\|_p^p + \|u_0^-\|_p^p \leqslant 0$$

(see (11)), so $u_0 \geqslant 0$, $u_0 \neq 0$. Then, we have

$$-\operatorname{div} a(\nabla u_0(z)) = f(z, u_0(z)) \quad \text{for almost all } z \in \Omega$$

(see (19) and (11)) and

$$\frac{\partial u}{\partial n_a} + \beta(z)u_0^{p-1} = 0$$

(see (22)) and thus $u_0$ is a positive solution of problem (1).

From Winkert [26], we know that $u_0 \in L^\infty(\Omega)$. So, we can apply the regularity result of Lieberman [20, p. 320] and infer that $u_0 \in C_+ \setminus \{0\}$. Let $\varrho = \|u_0\|_\infty$ and let $\xi_\varrho > 0$ be as in (10). Then

$$-\operatorname{div} a(\nabla u_0(z)) + \xi_\varrho u_0(z)^{p-1} = f(z, u_0(z)) + \xi_\varrho u_0(z)^{p-1} \geqslant 0$$

for almost all $z \in \Omega$ (see (10)), so

$$\operatorname{div} a(\nabla u_0(z)) \leqslant \xi_\varrho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega.$$

We set $\chi(t) = ta_0(t)$ for all $t > 0$. Hypothesis H($a$)(iii) and (7) ensure the following one-dimensional estimate

$$t\chi'(t) = t^2 a_0'(t) + ta_0(t) \geq c_1 t^{p-1} \quad \forall t > 0.$$

Integrating by parts yields

$$\int_0^t s\chi'(s)\, ds = t\chi(t) - \int_0^t \chi(s)\, ds = t^2 a_0(t) - G_0(t) \geq \frac{c_1}{p} t^p.$$

We set

$$H(t) \;=\; t^2 a_0(t) - G_0(t) \quad \text{and} \quad H_0(t) = \frac{c_1}{p} t^p \quad \forall t > 0.$$

Let $\delta \in (0,1)$ and $\varepsilon > 0$. We introduce the sets

$$C_1 = \big\{t \in (0,1) : \; H(t) \geqslant s\big\} \quad \text{and} \quad C_2 = \big\{t \in (0,1) : \; H_0(t) \geqslant s\big\}.$$

Clearly $C_2 \subseteq C_1$ and so $\inf C_1 \leq \inf C_2$. Hence from Leoni [18, p. 6] we have

$$H^{-1}(s) \leq H_0^{-1}(s).$$

It follows that

$$\int_0^\delta \frac{1}{H^{-1}(\frac{\xi_\varrho}{p}s^p)}ds \geq \int_0^\delta \frac{1}{H_0^{-1}(\frac{\xi_\varrho}{p}s^p)}ds = \frac{\xi_p}{c_1}\int_0^\delta \frac{ds}{s} = +\infty.$$

So, we can apply the strong maximum principle of Pucci-Serrin [24, pp. 111] and have $u_0(z) > 0$ for all $z \in \Omega$. The boundary point theorem of Pucci-Serrin [24, pp. 120] implies that $u_0 \in \operatorname{int} C_+$. □

In fact we can establish the existence of a smallest positive solution for problem (1). To this end, we need to do some preparatory work.

Note that hypotheses H($f$)(i) and (iii) imply that given any $\varepsilon > 0$ (to be fixed more precisely in the process of the proof), we can find $c_9 = c_9(\varepsilon) > 0$ such that

$$f(z,\zeta) \geqslant (\eta(z)-\varepsilon)\zeta^{q-1} - c_9\zeta^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{23}$$

We consider the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) = (\eta(z)-\varepsilon)u(z)^{q-1} - c_9 u(z)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0. \end{cases} \tag{24}$$

**Proposition 3.3.** *If hypotheses* H($a$) *and* H($\beta$) *hold, then problem* (24) *has a unique positive solution* $\overline{u} \in \operatorname{int} C_+$.

*Proof.* First we show that a positive solution exists.

To this end let $\psi\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the $C^1$-functional defined by

$$\begin{aligned} \psi(u) = &\int_\Omega G(\nabla u)\,dz + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma \\ &- \frac{1}{q}\int_\Omega (\eta(z)-\varepsilon)(u^+)^q\,dz + \frac{c_9}{r}\|u^+\|_r^r \quad \forall u \in W^{1,p}(\Omega). \end{aligned}$$

Using Corollary 2.4 and hypothesis H($\beta$), we have

$$\begin{aligned} \psi(u) &\geqslant \frac{c_1}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{c_9}{r}\|u^+\|_r^r - \frac{1}{q}\int_\Omega(\eta(z)+1)(u^+)^q\,dz - \frac{1}{p}\|u^+\|_p^p \\ &\geqslant c_{10}\|u\|^p + \Big(\frac{c_9}{r}\|u^+\|_r^{r-q} - c_{11}(\|u^+\|_r^{p-q}+1)\Big)\|u^+\|_r^q, \end{aligned}$$

for some $c_{10}, c_{11} > 0$ (recall that $r > p \geqslant q$). It follows that $\psi$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W^{1,p}(\Omega)$ such that

$$\psi(\overline{u}) = \inf_{u \in W^{1,p}(\Omega)} \psi(u). \tag{25}$$

As in the proof of Proposition 3.2, using hypothesis H$(a)$(iv) and the condition on $\eta$ (see hypothesis H$(f)$(iii)), we have that for $t \in (0,1)$ small,

$$\psi(t\widehat{u}_1(q,\beta)) \leqslant \frac{c_9 t^r}{r}\|\widehat{u}_1(q,\beta)\|_r^r - c_{12}t^q,$$

for some $c_{12} > 0$ (recall that $\|\widehat{u}_1(q,\beta)\|_q = 1$). Since $q < p$, by taking $t \in (0,1)$ even smaller, we have

$$\psi(t\widehat{u}_1(q,\beta)) < 0,$$

so $\psi(\overline{u}) < 0 = \psi(0)$ (see (25)), hence $\overline{u} \neq 0$.

From (25), we have $\psi'(\overline{u}) = 0$, so

$$\begin{aligned} &\langle A(\overline{u}), h\rangle - \int_\Omega (\overline{u}^-)^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)(\overline{u}^+)^{p-1} h\, d\sigma \\ &= \int_\Omega (\eta(z) - \varepsilon)(\overline{u}^+)^{q-1} h\, dz - c_9 \int_\Omega (\overline{u}^+)^{r-1} h\, dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{26}$$

Choosing $h = -\overline{u}^- \in W^{1,p}(\Omega)$ and using Lemma 2.3, we obtain

$$\frac{c_1}{p-1}\|\nabla \overline{u}^-\|_p^p + \|\overline{u}^-\|_p^p \leqslant 0,$$

so $\overline{u} \geqslant 0$ and $\overline{u} \neq 0$. Therefore (26) becomes

$$\langle A(\overline{u}), h\rangle + \int_{\partial\Omega} \beta(z)\overline{u}^{p-1} h\, d\sigma = \int_\Omega (\eta(z) - \varepsilon)\overline{u}^{q-1} h\, dz - c_9 \int_\Omega \overline{u}^{r-1} h\, dz,$$

for all $h \in W^{1,p}(\Omega)$. From this as in the proof of Proposition 3.2, via the nonlinear Green's identity, we infer that $\overline{u}$ is a positive solution of (24). The nonlinear regularity theory (see Lieberman [19]) and the nonlinear maximum principle (see Pucci-Serrin [24]), imply that $\overline{u} \in \operatorname{int} C_+$.

Next we show the uniqueness of this positive solution $\overline{u} \in \operatorname{int} C_+$. To this end we introduce the integral functional $\widehat{\sigma}\colon L^q(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\widehat{\sigma}(u) = \begin{cases} \displaystyle\int_\Omega G(\nabla(u^{\frac{1}{q}}))dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{\frac{p}{q}} d\sigma & \text{if } u \geqslant 0,\ u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

In what follows $\operatorname{dom}\widehat{\sigma} = \{u \in L^q(\Omega) :\ \widehat{\sigma}(u) < +\infty\}$ (the effective domain of $\widehat{\sigma}$). Let $u_1, u_2 \in \operatorname{dom}\widehat{\sigma}$ and let $v_1 = u_1^{\frac{1}{q}}$, $v_2 = u_2^{\frac{1}{q}}$. Then $v_1, v_2 \in W^{1,p}(\Omega)$. We set

$$v = \big(tu_1 + (1-t)u_2\big)^{\frac{1}{q}}, \quad \text{with } 0 \leqslant t \leqslant 1.$$

From Diaz-Saa [5, Lemma 1] we have

$$|\nabla v(z)| \leqslant \big(t|\nabla v_1(z)|^q + (1-t)|\nabla v_2(z)|^g\big)^{\frac{1}{q}},$$

so

$$G_0(|\nabla v(z)|) \leqslant G_0\Big(\big(t|\nabla v_1(z)|^q + (1-t)|\nabla v_2(z)|^q\big)^{\frac{1}{q}}\Big)$$
$$\leqslant tG_0\big(|\nabla u_1(z)^{\frac{1}{q}}|\big) + (1-t)G_0\big(|\nabla u_2(z)^{\frac{1}{q}}|\big)$$

for almost all $z \in \Omega$ (since $G_0$ is increasing and using hypothesis H($a$)(iv)), thus

$$G\big(|\nabla v(z)|\big) \leqslant tG\big(\nabla u_1(z)^{\frac{1}{q}}\big) + (1-t)G\big(\nabla u_2(z)^{\frac{1}{q}}\big) \quad \text{for almost all } z \in \Omega$$

and hence the functional $u \longmapsto \int_\Omega G(\nabla u^{\frac{1}{q}})\, dz$ is convex on $\operatorname{dom}\sigma$..

Also, since $q \leqslant p$ and because $\beta \geqslant 0$ (see hypothesis H($\beta$)), the functional $u \longmapsto \int_{\partial\Omega} \beta(z) u^{\frac{p}{q}} d\sigma$ is convex on $\operatorname{dom}\sigma$. Therefore the integral functional $\widehat{\sigma}$ is convex. Also by Fatou's lemma, we see that $\widehat{\sigma}$ is lower semicontinuous.

Suppose that $\overline{y} \in W^{1,p}(\Omega)$ is another positive solution of problem (24). Then as above (see the first part of the proof), we have that $\overline{y} \in \operatorname{int} C_+$. Therefore, for every $h \in C^1(\overline{\Omega})$ and for $|t| \leqslant 1$ small, we have

$$\overline{u}^q + th,\ \overline{y}^q + th \in \operatorname{dom}\widehat{\sigma} = \{u \in L^q(\Omega):\ \widehat{\sigma}(u) < +\infty\}.$$

Then it is easy to see that $\widehat{\sigma}$ is Gâteaux differentiable at $\overline{u}^q$ and at $\overline{y}^q$ in the direction $h$. Moreover, via the chain rule and the nonlinear Green's identity, we obtain

$$\widehat{\sigma}'(\overline{u}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla \overline{u})}{\overline{u}^{q-1}} h\, dz \quad \forall h \in W^{1,p}(\Omega)$$
$$\widehat{\sigma}'(\overline{y}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla \overline{y})}{\overline{y}^{q-1}} h\, dz \quad \forall h \in W^{1,p}(\Omega)$$

(recall that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$). The convexity of $\sigma$ implies that $\sigma'$ is monotone. Therefore

$$0 \leqslant \frac{1}{q}\int_\Omega\left(\frac{-\operatorname{div} a(\nabla\overline{u})}{\overline{u}^{q-1}} - \frac{-\operatorname{div} a(\nabla\overline{y})}{\overline{y}^{q-1}}\right)(\overline{u}^q - \overline{y}^q)\, dz = \frac{c_9}{q}\int_\Omega (\overline{y}^{r-q} - \overline{u}^{r-q})(\overline{u}^q - \overline{y}^q)\, dz \leqslant 0,$$

so $\overline{u} = \overline{y}$ (since $\zeta \longrightarrow \zeta^{r-q}$ is strictly increasing on $(0,+\infty)$). This proves the uniqueness of the solution $\overline{u} \in \operatorname{int} C_+$. □

In what follows by $S_+$ we denote the set of positive solutions for problem (1). From Proposition 3.2 we have $\emptyset \neq S_+ \subseteq \operatorname{int} C_+$. Moreover, as in Gasiński-Papageorgiou [13] (see the proof of Proposition 3.3), exploiting the monotonicity of $A$, we show that $S_+$ is downward directed, that is if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leqslant u_1$ and $u \leqslant u_2$.

**Proposition 3.4.** *If hypotheses* H($a$), H($\beta$) *and* H($f$) *hold, then* $\overline{u} \leqslant u$ *for all* $u \in S_+$.

*Proof.* Let $u \in S_+$ and consider the following Carathéodory function

$$\gamma(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ (\eta(z)-\varepsilon)\zeta^{q-1} - c_9\zeta^{r-1} + \zeta^{p-1} & \text{if } 0 \leqslant \zeta \leqslant u(z), \\ (\eta(z)-\varepsilon)u(z)^{q-1} - c_9 u(z)^{r-1} + u(z)^{p-1} & \text{if } u(z) < \zeta. \end{cases} \tag{27}$$

We set

$$\Gamma(z,\zeta) = \int_0^\zeta \gamma(z,s)\,ds$$

and consider the $C^1$-functional $\widehat{\psi}\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\psi}(u) = \int_\Omega G(\nabla u)\,dz + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\,d\sigma - \int_\Omega \Gamma(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$. From Corollary 2.4, hypothesis H($\beta$) and (27), we see that $\widehat{\psi}$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_* \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}(\overline{u}_*) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\psi}(u). \tag{28}$$

As in the proof of Proposition 3.3, for $t \in (0,1)$ small (at least such that $t\widehat{u}_1(q,\beta)(z) \leqslant \min_{\overline{\Omega}} u$ for all $z \in \overline{\Omega}$; recall that $\widehat{u}_1(q,\beta), u \in \operatorname{int} C_+$), we have

$$\widehat{\psi}(t\widehat{u}_1(q,\beta)) < 0,$$

so $\widehat{\psi}(\overline{u}_*) < 0 = \widehat{\psi}(0)$ (see (28)), hence $\overline{u}_* \neq 0$.

From (28) we have $\widehat{\psi}'(\overline{u}_*) = 0$, so

$$\langle A(\overline{u}_*), h\rangle + \int_\Omega |\overline{u}_*|^{p-2}\overline{u}_* h\,dz + \int_{\partial\Omega} \beta(z)(\overline{u}_*^+)^{p-1} h\,d\sigma = \int_\Omega \gamma(z,\overline{u}_*)h\,dz \tag{29}$$

for all $h \in W^{1,p}(\Omega)$. In (29) first we choose $h = -\overline{u}_*^- \in W^{1,p}(\Omega)$. Then using Lemma 2.3, we obtain

$$\frac{c_1}{p-1}\|\nabla \overline{u}_*^-\|_p^p + \|\overline{u}_*^-\|_p^p \leqslant 0,$$

so $\overline{u}_* \geqslant 0$, $\overline{u}_* \neq 0$. Also, in (29) we choose $h = (\overline{u}_* - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(\overline{u}_*), (\overline{u}_* - u)^+\rangle + \int_\Omega \overline{u}_*^{p-1}(\overline{u}_* - u)^+ dz + \int_{\partial\Omega} \beta(z)\overline{u}_*^{p-1}(\overline{u}_* - u)^+\,d\sigma \\ &= \int_\Omega \big((\eta(z)-\varepsilon)u^{q-1} - c_9 u^{r-1}\big)(\overline{u}_* - u)^+\,dz + \int_\Omega u^{p-1}(\overline{u}_* - u)^+ dz \\ &\leqslant \langle A(u), (\overline{u}_* - u)^+\rangle + \int_\Omega u^{p-1}(\overline{u}_* - u)^+ dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\overline{u}_* - u)^+\,\sigma \end{aligned}$$

(see (23)), so

$$\begin{aligned}
&\langle A(\overline{u}_*) - A(u), (\overline{u}_* - u)^+\rangle + \int_\Omega (\overline{u}_*^{p-1} - u^{p-1})(\overline{u}_* - u)^+ \, dz \\
&+ \int_{\partial\Omega} \beta(z)(\overline{u}_*^{p-1} - u^{p-1})(\overline{u}_* - u) d\sigma \\
&= 0,
\end{aligned}$$

thus $|\{\overline{u}_* > u\}|_N = 0$, hence $\overline{u}_* \leqslant u$.

So, we have proved that

$$\overline{u}_* \in [0, u] = \{v \in W^{1,p}(\Omega) : \ 0 \leqslant v(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}$$

and $\overline{u}_* \neq 0$. Then from (27) and (29), it follows that

$$\langle A(\overline{u}_*), h\rangle + \int_{\partial\Omega} \beta(z)\overline{u}_*^{p-1} h \, d\sigma = \int_\Omega \big((\eta(z) - \varepsilon)\overline{u}_*^{q-1} - c_9 \overline{u}_*^{r-1}\big) h \, dz \quad \forall\, h \in W^{1,p}(\Omega),$$

so $\overline{u}_*$ is a positive solution of (24) and $\overline{u}_* = \overline{u} \in \operatorname{int} C_+$ (see Proposition 3.3) with $\overline{u} \leqslant u$ for all $u \in S_+$. □

**Proposition 3.5.** *If hypotheses* H$(a)$, H$(\beta)$ *and* H$(f)$ *hold, then problem* (1) *admits the smallest positive solution* $u_* \in \operatorname{int} C_+$.

*Proof.* From Dunford-Schwartz [6, p. 336], we know that we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n.$$

In fact since $S_+$ is downward directed, we may assume that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is decreasing. So, $u_n \leqslant u_1 \in \operatorname{int} C_+$ for all $n \geqslant 1$. We have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h \, d\sigma = \int_\Omega f(z, u_n) h \, dz \quad \forall h \in W^{1,p}(\Omega), \ n \geqslant 1. \tag{30}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ and using Lemma 2.3, hypothesis H$(\beta)$ and the fact that $0 \leqslant u_n(z) \leqslant \|u_1\|_\infty$ for all $z \in \overline{\Omega}$, we infer that the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W^{1,p}(\Omega), \tag{31}$$

$$u_n \longrightarrow u_* \quad \text{in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \tag{32}$$

In (30), we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (31). Then

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_*\rangle = 0,$$

so

$$u_n \longrightarrow u_* \quad \text{in } W^{1,p}(\Omega) \tag{33}$$

(see Proposition 2.6). So, if in (30) we pass to the limit as $n \to +\infty$ and use (33), then

$$\langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h \, d\sigma = \int_\Omega f(z, u_*) h \, dz \quad \forall\, h \in W^{1,p}(\Omega). \tag{34}$$

Also, from Proposition 3.3, we have $\overline{u} \leqslant u_n$ for all $n \geqslant 1$, hence $\overline{u} \leqslant u_*$. This fact and (34) imply that $u_* \in S_+$ and $u_* = \inf S_+$. $\square$

If we strengthen the hypotheses on the reaction $f(z, \cdot)$, we can guarantee the uniqueness of the positive solution for problem (1). The new stronger conditions on $f(z, \zeta)$ are the following.

<u>H$(f)'$:</u> $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$ for almost all $z \in \Omega$, hypotheses H$(f)'$(i)–(iii) are the same as the corresponding hypotheses H$(f)$(i)–(iii) and

(iv) for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{f(z,\zeta)}{\zeta^{q-1}}$ is strictly decreasing on $(0, +\infty)$.

**Proposition 3.6.** *If hypotheses* H$(a)$, H$(\beta)$ *and* H$(f)'$ *hold, then problem* (1) *admits a unique positive solution.*

*Proof.* From Proposition 3.2 we already have one positive solution $u_0 \in \operatorname{int} C_+$. As in the proof of Proposition 3.3, we consider the integral functional

$$\widehat{\sigma}\colon L^q(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

defined by

$$\widehat{\sigma}(u) = \begin{cases} \displaystyle\int_\Omega G(\nabla(u^{\frac{1}{q}}))\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{\frac{p}{q}} d\sigma & \text{if } u \geqslant 0,\ u^{\frac{1}{q}} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From the proof of Proposition 3.3 we know that $\widehat{\sigma}$ is convex and lower semicontinuous. Moreover, from the nonlinear Green's identity, we have

$$\widehat{\sigma}'(u_0^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla u_0)}{u_0^{q-1}} h \, dz \quad \forall h \in W^{1,p}(\Omega). \tag{35}$$

Suppose that $y_0 \in W^{1,p}(\Omega)$ is another positive solution of problem (1). Then again we have $y_0 \in \operatorname{int} C_+$ and

$$\widehat{\sigma}'(y_0^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(\nabla y_0)}{y_0^{q-1}} h \, dz \quad \forall h \in W^{1,p}(\Omega). \tag{36}$$

Exploiting the monotonicity of $\widehat{\sigma}'$, from (35) and (36), we have

$$\begin{aligned} 0 &\leqslant \frac{1}{q}\int_\Omega \left(\frac{-\operatorname{div} a(\nabla u_0)}{u_0^{q-1}} - \frac{-\operatorname{div} a(\nabla y_0)}{y_0^{q-1}}\right)(u_0^q - y_0^q)\,dz \\ &= \int_\Omega \left(\frac{f(z,u_0)}{u_0^{q-1}} - \frac{f(z,y_0)}{y_0^{q-1}}\right)(u_0^q - y_0^q)\,dz \\ &\leqslant 0 \end{aligned}$$

(see (1) and hypothesis $\mathrm{H}(f)'$(iv)), so $u_0 = y_0$ since the function $\zeta \longmapsto \frac{f(z,\zeta)}{\zeta^{q-1}}$ is strictly decreasing on $(0,+\infty)$. Thus the positive solution of (1) is unique. $\square$

## 4. Particular cases

In this section, we show that the previous existence and uniqueness results can be applied to various nonlinear, nonhomogeneous parametric Robin problems.

We start with the following nonlinear $p$-logistic type equation with equidiffusive reaction:

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) = \lambda u(z)^{p-1} - h(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0,\ \lambda > 0. \end{cases} \tag{37}$$

The hypotheses on the perturbation $h(z,\zeta)$ are the following:

<u>$\mathrm{H}(h)$:</u> $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $h(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exist a function $a \in L^\infty(\Omega)_+$ and $r \in (p,p^*)$ such that

$$|h(z,\zeta)| \leqslant a(z)(1+\zeta^{r-1}) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0;$$

(ii) $\lim_{\zeta\to+\infty} \frac{h(z,\zeta)}{\zeta^{p-1}} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) $\lim_{\zeta\to 0^+} \frac{h(z,\zeta)}{\zeta^{q-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iv) for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{h(z,\zeta)}{\zeta^{q-1}}$ is strictly increasing on $(0,+\infty)$.

**Remark 4.1.** If $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ $(1 < p < +\infty)$ and $h(z,\zeta) = h(\zeta) = \zeta^{r-1}$ for all $\zeta \geqslant 0$ with $p < r < p^*$, then we have the classical equidiffusive $p$-logistic equation.

Note that, if $\lambda > \widehat{\lambda}_1(q,\beta)$, the hypotheses $\mathrm{H}(f)'$ are satisfied and so from Proposition 3.6, we have the following result.

**Proposition 4.2.** *If hypotheses* H($a$)*,* H($\beta$) *and* H($h$) *hold and* $\lambda > \widehat{\lambda}_1(q,\beta)$*, then problem* (37) *admits a unique positive solution* $u_\lambda \in \operatorname{int} C_+$*.*

We can also prove the monotonicity of the map $\lambda \longmapsto u_\lambda$.

**Proposition 4.3.** *If hypotheses* H($a$)*,* H($\beta$) *and* H($h$) *hold and* $\lambda > \mu > \widehat{\lambda}_1(q,\beta)$*, then* $u_\mu \leqslant u_\lambda$*.*

*Proof.* We consider the following truncation-perturbation of the reaction for problem (37):

$$k_\mu(z,\zeta) = \begin{cases} 0 & \text{if } \ \zeta < 0, \\ (\mu+1)\zeta^{p-1} - h(z,\zeta) & \text{if } \ 0 \leqslant \zeta \leqslant u_\lambda(z), \\ (\mu+1)u_\lambda(z)^{p-1} - h(z,u_\lambda(z)) & \text{if } \ u_\lambda(z) < \zeta. \end{cases} \tag{38}$$

We set

$$K_\mu(z,\zeta) = \int_0^\zeta k_\mu(z,s)\,ds$$

and consider the $C^1$-functional $\widehat{\varphi}_\mu \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\mu(u) = \int_\Omega G(\nabla u)\,dz + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\,dz - \int_\Omega K_\mu(z,\mu)\,dz,$$

for all $u \in W^{1,p}(\Omega)$. Clearly $\widehat{\varphi}_\mu$ is coercive (see Corollary 2.4 and (38)). Also it is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\mu(u_\mu) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_\mu(u).$$

As before (see for example the proof of Proposition 3.4), we show that

$$u_\mu \in [0,u_\lambda], \quad u_\mu \neq 0,$$

so $u_\mu \in \operatorname{int} C_+$ is the unique positive solution of (37), $u_\mu \leqslant u_\lambda$. □

In fact, we can improve this monotonicity property, provided we strengthen the requirements on the perturbation $h(z,\cdot)$. So, the new hypotheses on $h$ are the following:

H($h$)$'$: $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $h(z,0)=0$ for almost all $z \in \Omega$, hypotheses H($h$)$'$(i)–(iv) are the same as the corresponding hypotheses H($h$)(i)–(iv) and

(v) for every $\varrho > 0$, there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the function $\zeta \longmapsto \xi_\varrho \zeta^{p-1} - h(z,\zeta)$ is nondecreasing on $[0,\varrho]$.

**Remark 4.4.** Evidently the classical equidiffusive perturbation

$$h(z,\zeta) = h(\zeta) = \zeta^{r-1} \quad \forall \zeta \geqslant 0,$$

with $p < r < p^*$ satisfies condition $\mathrm{H}(h)'(v)$.

**Proposition 4.5.** *If hypotheses* $\mathrm{H}(a)$, $\mathrm{H}(\beta)$ *and* $\mathrm{H}(h)'$ *hold then* $\lambda > \widehat{\lambda}_1(q,\beta)$, *then the function* $\lambda \longmapsto u_\lambda$ *is strictly increasing from* $(\widehat{\lambda}_1(q,\beta), +\infty)$ *into* $C_+$, *that is, if* $\mu < \lambda$, *then* $u_\lambda - u_\mu \in \operatorname{int} C_+$.

*Proof.* From Proposition 4.2 we already have $u_\mu \leqslant u_\lambda$. Let $\varrho = \|u_\lambda\|_\infty$ and let $\xi_\varrho > 0$ be as postulated by hypothesis $\mathrm{H}(h)'(v)$. For $\lambda > 0$, let $u_\mu^\delta = u_\mu + \delta \in \operatorname{int} C_+$. Then we have

$$\begin{aligned}
&-\operatorname{div} a(\nabla u_\mu^\delta) + \xi_\varrho (u_\mu^\delta)^{p-1}\\
&\leqslant -\operatorname{div} a(\nabla u_\mu) + \xi_\varrho u_\mu^{p-1} + \chi(\delta)\\
&= \mu u_\mu^{p-1} - h(z,u_\mu) + \xi_\varrho u_\mu^{p-1} + \chi(\delta)\\
&\leqslant \mu u_\lambda^{p-1} - h(z,u_\lambda) + \xi_\varrho u_\lambda^{p-1} + \chi(\delta)\\
&= \lambda u_\lambda^{p-1} - h(z,u_\lambda) + \xi_\varrho u_\lambda^{p-1} - (\lambda-\mu) u_\lambda^{p-1} + \chi(\delta)\\
&\leqslant \lambda u_\lambda^{p-1} - h(z,u_\lambda) + \xi_\varrho u_\lambda^{p-1} - (\lambda-\mu) m_\lambda^{p-1} + \chi(\delta)\\
&\leqslant -\operatorname{div} a(\nabla u_\lambda) + \xi_\varrho u_\lambda^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \delta > 0 \text{ small},
\end{aligned}$$

with $\chi(\delta) \longrightarrow 0^+$ as $\delta \to 0^+$, $m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0$ (see hypothesis $\mathrm{H}(h)'(\mathrm{v})$ and recall that $u_\mu \leqslant u_\lambda$), so

$$u_\mu^\delta \leqslant u_\lambda,$$

hence $u_\lambda - u_\mu \in \operatorname{int} C_+$. □

Next we consider the following nonhomogeneous eigenvalue problem:

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) = \lambda u(z)^{\tau-1} & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n_a} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0,\ \lambda > 0. \end{cases} \tag{39}$$

We assume that $\tau < q \leqslant p$ ($q > 1$ is as in hypothesis $\mathrm{H}(a)(\mathrm{iv})$). Then we see that hypotheses $\mathrm{H}(f)'$ are satisfied and so from Proposition 3.6 and reasoning as in Proposition 4.3 we deduce the following result.

**Proposition 4.6.** *If hypotheses* $\mathrm{H}(a)$ *and* $\mathrm{H}(\beta)$ *hold, then for every* $\lambda > 0$ *problem* (39) *has a unique positive solution* $u_\lambda \in \operatorname{int} C_+$ *and* $\lambda \longmapsto u_\lambda$ *is strictly increasing from* $(0,+\infty)$ *into* $C_+$.

Finally we consider the following sublinear perturbation of the classical eigenvalue problem for $-\Delta_p^R$.

$$\begin{cases} -\Delta_p u(z) = \lambda u(z)^{p-1} + h(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0,\ \lambda \in \mathbb{R}. \end{cases} \tag{40}$$

In this case $a(y) = |y|^{p-2}y$ $(1 < p < +\infty)$ and so $c_1 = p-1$ and $q = p$. The hypotheses on the perturbation $h(z,\zeta)$ are the following:

$\underline{\mathrm{H}(h)''}$: $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $h(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exists a function $a \in L^\infty(\Omega)_+$ such that

$$|h(z,\zeta)| \leqslant a(z)(1+\zeta^{p-1}) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0;$$

(ii) $\lim_{\zeta\to+\infty} \frac{h(z,\zeta)}{\zeta^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iii) $\lim_{\zeta\to 0^+} \frac{h(z,\zeta)}{\zeta^{p-1}} = +\infty$ uniformly for almost all $z \in \Omega$;

(iv) for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{h(z,\zeta)}{\zeta^{p-1}}$ is strictly decreasing on $(0,+\infty)$;

(v) for every $\varrho > 0$ there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the function $\zeta \longmapsto h(z,\zeta) + \xi_\varrho \zeta^{p-1}$ is nondecreasing on $[0,\varrho]$.

Then as before we have the following result.

**Proposition 4.7.** *If hypotheses* $\mathrm{H}(\beta)$ *and* $\mathrm{H}(h)''$ *hold and* $\lambda < \widehat{\lambda}_1(p,\beta)$*, then problem* (40) *admits a unique solution* $u_\lambda \in \operatorname{int} C_+$ *and the map* $\lambda \longmapsto u_\lambda$ *is strictly increasing from* $(-\infty, \widehat{\lambda}_1(p,\beta))$ *into* $C_+$*.*

In fact, if we assume that $h(z,\zeta) > 0$ for almost all $z \in \Omega$ and all $\zeta > 0$, then we can show that the bound $\widehat{\lambda}_1(p,\beta) > 0$ is sharp.

**Proposition 4.8.** *If hypotheses* $\mathrm{H}(\beta)$, $\mathrm{H}(h)''$ *hold* $h(z,\zeta) > 0$ *for almost all* $z \in \Omega$*, all* $\zeta > 0$ *and* $\lambda \geqslant \widehat{\lambda}_1(p,\beta)$*, then problem* (40) *admits no positive solution.*

*Proof.* Let $\lambda \geqslant \widehat{\lambda}_1(p,\beta)$ and assume that problem (40) has a positive solution $u_\lambda \in W^{1,p}(\Omega)$. As in the proof of Proposition 3.2, we can show that $u_\lambda \in \operatorname{int} C_+$. Let $w \in \operatorname{int} C_+$ and consider the function

$$R(w,u_\lambda)(z) = |\nabla w(z)|^p - |\nabla u_\lambda(z)|^{p-2} \left( \nabla u_\lambda(z),\ \nabla \left( \frac{w^p}{u_\lambda^{p-1}} \right)(z) \right)_{\mathbb{R}^N}.$$

Using the generalized Poicone identity (see Allegretto-Huang [1]), we have

$$
\begin{aligned}
0 &\leqslant \int_\Omega R(w,u_\lambda)\,dz \\
&= \|\nabla w\|_p^p - \int_\Omega |\nabla u_\lambda|^{p-2}\left(\nabla u_\lambda(z),\ \nabla\left(\frac{w^p}{u_\lambda^{p-1}}\right)(z)\right)_{\mathbb{R}^N} dz \\
&= \|\nabla w\|_p^p - \int_\Omega (-\Delta_p u_\lambda)\frac{w^p}{u_\lambda^{p-1}} dz - \left\langle \frac{\partial u_\lambda}{\partial n_p}, \frac{w^p}{u_\lambda^{p-1}} \right\rangle_{\partial\Omega} \\
&= \|\nabla w\|_p^p - \lambda\|w\|_p^p - \int_\Omega h(z,u_\lambda)\frac{w^p}{u_\lambda^{p-1}} dz + \int_{\partial\Omega} \beta(z) w^p\, d\sigma \\
&< \|\nabla w\|_p^p + \int_{\partial\Omega} \beta(z) w^p\, d\sigma - \lambda\|w\|_p^p
\end{aligned}
$$

(where we have used Green's identity; see e.g., Gasiński-Papageorgiou [11, p. 210]). Choosing $w = \widehat{u}_1(p,\beta) \in \operatorname{int} C_+$, we have

$$0 < \widehat{\lambda}_1(p,\beta) - \lambda \leqslant 0,$$

a contradiction. Therefore (40) has no positive solution for $\lambda \geqslant \widehat{\lambda}_1(p,\beta)$. □

**Remark 4.9.** Is it possible to have a sharp existence-nonexistence result for the general nonhomogeneous problem (that is, if $\Delta_p u$ is replaced by $\operatorname{div} a(\nabla u)$)?

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