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Sharp Logarithmic Inequalities for Hardy Operators

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Abstract. Let $\ell \geq 1$ be a fixed number. We determine, for each $K > 0$, the best constant $L = L(K, \ell) \in (0, \infty]$ such that the following holds. If f is a function on $(0, 1]$ with $\int_0^1 |f(r)|\mathrm{d}r = 1$, then

$$\int_0^1 t^{\ell-1} \left(\frac{1}{t} \int_0^t |f(r)|\mathrm{d}r \right)^\ell \mathrm{d}t \leq K \int_0^1 |f(r)| \log |f(r)| \, \mathrm{d}r + L.$$

As an application, we derive a sharp local logarithmic estimate for n-dimensional fractional Hardy operator.

Keywords. Maximal operator, fractional maximal operator, LlogL inequality, best constant

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1. Introduction

The motivation for the results of this paper comes from natural questions concerning one-sided Hardy-Littlewood maximal operator on the positive halfline and the fractional Hardy operator on $\mathbb{R}^n$. To put these problems into an appropriate framework, let us start with some related statements from the literature. A classical Hardy inequality states that for any nonnegative function f on the positive halfline $(0, \infty)$ we have the sharp estimate

$$\int_0^\infty \left(\frac{1}{t} \int_0^t f(s)\mathrm{d}s \right)^k \mathrm{d}t \leq \left(\frac{k}{k-1} \right)^k \int_0^\infty f^k(s)\mathrm{d}s, \tag{1}$$

for any exponent $k > 1$. A convenient reference is the monograph [7] by Hardy, Littlewood and Pólya. This inequality is of fundamental importance for analysis and PDEs, and, by now, there are numerous proofs and modifications of this

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significant result. We will be particularly interested in the following extension of (1), established by Hardy and Littlewood in [6]:

$$\int_0^\infty t^\alpha \left(\frac{1}{t}\int_0^t f(s)\mathrm{d}s\right)^\ell \mathrm{d}t \leq \left(\frac{k}{k-1}\right)^k \left(\int_0^\infty f^k(x)\mathrm{d}x\right)^{\frac{\ell}{k}},$$

where $\ell \geq k > 1$ and $\alpha = \frac{\ell}{k} - 1$. However, as Hardy and Littlewood observed, the constant $(k/(k-1))^k$ above is no longer optimal when ℓ is strictly larger than k. Nevertheless, they managed to guess what the best value is, and their conjecture was confirmed a few years later by Bliss [2]. Here is the precise statement.

Theorem 1.1. *Suppose that $1 < k < \ell$ are fixed constants, put $\alpha = \frac{\ell}{k} - 1$ and let f be a nonnegative function on $(0,\infty)$. Then we have*

$$\int_0^\infty t^\alpha \left(\frac{1}{t}\int_0^t f(s)ds\right)^\ell dt \leq C_{k,\ell} \left(\int_0^\infty f^k(x)dx\right)^{\frac{\ell}{k}}, \tag{2}$$

where

$$C_{k,\ell} = \frac{1}{\ell-\alpha-1}\left[\frac{\alpha\Gamma\left(\frac{\ell}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{\ell-1}{\alpha}\right)}\right]^\alpha. \tag{3}$$

The inequality is sharp.

There is a vast literature concerning various extensions and applications of the above results, and it is absolutely impossible to give even a short review here. We refer the interested reader to the monographs [8] by Kufner and Opic, and [9] by Kufner and Persson for an overview of related results. Let us just mention here the works of Aubin [1] and Talenti [19], who linked the above estimates with a sharp version of Sobolev inequality, and the recent papers of Lu, Yan and Zhao [10, 11] for applications concerning the so-called Hardy fractional maximal operator. We will continue the research in the direction of the latter two papers, but we postpone the definition of the fractional operators and the statements of the results to Section 4 below.

The main purpose of this paper is to take a look at the case $k = 1$ in (2). For this value of k, the constant $C_{k,\ell}$ is infinite; hence, it is natural to consider a slight modification of the estimate, which involves an appropriate logarithmic (or rather entropic) term:

$$\int_0^1 t^{\ell-1}\left(\frac{1}{t}\int_0^t f(r)\mathrm{d}r\right)^\ell \mathrm{d}t \leq K\left(\int_0^1 f(r)\mathrm{d}r\right)^{\ell-1} \mathrm{Ent}\, f + L\left(\int_0^1 f(r)\mathrm{d}r\right)^\ell. \tag{4}$$

Here $\mathrm{Ent}\, f$, the entropy of a function f over $[0,1]$, is defined by

$$\mathrm{Ent}\, f = \int_0^1 f(r)\log f(r)\mathrm{d}r - \int_0^1 f(r)\mathrm{d}r\,\log\int_0^1 f(r)\mathrm{d}r.$$

Note that the assumption $\int_0^1 f(r)\mathrm{d}r = 1$, which can be imposed due to the homogeneity of both sides, transforms this bound into the simpler form presented in the abstract.

An important remark is in order. Note that there is no hope for the above inequality to hold on the whole halfline $[0,\infty)$ (i.e., we cannot replace the interval $(0,1]$ by $(0,\infty)$): this can be seen by inserting $f = \chi_{(0,1]}$, for which the left-hand side would become infinite. However, using the substitution $t := Mt$ and $r := Mr$, we easily transform (4) into the following version on an interval $(0, M]$:

$$\int_0^M t^{\ell-1}\left(\frac{1}{t}\int_0^t f(r)\mathrm{d}r\right)^{\ell}\mathrm{d}t \leq K\left(\int_0^M f(r)\mathrm{d}r\right)^{\ell-1}\operatorname{Ent} f+(L+\log M)\left(\int_0^M f(r)\mathrm{d}r\right)^{\ell},$$

where this time Ent denotes the entropy over the interval $(0, M]$.

We come back to (4). As with any estimate of this type, the following two natural problems can be studied (cf. Zygmund [24]):

(I) For which $K > 0$ there is a universal $L < \infty$ such that the estimate holds?

(II) For K as in (I), what is the best (i.e., the smallest) value $L(K,\ell)$ of L?

Our principal goal is to answer both these questions. Here is the main result.

Theorem 1.2. (i) *If $\ell = 1$, then*

$$L(K,\ell) = \begin{cases} \infty & \text{if } K \leq 1, \\ K\log\frac{K}{K-1} & \text{if } K > 1. \end{cases}$$

(ii) *If $\ell > 1$, then*

$$L(K,\ell) = \begin{cases} \infty & \text{if } K < 1, \\ \frac{K^{\frac{\ell}{\ell-1}}}{\ell-1}\int_0^{K^{-1}} s^{\frac{1}{\ell-1}-1}\log\frac{1}{1-s}\,\mathrm{d}s & \text{if } K \geq 1. \end{cases}$$

A few words about the proof and the organization of the paper are in order. Though a natural idea is to try some calculus of variation arguments, we will not choose this path. One of the important contributions of this work is the novel approach, which rests on the construction of a certain special function, having appropriate majorization and monotonicity properties. Thus, it can be regarded as a modification of the so-called Bellman function method, a powerful technique which has gathered a lot of interest in the recent literature on probability and harmonic analysis. Consult, for instance, the works of Burkholder [3,4], Nazarov, Treil and Volberg [12–14], Osękowski [15], Slavin and Vasyunin [16,17], Vasyunin [20], and Vasyunin and Volberg [21–23], and the references therein. This interesting connection certainly deserves a further exploration and, as we believe, can be exploited to prove a number of significant results.

Theorem 1.2 will be proved in the next two sections. Section 2 handles the case $\ell = 1$, which is slightly easier; Section 3 studies the case $\ell > 1$. The final part of the paper is devoted to applications: we prove there related sharp estimates for Hardy fractional maximal operator.

2. The case $\ell = 1$

2.1. Proof of (4) **with** $L = L(K, 1)$**.** Clearly, we may assume that $K > 1$, since otherwise there is nothing to prove. For the sake of convenience and clarity, we have decided to split the reasoning into a few intermediate steps.

Step 1. *Some definitions and reductions.* As announced in the previous section, the key role in the proof is played by a certain special function. To introduce it, we need an auxiliary technical object. We start with the observation that the function $x \mapsto x - \log x$ is strictly increasing on the interval $[1, \infty)$ and takes value 1 at 1. Consequently, for any $s \geq 0$ there is a unique $\varphi = \varphi(s) \in [1, \infty)$ satisfying

$$\varphi(s) - \log \varphi(s) = s + 1. \tag{5}$$

By some standard theorem on regularity of implicit functions, we infer that φ is of class C^1. Furthermore, by the direct differentiation of the above equality, we get

$$\varphi'(s) - \frac{\varphi'(s)}{\varphi(s)} = 1. \tag{6}$$

Now we are ready to introduce the special function on which the whole argumentation will be based. Namely, consider the function B, defined on the set $\{(x, y) \in (0, \infty) \times \mathbb{R} : \ y \geq x \log x\}$ by the formula

$$B(x, y) = x\varphi\left(\frac{y}{x} - \log x\right).$$

Before we turn to the proof of the entropic estimate, observe that it is enough to study the estimate for continuous and strictly positive functions f only; this follows at once from some standard approximation arguments. Given such an f, we introduce the associated operators X and Y, given by

$$X_t(f) = \frac{1}{1-t}\int_0^{1-t} f(s)\mathrm{d}s \quad \text{and} \quad Y_t(f) = \frac{1}{1-t}\int_0^{1-t} f(s)\log f(s)\mathrm{d}s, \tag{7}$$

for $t \in [0, 1)$. These two objects have a very nice probabilistic interpretation; since this interpretation will not be used in any arguments below, we have decided to postpone its description to Remark 2.2 below. For now, observe that by Jensen's inequality, the pair $(X_t(f), Y_t(f))$ takes values in the domain of the function B (note that $X_t(f) > 0$ since f is assumed to be positive).

Step 2. *A key lemma.* The crucial interplay between X, Y and the special function B is studied in the statement below.

Lemma 2.1. *For any $t \in [0,1)$ we have*

$$B(X_0(f), Y_0(f)) \geq (1-t)B(X_t(f), Y_t(f)) + \int_{1-t}^{1} \frac{1}{x} \int_0^x f(r)\mathrm{d}r \, \mathrm{d}x.$$

Proof. Denote the right-hand side by $F(t)$. We will show that the function F is nonincreasing: this will clearly prove the claim, as the left hand side is equal to $F(0)$. Since f is continuous, we see that F is of class C^1 and hence it suffices to show that $F'(t) \leq 0$ for $t \in (0,1)$. A direct differentiation yields

$$F'(t) = \varphi'(s)\left[-f(1-t)\log f(1-t) + \frac{f(1-t)Y_t(f)}{X_t(f)} - X_t(f) + f(1-t)\right] - f(1-t)\varphi(s) + X_t(f),$$

where

$$s = \frac{Y_t(f)}{X_t(f)} - \log X_t(f). \tag{8}$$

Substituting $x = X_t(f)$, $y = Y_t(f)$ and $d = f(1-t)$, the inequality $F'(t) \leq 0$ reduces to

$$\varphi'(s)\left[-d\log d + d\frac{y}{x} - x + d\right] - d\varphi(s) + x \leq 0.$$

We have $\varphi'(s) > 0$, so the left-hand side, considered as a function of d, attains its maximum for d satisfying $\varphi'(s)(-\log d + \frac{y}{x}) - \varphi(s) = 0$, or

$$d = \exp\left(\frac{y}{x} - \frac{\varphi(s)}{\varphi'(s)}\right) = x\exp\left(s - \frac{\varphi(s)}{\varphi'(s)}\right). \tag{9}$$

Plugging this above, we compute that the maximum is equal to

$$x\left(\varphi'(s)\exp\left(s - \frac{\varphi(s)}{\varphi'(s)}\right) - \varphi'(s) + 1\right), \tag{10}$$

which is zero. Indeed, by (6), we have $\frac{\varphi(s)}{\varphi'(s)} = \varphi(s) - 1$ and

$$\exp\left(s - \frac{\varphi(s)}{\varphi'(s)}\right) = \exp\left(s - \varphi(s) + 1\right) = \frac{1}{\varphi(s)}, \tag{11}$$

where in the last passage we have exploited (5). Thus, the expression in (10) is

$$x\left(\frac{\varphi'(s)}{\varphi(s)} - \varphi'(s) + 1\right) = 0,$$

where the equality follows from (6). This shows that F is indeed nonincreasing, and the proof of the lemma is finished. □

Step 3. *The completion of the proof.* Note that the function φ is concave: this follows at once from the fact that φ is strictly increasing and the identity

$$\varphi'(s) = 1 + \frac{1}{\varphi(s)-1},$$

which, in turn, is a consequence of (6). Moreover, this identity and the facts that $\lim_{s\downarrow 0}\varphi(s) = 1$ and $\lim_{s\to\infty}\varphi(s) = \infty$, also give $\lim_{s\downarrow 0}\varphi'(s) = \infty$ and $\lim_{s\to\infty}\varphi'(s) = 1$. Thus, for any $K \in (1,\infty)$ there is a unique line γ_K of slope K, tangent to the graph of φ. If $(s_0, \varphi(s_0))$ denotes the tangency point, we compute that $\varphi(s_0) = \frac{K}{K-1}$ by means of (6), and hence, by (5),

$$s_0 = \frac{1}{K-1} - \log\frac{K}{K-1}.$$

Consequently, the formula for the tangent line reads

$$\gamma_K(s) = K\left(s - \frac{1}{K-1} + \log\frac{K}{K-1}\right) + \frac{K}{K-1} = Ks + K\log\frac{K}{K-1}.$$

Therefore, since $\frac{\int_0^1 f(r)\log f(r)\mathrm{d}r}{\int_0^1 f(r)\mathrm{d}r} - \log\int_0^1 f(r)\mathrm{d}r \geq 0$, we can proceed as follows:

$$\begin{aligned}
&K \operatorname{Ent} f + K\log\frac{K}{K-1}\int_0^1 f(r)\mathrm{d}r\\
&= \int_0^1 f(r)\mathrm{d}r\cdot\gamma_K\left(\frac{\int_0^1 f(r)\log f(r)\mathrm{d}r}{\int_0^1 f(r)\mathrm{d}r} - \log\int_0^1 f(r)\mathrm{d}r\right)\\
&\geq \int_0^1 f(r)\mathrm{d}r\cdot\varphi\left(\frac{\int_0^1 f(r)\log f(r)\mathrm{d}r}{\int_0^1 f(r)\mathrm{d}r} - \log\int_0^1 f(r)\mathrm{d}r\right)\\
&= B\left(X_0(f), Y_0(f)\right)\\
&\geq \int_{1-t}^1 \frac{1}{x}\int_0^x f(r)\mathrm{d}r\,\mathrm{d}x,
\end{aligned} \tag{12}$$

for any $t \in (0,1)$. Here in the last passage we have exploited Lemma 2.1 and the fact that B is nonnegative. Letting t go to 1 and applying Fatou's lemma, we get the desired estimate.

Remark 2.2. The operators X and Y have an important interpretation in terms of martingales (cf. Doob [5]). Namely, let $([0,1], \mathcal{B}([0,1]), |\cdot|)$ be the Lebesgue probability space and for each $t \in [0,1]$, let $\mathcal{F}_t$ be the σ-algebra generated by $[0,1-t]$ and all the Borel subsets of $(1-t,1]$. Then $(\mathcal{F}_t)_{t\in[0,1]}$ forms a filtration, i.e., we have $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leq s < t \leq 1$. Let $f : [0,1] \to \mathbb{R}$ be a function satisfying the LlogL integrability condition $\int_0^1 f(r)\log f(r)\mathrm{d}r < \infty$, and consider the associated martingales

$$M_t = \mathbb{E}\left(f|\mathcal{F}_t\right) \quad \text{and} \quad N_t = \mathbb{E}\left(f\log f|\mathcal{F}_t\right), \quad t\in[0,1].$$

Then for any $\omega\in[0,1]$ and any $t\in[0,1]$ we have, with probability 1,

$$M_t(\omega) = \begin{cases} \frac{1}{1-t}\int_0^{1-t} f(r)\mathrm{d}r & \text{if } t\le 1-\omega, \\ f(\omega) & \text{if } t > 1-\omega \end{cases}$$

and

$$N_t(\omega) = \begin{cases} \frac{1}{1-t}\int_0^{1-t} f(r)\log f(r)\mathrm{d}r & \text{if } t\le 1-\omega, \\ f(\omega)\log f(\omega) & \text{if } t > 1-\omega. \end{cases}$$

Therefore, $X_t(f)$, $Y_t(f)$ can be regarded as “nontrivial” or “running” parts of the martingales corresponding to f and $f\log f$. This observation also sheds some additional light on Lemma 2.1 and exhibit its connection with Bellman function method. Roughly speaking, the lemma states that the composition of the special function B with the appropriate martingales $X(f)$, $Y(f)$ has a monotonicity property. This is the underlying concept of Bellman function method: see [15] for the general exposition on the subject from the probabilistic point of view.

2.2. Optimality of $L(K,1)$ and the search for extremal functions. Now we will prove that the entropic bound we have just proved above is sharp for each K; this will be done by exhibiting appropriate examples. Assume first that $K>1$. To give the reader some ideas how the extremal functions can be discovered, let us inspect carefully the above argumentation. The first thought which comes into one's mind is to search for those f, which give equality in Lemma 2.1; that is, for those which produce a constant function F defined there. To accomplish this, we must ensure that for any $t\in(0,1)$ we have

$$f(1-t) = \frac{1}{1-t}\int_0^{1-t} f(r)\mathrm{d}r\cdot\exp\left(s-\frac{\varphi(s)}{\varphi'(s)}\right), \tag{13}$$

see (9). Passing from t to $1-t$ in (13) we obtain, using (8) and (11),

$$\varphi(s) = \frac{\frac{1}{t}\int_0^t f(r)\mathrm{d}r}{f(t)}, \quad t\in(0,1),$$

with

$$s = \frac{\int_0^{1-t} f(r)\log f(r)\mathrm{d}r}{\int_0^{1-t} f(r)\mathrm{d}r} - \log\left(\frac{1}{1-t}\int_0^{1-t} f(r)\mathrm{d}r\right)$$

(by Jensen's inequality, s is nonnegative). Hence, by (5),

$$\frac{\frac{1}{t}\int_0^t f(r)\mathrm{d}r}{f(t)} - \log\left(\frac{\frac{1}{t}\int_0^t f(r)\mathrm{d}r}{f(t)}\right) = \frac{\int_0^t f(r)\log f(r)\mathrm{d}r}{\int_0^t f(r)\mathrm{d}r} - \log\left(\frac{1}{t}\int_0^t f(r)\mathrm{d}r\right) + 1,$$

for all $t \in (0,1)$. This is further equivalent to saying that

$$\frac{\left(\int_0^t f(r)\mathrm{d}r\right)^2}{tf(t)} + \log f(t)\int_0^t f(r)\mathrm{d}r = \int_0^t f(r)\log f(r)\mathrm{d}r + \int_0^t f(r)\mathrm{d}r.$$

Assume for a moment that f is of class C^1 and let us differentiate both sides above. We obtain

$$\frac{2}{t}\int_0^t f(r)\mathrm{d}r - \frac{\left(\frac{1}{t}\int_0^t f(r)\mathrm{d}r\right)^2}{f(t)} - \frac{\left(\int_0^t f(r)\mathrm{d}r\right)^2}{tf(t)^2}f'(t) + \frac{f'(t)}{f(t)}\int_0^t f(r)\mathrm{d}r = f(t),$$

which, after some manipulations, can be rewritten as

$$\left[1 - \frac{1}{tf(t)}\int_0^t f(r)\mathrm{d}r\right]\left[\int_0^t f(r)\mathrm{d}r - tf(t) + \frac{tf'(t)}{f(t)}\int_0^t f(r)\mathrm{d}r\right] = 0.$$

Let us search for f such that the expression in the second square bracket is zero. This is equivalent to $\left(\frac{tf(t)}{\int_0^t f(r)\mathrm{d}r}\right)' = 0$, or $ctf(t) = \int_0^t f(r)\mathrm{d}r$ for some constant c and all $t \in (0,1)$. The latter equation implies that f is of the form $f(t) = \beta t^\lambda$ for some parameters β and λ.

Thus, the above reasoning suggests that the power functions are extremal in (4), and all we need to make appropriate choices for β and λ. Since (4) is homogeneous, we may take $\beta = 1$; to get the value of λ, we simply plug $f(t) = t^\lambda$ into the estimate. Then the left-hand side is $(\lambda+1)^{-2}$; furthermore, we have $\int_0^1 f(r)\mathrm{d}r = (\lambda+1)^{-1}$ and $\int_0^1 f(r)\log f(r)\mathrm{d}r = -\lambda(\lambda+1)^{-2}$, so the right-hand side is equal to

$$K\left[-\frac{\lambda}{(\lambda+1)^2} + \frac{1}{\lambda+1}\log(\lambda+1)\right] + \frac{K}{\lambda+1}\log\frac{K}{K-1}$$

and it is easy to see that the choice $\lambda = -\frac{1}{K}$ makes both sides equal.

This shows that for $K>1$, the estimate (4) is indeed sharp with $L=L(K,1)$: the equality holds for a nontrivial choice of f. To handle the remaining values of the parameter K, note that by the very definition, the function $L(\cdot,1)$ is nonincreasing. In consequence, for any $K \leq 1$ we get

$$L(K,1) \geq \lim_{K'\downarrow 1} L(K',1) = \infty.$$

This completes the proof. $\square$

3. The case $\ell > 1$

Now we turn to the more challenging part of Theorem 1.2. Throughout this section, $\ell > 1$ is given and fixed. Again, we have to split the reasoning into several parts.

3.1. An auxiliary function. We begin our analysis by proving the following crucial technical statement.

Lemma 3.1. *For any $s > 0$ there is a unique $u = u(s) \in (-\infty, 0)$ satisfying*

$$\frac{\ell-1}{\ell}(s-u)(1-e^u)^{\frac{1}{\ell-1}} = \int_u^0 (1-e^r)^{\frac{1}{\ell-1}}\,\mathrm{d}r. \tag{14}$$

Furthermore, u is of class C^∞ on $(0,\infty)$, $\lim_{s\downarrow 0} u(s) = 0$ and $\lim_{s\to\infty} u(s) = -\infty$.

Proof. Consider the function

$$\mathcal{F}(s,u) = \frac{\ell-1}{\ell}(s-u)(1-e^u)^{\frac{1}{\ell-1}} - \int_u^0 (1-e^r)^{\frac{1}{\ell-1}}\,\mathrm{d}r, \quad (s,u) \in \mathbb{R} \times (-\infty, 0].$$

Obviously, $\mathcal{F} \in C^\infty(\mathbb{R} \times (-\infty, 0])$ and the equality (14) can be written in the form

$$\mathcal{F}(s,u) = 0, \quad (s,u) \in [0,\infty) \times (-\infty, 0].$$

Fix $s > 0$ and consider the function $F(u) = \mathcal{F}(s,u)$, $u \in (-\infty, 0]$. We derive that

$$F'(u) = \frac{1}{\ell}(1-e^u)^{\frac{1}{\ell-1}-1}\big[1 - e^u - (s-u)e^u\big].$$

Denote the expression in the square brackets by $G(u)$. We have $G(0) = -s < 0$, $\lim_{u\to-\infty} G(u) = 1$ and $G'(u) = (u-s)e^u < 0$ for $u \in (-\infty, 0)$. Consequently, there is $u_0(s) < 0$ such that G is positive on $(-\infty, u_0)$ and negative on $(u_0, 0)$; therefore, F is increasing on $(-\infty, u_0)$ and decreasing on $(u_0, 0)$. Furthermore, note that $F(0) = 0$ and $\lim_{u\to-\infty} F(u) = -\infty$. The latter identity follows at once from

$$\lim_{u\to-\infty} \frac{\int_u^0 (1-e^t)^{\frac{1}{\ell-1}}\,\mathrm{d}t}{(s-u)(1-e^u)^{\frac{1}{\ell-1}}} = 1,$$

which can be verified with the use of de l'Hospital rule. This proves the existence and uniqueness of $u(s)$. It is clear from the above arguments that $u(s) < u_0(s)$, so

$$1 - e^{u(s)} - (s - u(s))e^{u(s)} > 0. \tag{15}$$

The fact that u is of class C^∞ follows from standard statements concerning regularity of implicit functions. The assertion that $\lim_{s\downarrow 0} u(s) = 0$ is a consequence of $\mathcal{F}(0,0)=0$. To study the limit of u at infinity, differentiate both sides of $\mathcal{F}(s,u)=0$ with respect to s to obtain

$$\frac{\ell-1}{\ell}(1-u')(1-e^u)^{\frac{1}{\ell-1}}-\frac{1}{\ell}(s-u)(1-e^u)^{\frac{1}{\ell-1}-1}e^u u'+(1-e^u)^{\frac{1}{\ell-1}}u'=0,$$

or

$$u'(s)\big[(s-u(s))e^{u(s)}-1+e^{u(s)}\big]=(\ell-1)(1-e^{u(s)}). \tag{16}$$

Together with (15), it implies that $u'(s)<0$, so u is decreasing on $[0,\infty)$ and the limit $\lim_{s\to\infty}u(s)$ exists. It must be equal to $-\infty$, since other possibilities lead to a contradiction with (15). □

Lemma 3.2. *There is a strictly increasing, continuous function $\varphi=\varphi^{(\ell)}$ on $[0,\infty)$, satisfying the differential equation*

$$\exp\left(s-\frac{\ell\varphi(s)}{\varphi'(s)}\right)=1-\frac{1}{\varphi'(s)},\qquad s\in(0,\infty), \tag{17}$$

and the initial condition $\varphi(0)=\frac{1}{\ell}$. Furthermore, φ is strictly concave and satisfies

$$\lim_{s\downarrow 0}\varphi'(s)=\infty \qquad \textit{and} \qquad \lim_{s\uparrow\infty}\varphi'(s)=1.$$

Proof. Let u be the function from the previous lemma and put

$$\varphi(s)=\frac{s-u(s)}{\ell(1-e^{u(s)})},\qquad s\in(0,\infty). \tag{18}$$

Of course, φ is of class C^∞. Let us compute the derivative of φ. Using (16), we see that

$$\begin{aligned}\varphi'(s)&=\frac{1}{\ell(1-e^{u(s)})}+u'(s)\left[\frac{(s-u(s))e^{u(s)}}{\ell(1-e^{u(s)})^2}-\frac{1}{\ell(1-e^{u(s)})}\right]\\&=\frac{1}{\ell(1-e^{u(s)})}+\frac{(\ell-1)(1-e^{u(s)})}{\ell(1-e^{u(s)})^2}\\&=\frac{1}{1-e^{u(s)}}\end{aligned} \tag{19}$$

and hence, in particular, φ is strictly increasing. The latter identity can be rewritten as $e^{u(s)}=1-\frac{1}{\varphi'(s)}$; thus, (17) will be proved if we show that $u(s)=s-\frac{\ell\varphi(s)}{\varphi'(s)}$. But this follows from (18) and (19): indeed,

$$s-\frac{\ell\varphi(s)}{\varphi'(s)}=s-(s-u(s))=u(s).$$

Now we will prove that the function φ satisfies $\lim_{s\downarrow 0}\varphi(s)=\ell^{-1}$. To do this, note that $u(s)\uparrow 0$ as $s\downarrow 0$, by Lemma 3.1. Consequently, by (14) and (18), we may write that

$$\lim_{s\downarrow 0}\varphi(s)=\lim_{u\uparrow 0}\frac{\int_u^0(1-e^t)^{\frac{1}{\ell-1}}\mathrm{d}t}{(\ell-1)(1-e^u)^{\frac{1}{\ell-1}+1}}=\lim_{u\uparrow 0}\frac{1}{\ell e^u}=\frac{1}{\ell}.$$

Next, let us show that φ is concave. This is simple: we have shown in the proof of the previous lemma that u is decreasing and hence, by (19), so is φ'. Finally, let us address the behavior of $\varphi'(s)$ for $s\downarrow 0$ and $s\to\infty$. Since $\varphi(0)=\frac{1}{\ell}$, the equality $\lim_{s\downarrow 0}\varphi'(s)=\infty$ follows directly from (17). To prove that $\lim_{s\to\infty}\varphi'(s)=1$, use (19) and the equation $\lim_{s\to\infty}u(s)=-\infty$ shown in the preceding lemma. This completes the proof. □

3.2. Proof of (4)**.** We will obtain the desired bound in two steps.

Step 1. *A key lemma.* Let $\varphi=\varphi^{(\ell)}$ be the function introduced in the previous subsection. As in the case $\ell=1$, the proof of (4) will exploit the properties of a certain special function. This time $B:\{(x,y)\in(0,\infty)\times\mathbb{R}:y\geq x\log x\}\to\mathbb{R}$ is given by the formula

$$B(x,y)=x^\ell\varphi\left(\frac{y}{x}-\log x\right).$$

For a given continuous function $f:[0,1]\to(0,\infty)$ and any $t\in[0,1)$, let $X_t(f)$, $Y_t(f)$ be given by (7). Here is the analogue of Lemma 2.1.

Lemma 3.3. *For any* $t\geq 0$ *we have*

$$B(X_0(f),Y_0(f))\geq(1-t)^\ell B(X_t(f),Y_t(f))+\int_{1-t}^1\frac{1}{x}\left(\int_0^x f(r)\mathrm{d}r\right)^\ell\mathrm{d}x.$$

Proof. The argument is the same as in the case $\ell=1$. Denote the right-hand side by $F(t)$ and compute that

$$\begin{aligned}&F'(t)\\&=-\ell(1-t)^{\ell-1}B(X_t(f),Y_t(f))+(1-t)^\ell\frac{\partial B}{\partial x}(X_t(f),Y_t(f))\left[\frac{X_t(f)}{1-t}-\frac{f(1-t)}{1-t}\right]\\&\quad+(1-t)^\ell\frac{\partial B}{\partial y}(X_t(f),Y_t(f))\left[\frac{Y_t(f)}{1-t}-\frac{f(1-t)\log f(1-t)}{1-t}\right]+\frac{\left(\int_0^{1-t}f\right)^\ell}{1-t}.\end{aligned}$$

Clearly, we will be done if we show that this derivative is nonpositive. To do this, substitute $x=X_t(f)$, $y=Y_t(f)$, $d=f(1-t)$ and note that $F'(t)\leq 0$ is equivalent to saying that

$$-\ell B(x,y)+\frac{\partial B}{\partial x}(x,y)[x-d]+\frac{\partial B}{\partial y}(x,y)[y-d\log d]+x^\ell\leq 0.$$

By the definition of B, this amounts to saying that

$$-\ell x^\ell \varphi(s)+\left[\ell x^{\ell-1}\varphi(s)+x^\ell\left(-\frac{y}{x^2}-\frac{1}{x}\right)\varphi'(s)\right](x-d)+x^{\ell-1}\varphi'(s)(y-d\log d)+x^\ell$$

is nonpositive, where $s=\frac{y}{x}-\log x$. After some computations, this can be further transformed into the estimate

$$-\ell\varphi(s)+\varphi'(s)\cdot\frac{y}{x}+\left[\ell\varphi(s)-\left(\frac{y}{x}+1\right)\varphi'(s)\right]\left(1-\frac{d}{x}\right)-\varphi'(s)\frac{d\log d}{x}+1\leq 0.$$

As we have already proved, the function φ is strictly increasing. Consequently, the left-hand side above, considered as a function of d, attains its maximum for d satisfying $-\ell\varphi(s)+\left(\frac{y}{x}+1\right)\varphi'(s)=\varphi'(s)(1+\log d)$, or

$$\frac{d}{x}=\exp\left(s-\frac{\ell\varphi(s)}{\varphi'(s)}\right).$$

Furthermore, plugging this extremal d above, we easily check that the maximal value is equal to

$$\varphi'(s)\exp\left(s-\frac{\ell\varphi(s)}{\varphi'(s)}\right)-\varphi'(s)+1,$$

which is zero, in view of (17). This completes the proof of the lemma. □

Step 2. Completion of the proof. As we have shown in Lemma 3.2, the function φ is strictly concave and its derivative behaves appropriately for $s\downarrow 0$ and $s\to\infty$. Consequently, for any $K>1$ there is a unique line γ_K of slope K tangent to the graph of φ. Denoting by $(s_0,\varphi(s_0))$ the tangency point, we have $\varphi'(s_0)=K$ and hence, by (19), $u(s_0)=\log(1-K^{-1})$. Consequently, (14) implies that

$$s_0=\log(1-K^{-1})+\frac{\ell K^{\frac{1}{\ell-1}}}{\ell-1}\int_{\log(1-K^{-1})}^0(1-e^r)^{\frac{1}{\ell-1}}\mathrm{d}r$$

and, by (18),

$$\varphi(s_0)=\frac{K^{\frac{\ell}{\ell-1}}}{\ell-1}\int_{\log(1-K^{-1})}^0(1-e^r)^{\frac{1}{\ell-1}}\mathrm{d}r.$$

Therefore, the line γ_K is given by

$$\gamma_K(s)=Ks-K\log(1-K^{-1})-K^{\frac{\ell}{\ell-1}}\int_{\log(1-K^{-1})}^0(1-e^r)^{\frac{1}{\ell-1}}\mathrm{d}r=Ks+L(K,\ell),$$

where in the last passage we have used integration by parts. It suffices to repeat the argumentation from (12) to obtain (4) in the case $K>1$ (in particular, the reasoning exploits the condition $B\geq 0$, which holds true also when $\ell>1$). The case $K=1$ follows at once by continuity.

3.3. Sharpness and the search for extremal examples. The reasoning is similar to that in the case $\ell = 1$. The main idea is to search for a function f satisfying

$$\frac{f(1-t)}{\frac{1}{1-t}\int_0^{1-t} f(r)\mathrm{d}r} = \exp\left(s - \frac{\ell\varphi(s)}{\varphi'(s)}\right) = e^u$$

for each $t \in (0,1)$. Here, as previously, $s = \frac{\int_0^{1-t} f(r)\log f(r)\mathrm{d}r}{\int_0^{1-t} f(r)\mathrm{d}r} - \log\left(\frac{1}{1-t}\int_0^{1-t} f(r)\mathrm{d}r\right)$ and $u = u(s)$ is given by (14). Passing from t to $1-t$ and applying (14), our goal is to find f such that

$$\begin{aligned}
&\frac{\ell-1}{\ell}\left(\frac{\int_0^t f(r)\log f(r)\mathrm{d}r}{\int_0^t f(r)\mathrm{d}r} - \log f(t)\right)\left(1 - \frac{f(t)}{\frac{1}{t}\int_0^t f(r)\mathrm{d}r}\right)^{\frac{1}{\ell-1}} \\
&= \int_{\log\left[\frac{tf(t)}{\int_0^t f(r)\mathrm{d}r}\right]}^0 (1-e^r)^{\frac{1}{\ell-1}}\mathrm{d}r.
\end{aligned} \tag{20}$$

This equality seems complicated and at the first glance it is not quite clear how to solve it. The key observation, which comes after some experiments, is to reduce it to a simpler statement.

Lemma 3.4. *If f is continuous on $[0,1]$ and satisfies*

$$\left(1 - \frac{f(t)}{\frac{1}{t}\int_0^t f(r)\mathrm{d}r}\right)^{\frac{1}{\ell-1}} = c\int_0^t f(r)\mathrm{d}r, \quad t \in (0,1), \tag{21}$$

for some constant c, then (20) *holds true.*

Proof. Differentiating both sides of (21) gives

$$\frac{1}{\ell-1}\left(1 - \frac{f(t)}{\frac{1}{t}\int_0^t f(r)\mathrm{d}r}\right)^{\frac{2-\ell}{\ell-1}}\left(-\frac{f(t)+tf'(t)}{\int_0^t f(r)\mathrm{d}r} + \frac{tf(t)^2}{\left(\int_0^t f(r)\mathrm{d}r\right)^2}\right) = cf(t),$$

which combined with (21) again, yields the equality

$$\frac{1}{\ell-1}\left(-f(t) - tf'(t) + \frac{tf(t)^2}{\int_0^t f(r)\mathrm{d}r}\right) = f(t)\left(1 - \frac{tf(t)}{\int_0^t f(r)\mathrm{d}r}\right).$$

After some easy manipulations, we get that the above identity is equivalent to

$$\frac{\ell-1}{\ell}f'(t) = \frac{f(t)}{t} + f'(t) - \frac{f(t)^2}{\int_0^t f(r)\mathrm{d}r}. \tag{22}$$

We turn to (20). By (21), we can rephrase it as

$$c\,\frac{\ell-1}{\ell}\left(\int_0^t f(r)\log f(r)\mathrm{d}r-\log f(t)\int_0^t f(r)\mathrm{d}r\right)=\int_{\log\left[\frac{tf(t)}{\int_0^t f(r)\mathrm{d}r}\right]}^0(1-e^r)^{\frac{1}{\ell-1}}\mathrm{d}r. \quad (23)$$

Both sides of (20) are equal to 0 if we let t go to 0, so it is enough to prove that the derivatives are equal. However, if we differentiate (23), we get an equality equivalent to (22). This gives the claim. □

Thus, in the proof of the sharpness, it is reasonable to consider functions f enjoying (21). It is not difficult to solve this equation: it is satisfied by the functions of the form

$$f(t)=\frac{b}{\left(1+(bct)^{\ell-1}\right)^{\frac{\ell}{\ell-1}}},$$

where b is an arbitrary constant. To find the appropriate b and c, let us restrict ourselves to these values, for which we have $b=\left(1+(bc)^{\ell-1}\right)^{\frac{1}{\ell-1}}$. For this choice of b and c, we have $\int_0^1 f(r)\mathrm{d}r=1$. Furthermore, denoting $(bc)^{\ell-1}=d$, we compute that

$$\int_0^1\frac{1}{t}\left(\int_0^t f(r)\mathrm{d}r\right)^\ell\mathrm{d}t=(1+d)^{\frac{\ell}{\ell-1}}\int_0^1\frac{x^{\ell-1}}{(1+dx^{\ell-1})^{\frac{\ell}{\ell-1}}}\mathrm{d}x \quad (24)$$

and, integrating by parts,

$$\int_0^1 f(r)\log f(r)\mathrm{d}r=-\log(1+d)+\ell(1+d)^{\frac{1}{\ell-1}}d\int_0^1\frac{x^{\ell-1}}{(1+dx^{\ell-1})^{\frac{\ell}{\ell-1}}}\mathrm{d}x.$$

Consequently, we see that

$$\begin{aligned}&\int_0^1\frac{1}{t}\left(\int_0^t f(r)\mathrm{d}r\right)^\ell\mathrm{d}t-K\int_0^1 f(r)\log f(r)\mathrm{d}r\\&=K\log(1+d)+(1+d)^{\frac{1}{\ell-1}}(1+d-K\ell d)\int_0^1\frac{x^{\ell-1}}{(1+dx^{\ell-1})^{\frac{\ell}{\ell-1}}}\mathrm{d}x.\end{aligned}$$

Suppose now that $K>1$. Comparing the above expression to the formula for $L(K)$, it is natural to try $d=(K-1)^{-1}$. Then

$$\begin{aligned}&K\log(1+d)+(1+d)^{\frac{1}{\ell-1}}(1+d-K\ell d)\int_0^1\frac{x^{\ell-1}}{(1+dx^{\ell-1})^{\frac{\ell}{\ell-1}}}\mathrm{d}x\\&=K\log\frac{K}{K-1}+\left(\frac{K}{K-1}\right)^{\frac{\ell}{\ell-1}}(1-\ell)\int_0^1\frac{x^{\ell-1}}{\left(1+\frac{x^{\ell-1}}{K-1}\right)^{\frac{\ell}{\ell-1}}}\mathrm{d}x\end{aligned}$$

and the substitution $r = 1 - \left(1 + \frac{x^{\ell-1}}{K-1}\right)^{-1}$ under the integral transforms the above expression into

$$K \log \frac{K}{K-1} - K^{\frac{\ell}{\ell-1}} \int_0^{K-1} r^{\frac{1}{\ell-1}} \frac{\mathrm{d}r}{1-r}.$$

Integrating by parts, we see that this is precisely $L(K,\ell)$ and hence we are done with the case $K > 1$. To get the sharpness for $K = 1$, we argue as in the case $\ell = 1$: the function $L(\cdot,\ell)$ is nonincreasing, so

$$L(1,\ell) \geq \lim_{K\downarrow 1} L(K,\ell) = \frac{1}{\ell-1} \int_0^1 s^{\frac{1}{\ell-1}-1} \log \frac{1}{1-s} \mathrm{d}s,$$

as desired. It remains to handle the case $K<1$. Fix an arbitrary $K'>1$ and let f be the extremal function for this parameter: $f(t) = (1+d)^{\frac{1}{\ell-1}}(1+dt^{\ell-1})^{-\frac{\ell}{\ell-1}}$, where $d = (K'-1)^{-1}$. For this f, we have equality in (4):

$$\int_0^1 \frac{1}{t}\left(\int_0^t f(r)\mathrm{d}r\right)^\ell \mathrm{d}t = K' \int_0^1 f(r)\log f(r)\mathrm{d}r + L(K',\ell)$$

and hence

$$\begin{aligned} L(K,\ell) &\geq \int_0^1 \frac{1}{t}\left(\int_0^t f(r)\mathrm{d}r\right)^\ell \mathrm{d}t - K \int_0^1 f(r)\log f(r)\mathrm{d}r \\ &= \left(1 - \frac{K}{K'}\right)\int_0^1 \frac{1}{t}\left(\int_0^t f(r)\mathrm{d}r\right)^\ell \mathrm{d}t + \frac{K}{K'}L(K',\ell). \end{aligned}$$

But the latter expression converges to infinity as $K' \downarrow 1$. Indeed, substituting $y = dx^{\ell-1}$ in (24), we get

$$\int_0^1 \frac{1}{t}\left(\int_0^t f(r)\mathrm{d}r\right)^\ell \mathrm{d}t = (\ell-1)^{-1}\left(1+\frac{1}{d}\right)^{\frac{\ell}{\ell-1}} \int_0^d \frac{y^{\frac{1}{\ell-1}}}{(1+y)^{\frac{\ell}{\ell-1}}}\mathrm{d}y$$

and it suffices to note that $d \to \infty$ as K' approaches 1. This shows that $L(K,\ell)$ must be infinite and the proof of the sharpness is complete. □

4. Estimates for fractional maximal operators on $\mathbb{R}^n$

This section contains an application of the results obtained above. Let us start with the necessary definitions. For any positive integer n and any $\beta \in (0,n)$, we define Hardy fractional maximal operator $\mathbb{H}_\beta$, acting on locally integrable functions f on $\mathbb{R}^n$, by the formula

$$\mathbb{H}_\beta f(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}} \int_{|y|<|x|} |f(y)|\mathrm{d}y, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

This operator is closely related to other classical objects in analysis: the so-called Hardy-Littlewood fractional maximal operator M_β, which is given by

$$M_\beta f(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\beta}{n}}}\int_{|y-x|<r}|f(y)|\mathrm{d}y,\quad x\in\mathbb{R}^n,$$

and to the Riesz potential I_β, defined by the formula

$$I_\beta f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\beta}}\mathrm{d}y,\quad x\in\mathbb{R}^n.$$

The relation between these operators is the following: as one easily checks,

$$M_\beta f(x)=\sup_{y\in\mathbb{R}^n}(\mathbb{H}_\beta(f(\cdot+x)))(y),\quad x\in\mathbb{R}^n,$$

and $\mathbb{H}_\beta(f)(x)\leq 2^{n-\beta}M_\beta(f)(x)\leq 2^{n-\beta}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{\beta}{n}-1}I_\beta(f)(x)$, $x\in\mathbb{R}^n\setminus\{0\}$, where $w_{n-1}=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ denotes the measure of the unit sphere in $\mathbb{R}^n$. Thus, it of interest to study the action of $\mathbb{H}_\beta$ on L^p spaces. It is not difficult to show that if $1<p<q<\infty$ satisfy $\frac{1}{p}=\frac{1}{q}+\frac{\beta}{n}$, then $\mathbb{H}_\beta$ is bounded as an operator from L^p to L^q: this follows immediately from the above relation between $\mathbb{H}_\beta$ and I_β, and the corresponding result for Riesz potentials (see e.g. Stein [18]). The precise value of the norm has been recently identified by Lu and Zhao in [11]: here is the statement.

Theorem 4.1. *If $0<\beta<n$ and $1<p<q<\infty$ satisfy $\frac{1}{p}=\frac{1}{q}+\frac{\beta}{n}$, then*

$$||\mathbb{H}_\beta||_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}=\left(\frac{p}{q(p-1)}\right)^{\frac{1}{q}}\left(\frac{n}{q\beta}\cdot\mathcal{B}\left(\frac{n}{q\beta},\frac{n(q-1)}{q\beta}\right)\right)^{-\frac{\beta}{n}},$$

where $\mathcal{B}(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt$ is the usual beta function.

The estimates obtained in the previous sections will allow us to prove a certain version of this theorem for $p=1$, which can be regarded as a local boundedness from $L\log L$ to $L^{\frac{n}{n-\beta}}$. The main result of this section can be stated as follows.

Theorem 4.2. *Let n be a positive integer, let $\beta\in(0,n)$ and put $q=\frac{n}{n-\beta}$. Then for any $K>0$ and any locally integrable function f on $\mathbb{R}^n$ satisfying*

$$\frac{1}{|B(0,1)|}\int_{B(0,1)}|f(x)|dx=1,\tag{25}$$

we have the bound

$$\int_{B(0,1)}(\mathbb{H}_\beta f(x))^q dx\leq K|B(0,1)|^{q-1}\int_{B(0,1)}|f(x)|\log|f(x)|dx+L(K,q)|B(0,1)|^q.$$

For any $K>0$, the constant $L(K,q)$ cannot be replaced by a smaller number.

By a standard homogenization argument, we may get rid of the normalization assumption (25) and obtain the following result. Here, for a nonnegative function g on $B(0,1)$,

$$\mathrm{Ent}_{B(0,1)}(g)=\frac{1}{|B(0,1)|}\int_{B(0,1)} g\log g-\left(\frac{1}{|B(0,1)|}\int_{B(0,1)} g\right)\log\left(\frac{1}{|B(0,1)|}\int_{B(0,1)} g\right)$$

denotes the entropy of g over $B(0,1)$.

Corollary 4.3. *Let n be a positive integer, let $\beta\in(0,n)$ and put $q=\frac{n}{n-\beta}$. Then for any $K>0$ and any locally integrable function f on $\mathbb{R}^n$, we have the sharp bound*

$$\begin{aligned}&\int_{B(0,1)}(\mathbb{H}_\beta f(x))^q \mathrm{d}x\\ &\leq K|B(0,1)|\left(\int_{B(0,1)}|f(x)|\mathrm{d}x\right)^{q-1}\mathrm{Ent}_{B(0,1)}(|f|)+L(K,q)\left(\int_{B(0,1)}|f(x)|\mathrm{d}x\right)^{q}.\end{aligned}$$

In the proof of the above theorem, we will need the following lemma, a slight modification of the corresponding fact from [10].

Lemma 4.4. *For a locally integrable function f on $\mathbb{R}^n$, let*

$$g_f(x)=\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}|f(|x|\xi)|\mathrm{d}\xi,\quad x\in\mathbb{R}^n.$$

Then $\mathbb{H}_\beta g_f(x)=\mathbb{H}_\beta(|f|)(x)$ for all x,

$$\int_{B(0,1)} g_f(x)\mathrm{d}x=\int_{B(0,1)}|f(x)|\mathrm{d}x$$

and

$$\int_{B(0,1)} g_f(x)\log g_f(x)\mathrm{d}x\leq\int_{B(0,1)}|f(x)|\log|f(x)|\mathrm{d}x.$$

Proof. The equality $\mathbb{H}_\beta g_f(x)=\mathbb{H}_\beta(|f|)(x)$ can be found in [10]. The second identity follows immediately from the passage to polar coordinates. The last inequality follows from the passage to polar coordinates and Jensen's inequality. Indeed, denoting by e_1 the vector $(1,0,0,\ldots,0)\in\mathbb{R}^n$, we see that

$$\begin{aligned}\int_{B(0,1)} g_f(x)\log g_f(x)\mathrm{d}x&=\int_0^1\int_{\mathbb{S}^{n-1}} g_f(r\xi)\log g_f(r\xi)r^{n-1}\mathrm{d}\xi\mathrm{d}r\\ &=\omega_{n-1}\int_0^1 g_f(re_1)\log g_f(re_1)r^{n-1}\mathrm{d}r.\end{aligned}$$

Now, the function $t \mapsto t \log t$ is convex on $[0, \infty)$ and $g_f(x)$ is the average of the function $\xi \mapsto |f(|x|\xi)|$ over $\mathbb{S}^{n-1}$. Consequently, Jensen's inequality implies

$$\begin{aligned}\int_{B(0,1)} g_f(x) \log g_f(x) \mathrm{d}x &\leq \omega_{n-1} \int_0^1 \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} |f(r\xi)| \log |f(r\xi)| r^{n-1} \mathrm{d}\xi \mathrm{d}r \\ &= \int_{B(0,1)} |f(x)| \log |f(x)| \mathrm{d}x.\end{aligned}$$

This completes the proof. □

Proof of Theorem 4.1. By the above lemma, we see that we may restrict ourselves to functions f which are radial and nonnegative. For such a function, we easily compute that

$$\begin{aligned}\mathbb{H}_\beta f(x) &= \frac{1}{|B(0,1)|^{1-\frac{\beta}{n}} |x|^{n-\beta}} \int_0^{|x|} \int_{\mathbb{S}^{n-1}} f(r\xi) r^{n-1} \mathrm{d}\xi \mathrm{d}r \\ &= |B(0,1)|^{\frac{\beta}{n}} |x|^{\beta-n} \int_0^{|x|^n} f(r^{\frac{1}{n}} e_1) \mathrm{d}r.\end{aligned}$$

Therefore, passing to polar coordinates again, we get

$$\begin{aligned}\int_{B(0,1)} (\mathbb{H}_\beta f(x))^q \mathrm{d}x &= |B(0,1)|^{q\frac{\beta}{n}} \int_0^1 \int_{\mathbb{S}^{n-1}} s^{q(\beta-n)} \left[\int_0^{s^n} f(r^{\frac{1}{n}} e_1) \mathrm{d}r \right]^q s^{n-1} \mathrm{d}\xi \mathrm{d}s \\ &= |B(0,1)|^q \int_0^1 s^{q-1} \left(\frac{1}{s} \int_0^s f(r^{\frac{1}{n}} e_1) \mathrm{d}r \right)^q \mathrm{d}s.\end{aligned}$$

Now, the function $\tilde{f} : [0, \infty) \to [0, \infty)$, given by $\tilde{f}(t) = f(t^{\frac{1}{n}} e_1)$, satisfies $\int_0^1 \tilde{f} \mathrm{d}t = 1$, so (4) yields

$$\begin{aligned}\int_{B(0,1)} (\mathbb{H}_\beta f(x))^q \mathrm{d}x &\leq |B(0,1)|^q \left(K \int_0^1 \tilde{f}(r) \log \tilde{f}(r) \mathrm{d}r + L(K,q) \right) \\ &= |B(0,1)|^q \left(\frac{K}{|B(0,1)|} \int_{B(0,1)} f(x) \log f(x) \mathrm{d}x + L(K,q) \right),\end{aligned}$$

where in the latter passage we have again exploited polar coordinates. This is precisely the estimate of Theorem 4.1. It is also clear that for any $K > 0$, the constant $L(K,q)$ cannot be improved: this follows at once from the sharpness of (4). The proof is complete. □

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