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Nonautonomous Dynamics with Discrete Time and Topological Equivalence

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Abstract. For evolution families with discrete time, we show that any exponential dichotomy is topologically equivalent to a certain normal form, in which the exponential behavior along the stable and unstable directions are multiples of the identity. We consider the general case of a generalized exponential dichotomy in which the usual exponential behavior is replaced by an arbitrary growth rate. In addition, we show that the topological equivalence between two evolution families with generalized exponential dichotomies can be completely characterized in terms of a notion of equivalence between the growth rates.

Keywords. Strong exponential dichotomies, topological equivalence

Mathematics Subject Classification (2010). 37D20

1. Introduction

Our main objective is to study the topological equivalence between evolution families with discrete time that admit an exponential dichotomy. We emphasize that the dynamics may be nonautonomous. Given invertible linear operators A_n and B_n for $n \in \mathbb{Z}$, the two nonautonomous dynamics with discrete time

$$x_{n+1} = A_n x_n \quad \text{and} \quad y_{n+1} = B_n y_n$$

are said to be *topologically equivalent* if there exist homeomorphisms h_m for $m \in \mathbb{Z}$ such that

$$A(m,n) \circ h_n = h_m \circ B(m,n)$$

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for $m, n \in \mathbb{Z}$, where

$$A(m,n) = \begin{cases} A_{m-1} \cdots A_n, & m > n, \\ \mathrm{Id}, & m = n, \\ A_m^{-1} \cdots A_{n-1}^{-1}, & m < n, \end{cases}$$

with a similar definition for $B(m,n)$ with each operator A_n replaced by B_n. In addition, unlike in other works, we consider a condition that allows controlling the asymptotic behavior of the maps h_m at zero and at infinity. Namely, we assume that there exists an increasing continuous map L such that

$$\|h_m(x)\| \le L(\|x\|) \quad \text{and} \quad \|h_m^{-1}(x)\| \le L(\|x\|)$$

for $m \in \mathbb{Z}$. These inequalities ensure that not only the type of stability of the two dynamics coincides, which is a statement essentially of qualitative nature, but also that their quantitative behavior is the same at zero and at infinity.

Moreover, we also consider the general case of a contraction and an expansion with an arbitrary growth rate, instead of only the usual exponential behavior. This type of generalized exponential behavior was considered earlier for example in [1–3] and corresponds to situations when the Lyapunov exponents are all infinite or are all zero, such as when the growth is polynomial.

In the above general setting, that is, for nonautonomous dynamics defined by two-sided sequences of linear operators and possibly with an arbitrary growth rate in the stable and unstable directions, we show that:

1. Any evolution family with a generalized exponential dichotomy is topologically equivalent to a normal form, in the which the behaviors in the stable and unstable directions are multiples of the identity.
2. The topological equivalence between two evolution families admitting generalized exponential dichotomies with different growth rates can be completely characterized by a notion of equivalence between growth rates.

The proof of the first result consists of constructing explicitly the conjugacy maps in the notion of a topological equivalence. For a nonautonomous linear differential equation, it was shown in [4] for finite-dimensional spaces and in [5] for arbitrary Banach spaces that if the equation admits an exponential dichotomy, with the usual growth rate, then the associated evolution family is topologically conjugate to the evolution family of a normal form.

The second result addresses the problem of how one can characterize the notion of topological equivalence between evolution families in terms of a notion of equivalence between their growth rates. Two (increasing sequences of) growth rates ρ_n and ρ'_n are said to be *equivalent* if there exist constants $\alpha, \beta > 0$ such that

$$\rho_m - \rho_n \ge \alpha(\rho'_m - \rho'_n) - \beta \quad \text{and} \quad \rho'_m - \rho'_n \ge \alpha(\rho_m - \rho_n) - \beta$$

for $m \geq n$. It turns out that this notion characterizes completely (and is in fact equivalent to) the notion of topological equivalence between evolution families with generalized exponential dichotomies, possibly with different growth rates.

2. Exponential dichotomies and topological equivalence

2.1. Main result. Let $\mathcal{B}(E)$ be the set of all bounded linear operators acting on a Banach space E. Given a sequence of invertible linear operators $A_m \in \mathcal{B}(E)$, for $m \in \mathbb{Z}$, we consider the *evolution family* $\mathcal{A} = \{A(m,n)\}_{m,n\in\mathbb{Z}}$ formed by the linear operators

$$A(m,n) = \begin{cases} A_{m-1} \cdots A_n, & m > n, \\ \mathrm{Id}, & m = n, \\ A_m^{-1} \cdots A_{n-1}^{-1}, & m < n. \end{cases}$$

Moreover, let $\mathcal{X}$ be the set of all increasing sequences $\rho = (\rho_m)_{m\in\mathbb{Z}}$ of real numbers such that $\rho_0 = 0$ and

$$c = \inf_{m\in\mathbb{Z}} (\rho_{m+1} - \rho_m) > 0.$$

Notice that

$$\lim_{m\to-\infty} \rho_m = -\infty \quad \text{and} \quad \lim_{m\to+\infty} \rho_m = +\infty.$$

Given $\rho \in \mathcal{X}$, we say that a sequence of invertible linear operators $(A_m)_{m\in\mathbb{Z}}$ in $\mathcal{B}(E)$ has a *ρ-strong exponential dichotomy* if there exist projections P_m, for $m \in \mathbb{Z}$, and constants $N, \mu, \nu > 0$ satisfying

$$\begin{gathered} P_m A(m,n) = A(m,n) P_n \\ N^{-1} e^{-\mu(\rho_{m-1}-\rho_{n+1})} \leq \|A(m,n)P_n\| \leq N e^{-\nu(\rho_{m+1}-\rho_{n-1})}, \\ N^{-1} e^{-\mu(\rho_{m-1}-\rho_{n+1})} \leq \|A(n,m)Q_m\| \leq N e^{-\nu(\rho_{m+1}-\rho_{n-1})} \end{gathered}$$

for $m \geq n$, where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$.

Now we introduce a notion of topological equivalence for a nonautonomous dynamics with discrete time that takes into account the growth of the conjugacies at infinity. We say that two evolution families $\mathcal{A} = \{A(m,n)\}$ and $\mathcal{B} = \{B(m,n)\}$ are *topologically equivalent* if there exist homeomorphisms $h_m\colon E \to E$ for $m \in \mathbb{Z}$ and an increasing continuous map $L\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$$L(0) = 0 \quad \text{and} \quad \lim_{\theta\to+\infty} L(\theta) = +\infty$$

such that:

1. $A(m,n) \circ h_n = h_m \circ B(m,n)$ for $m, n \in \mathbb{Z}$;
2. $\|h_m(x)\| \leq L(\|x\|)$ and $\|h_m^{-1}(x)\| \leq L(\|x\|)$ for $m \in \mathbb{Z}$.

The following is our main result. It shows that any strong exponential dichotomy is topologically equivalent to a standard normal form.

Theorem 2.1. *If $\mathcal{A}$ is an evolution family with a ρ-strong exponential dichotomy, then it is topologically equivalent to the evolution family $\mathcal{B}$ defined by*

$$B(m,n) = e^{\rho_n - \rho_m} P_0 + e^{\rho_m - \rho_n} Q_0.$$

2.2. Proof of Theorem 2.1. For each $m \in \mathbb{Z}$ and $x \in E$, let

$$\|x\|_m = \sum_{k \ge m} \|A_P(k,m)x\| + \sum_{k \le m} \|A_Q(k,m)x\|,$$

where

$$A_P(k,m) = A(k,m)P_m \quad \text{and} \quad A_Q(k,m) = A(k,m)Q_m.$$

It is easy to verify that $\|\cdot\|_m$ is a norm on E.

Lemma 2.2. *For each $m \in \mathbb{Z}$ and $x \in E$, we have*

$$\|x\| \le \|x\|_m \le \frac{2N}{1 - e^{-\nu c}} \|x\|. \tag{1}$$

Proof. We have

$$\|x\|_m = \|P_m x\|_m + \|Q_m x\|_m, \tag{2}$$

with

$$\|P_m x\|_m = \sum_{k \ge m} \|A_P(k,m)x\| \quad \text{and} \quad \|Q_m x\|_m = \sum_{k \le m} \|A_Q(k,m)x\|.$$

Since $\|P_m x\|_m \ge \|P_m x\|$ and $\|Q_m x\|_m \ge \|Q_m x\|$, by (2) we have $\|x\|_m \ge \|x\|$.

For the second inequality in (1), we note that

$$\|P_m x\|_m \le \sum_{k \ge m} N e^{-\nu(\rho_k - \rho_m)} \|x\| \le \sum_{k \ge m} N e^{-\nu c(k-m)} \|x\| = \frac{N}{1 - e^{-\nu c}} \|x\|$$

and analogously,

$$\|Q_m x\|_m \le \frac{N}{1 - e^{-\nu c}} \|x\|,$$

This completes the proof of the lemma. □

For each $x \in E$ and $m \in \mathbb{Z}$, let

$$p_m(x) = \|A(m,0)Px\|_m \quad \text{and} \quad q_m(x) = \|A(m,0)Qx\|_m,$$

where $P = P_0$ and $Q = Q_0$.

Lemma 2.3. *The following properties hold:*

1. *for $x \in PE \setminus \{0\}$, the sequence $(p_m(x))_{m\in\mathbb{Z}}$ is strictly decreasing and for $y \in QE \setminus \{0\}$, the sequence $(q_m(x))_{m\in\mathbb{Z}}$ is strictly increasing;*
2. *for each $x \in PE \setminus \{0\}$ there exists a unique $\tau(x) \in \mathbb{Z}$ such that*

$$p_{\tau(x)}(x) \le 1 < p_{\tau(x)-1}(x)$$

and for each $x \in QE \setminus \{0\}$ there exists a unique $\eta(x) \in \mathbb{Z}$ such that

$$q_{\eta(y)-1}(x) < 1 \le q_{\eta(x)}(x).$$

Proof. Note that

$$p_{m-1}(x)-p_m(x)=\|A_P(m-1,0)x\| > 0, \quad q_m(x)-q_{m-1}(x)=\|A_Q(m,0)x\| > 0.$$

This establishes property 1.

Now take $x \in PE \setminus \{0\}$. For $m > 0$, we have

$$p_m(x) \le \sum_{k\ge m} Ne^{-\nu\rho_k}\|x\| \le \sum_{k\ge m} Ne^{-\nu kc}\|x\| = \frac{Ne^{-\nu mc}}{1-e^{-\nu c}}\|x\|$$

and so $p_m(x) \to 0$ when $m \to +\infty$. Moreover, for $k < 0$ we have

$$\|x\| \le \|A_P(0,k)\| \cdot \|A_P(k,0)x\| \le Ne^{\nu\rho_k}\|A_P(k,0)x\| \le Ne^{\nu kc}\|A_P(k,0)x\|$$

and thus, $\|A_P(k,0)x\| \ge \frac{1}{N}e^{-\nu kc}\|x\|$. Hence,

$$p_m(x) = \sum_{k\ge m}\|A_P(k,0)x\| \ge \frac{\|x\|}{N}\sum_{k\ge m} e^{-\nu kc} \ge \frac{\|x\|}{N}e^{-\nu mc}$$

for $m < 0$, which implies that $p_m(x) \to +\infty$ when $m \to -\infty$. Together with property 1, this guarantees the existence and uniqueness of the integer $\tau(x)$ is guaranteed. One can use similar arguments to establish the existence and uniqueness of $\eta(x)$. □

We proceed with the proof of the theorem. Given $x \in PE\setminus\{0\}$ and $m \in \mathbb{Z}$, let $a = a(m,x)$ be the unique integer in $\mathbb{Z}$ such that

$$e^{\rho_{a-1}-\rho_m} < \|x\| \le e^{\rho_a-\rho_m}. \tag{3}$$

We note that there exists a unique $t = t(m,x) \in [0,1)$ such that

$$\|x\| = te^{\rho_{a-1}-\rho_m} + (1-t)e^{\rho_a-\rho_m} \tag{4}$$

Similarly, given $x \in QE \setminus \{0\}$ and $m \in \mathbb{Z}$, let $b = b(m,x)$ be the unique integer in $\mathbb{Z}$ such that

$$e^{\rho_m-\rho_b} \le \|x\| < e^{\rho_m-\rho_{b-1}}.$$

We note that there exists a unique $s = s(m, x) \in [0, 1)$ such that

$$\|x\| = se^{\rho_m - \rho_{b-1}} + (1 - s)e^{\rho_m - \rho_b}.$$

Clearly,

$$a(m, e^{-\rho_m} x) = a(0, x), \quad b(m, e^{\rho_m} x) = b(0, x) \tag{5}$$

and so

$$t(m, e^{-\rho_m} x) = t(0, x), \quad s(m, e^{\rho_m} x) = s(0, x). \tag{6}$$

For each $m \in \mathbb{Z}$, we define a map $f_m \colon PE \to P_m E$ by $f_m(0) = 0$ and

$$f_m(x) = \frac{A(m, 0)x}{tp_{a-1}(x) + (1 - t)p_a(x)}$$

for $x \neq 0$, and a map $g_m \colon QE \to Q_m E$ by $g_m(0) = 0$ and

$$g_m(x) = \frac{A(m, 0)x}{sq_{b-1}(x) + (1 - s)q_b(x)}$$

for $x \neq 0$. Finally, we define $h_m \colon E \to E$ by

$$h_m(x) = f_m(Px) + g_m(Qx).$$

One can easily verify that each map h_m is continuous outside the origin. We consider only f_m (the argument for g_m is analogous). Since the map $x \mapsto A(m, 0)x$ is continuous, in order to show that f_m is continuous we must prove that the function

$$F(x) = tp_{a-1}(x) + (1 - t)p_a(x)$$

is also continuous. Given $m, c \in \mathbb{Z}$, for each $x \in PE \setminus \{0\}$ satisfying

$$e^{\rho_{c-1} - \rho_m} < \|x\| < e^{\rho_c - \rho_m} \tag{7}$$

we have $a(m, x) = c$ and so $t(m, x) = \frac{\|x\| - e^{\rho_c - \rho_m}}{e^{\rho_{c-1} - \rho_m} - e^{\rho_c - \rho_m}}$. Hence, in the set of all x satisfying (7), for some $c \in \mathbb{Z}$, the function

$$F(x) = \frac{\|x\| - e^{\rho_c - \rho_m}}{e^{\rho_{c-1} - \rho_m} - e^{\rho_c - \rho_m}} p_{c-1}(x) + \frac{\|x\| - e^{\rho_{c-1} - \rho_m}}{e^{\rho_{c-1} - \rho_m} - e^{\rho_c - \rho_m}} p_c(x)$$

is continuous (since all norms are continuous). Now take $y \in PE \setminus \{0\}$ with $\|y\| = e^{\rho_{c-1} - \rho_m}$. Letting $x \to y$ with x satisfying (7), we have $a(m, x) = c$ and so $t(m, x) \to 1$, since $\|x\| \to e^{\rho_{c-1} - \rho_m}$. Therefore,

$$F(x) = t(m, x)p_{c-1}(x) + (1 - t(m, x))p_c(x) \to p_{c-1}(y).$$

On the other hand, since $a(m, y) = c - 1$ and $t(m, y) = 0$ we have

$$F(y) = t(m, y)p_{a(m,y)-1}(y) + (1 - t(m, y))p_{a(m,y)}(y) = p_{c-1}(y).$$

Similarly, one can show that $F(x) \to F(y)$ when $x \to y$ with $\|x\| < e^{\rho_{c-1}-\rho_m}$. Therefore, the function F is continuous outside the origin and so the same happens to the map f_m. The continuity of f_m at the origin follows readily from Step 4 below, where it is shown that there exists a continuous map $L\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $L(0) = 0$ such that $\|f_m(x)\| \le L(\|x\|)$.

The remainder of the proof consists of showing that the maps h_m have the required properties in the notion of a topological equivalence.

Step 1: Conjugacy. We prove that

$$h_m \circ (e^{\rho_n-\rho_m}P + e^{\rho_m-\rho_n}Q) = A(m,n) \circ h_n.$$

For this, it suffices to show that

$$f_m(e^{-\rho_m}x) = A(m,0)f_0(x) \quad \text{for } x \in PE$$

and

$$g_m(e^{\rho_m}x) = A(m,0)g_0(x) \quad \text{for } x \in QE. \tag{8}$$

For $t = t(m, e^{-\rho_m}x)$ and $a = a(m, e^{-\rho_m}x)$, by (5) and (6) we obtain

$$\begin{aligned} f_m(e^{-\rho_m}x) &= \frac{A(m,0)e^{-\rho_m}x}{tp_{a-1}(e^{-\rho_m}x) + (1-t)p_a(e^{-\rho_m}x)} \\ &= \frac{A(m,0)e^{-\rho_m}x}{t_0p_{a_0-1}(e^{-\rho_m}x) + (1-t_0)p_{a_0}(e^{-\rho_m}x)} \\ &= A(m,0)f_0(x), \end{aligned}$$

where $t_0 = t(0,x)$ and $a_0 = a(0,x)$. Identity (8) can be obtained in a similar manner.

Step 2: The maps h_m are one-to-one. Assume that $f_m(x) = f_m(y)$ and write $a = a(m,x)$, $a' = a(m,y)$, $t = t(m,x)$ and $t' = t(m,y)$. Then

$$\frac{x}{tp_{a-1}(x) + (1-t)p_a(x)} = \frac{y}{t'p_{a'-1}(y) + (1-t')p_{a'}(y)} = \xi \in PE. \tag{9}$$

Since $p_{a-1} > p_a$, we have

$$p_a(\xi) = \frac{p_a(x)}{tp_{a-1}(x) + (1-t)p_a(x)} \le \frac{p_a(x)}{tp_a(x) + (1-t)p_a(x)} = 1$$

and

$$p_{a-1}(\xi) = \frac{p_{a-1}(x)}{tp_{a-1}(x) + (1-t)p_a(x)} > \frac{p_{a-1}(x)}{tp_{a-1}(x) + (1-t)p_{a-1}(x)} = 1.$$

It follows from Lemma 2.3 that $a = \tau(\xi)$ and one can show in a similar manner that $a' = \tau(\xi)$. Hence $a = a'$ and

$$\frac{x}{tp_{a-1}(x) + (1-t)p_a(x)} = \frac{y}{t'p_{a-1}(y) + (1-t')p_a(y)}.$$

Consequently,

$$\frac{p_a(x)}{tp_{a-1}(x) + (1-t)p_a(x)} = \frac{p_a(y)}{t'p_{a-1}(y) + (1-t')p_a(y)}$$

and

$$\frac{p_{a-1}(x)}{tp_{a-1}(x) + (1-t)p_a(x)} = \frac{p_{a-1}(y)}{t'p_{a-1}(y) + (1-t')p_a(y)}.$$

Therefore,

$$d := \frac{p_a(y)}{p_a(x)} = \frac{p_{a-1}(y)}{p_{a-1}(x)} = \frac{t'p_{a-1}(y) + (1-t')p_a(y)}{tp_{a-1}(x) + (1-t)p_a(x)}$$

and so,

$$1 = \frac{1}{d} \cdot \frac{t'p_{a-1}(y) + (1-t')p_a(y)}{tp_{a-1}(x) + (1-t)p_a(x)} = \frac{t'p_{a-1}(x) + (1-t')p_a(x)}{tp_{a-1}(x) + (1-t)p_a(x)}.$$

Since $p_{a-1}(x) \neq p_a(x)$, this implies that $t = t'$ and by the definition of t and t' we obtain $\|x\| = \|y\|$. Finally, equality (9) implies that $x = \lambda y$ for some $\lambda > 0$, which together with $\|x\| = \|y\|$ yields that $x = y$. Hence, f_m is one-to-one. A similar argument shows that g_m is one-to-one and so the same happens to the map h_m.

Step 3: The maps h_m are onto. If $f_m(x) = y \in P_m E$ for some $x \in E$, then

$$\frac{A(m,0)x}{tp_{a-1}(x) + (1-t)p_a(x)} = y$$

and thus,

$$\frac{x}{tp_{a-1}(x) + (1-t)p_a(x)} = A(0,m)y.$$

Since $p_{a-1} > p_a$, this implies that

$$p_a(A(0,m)y) = \frac{p_a(x)}{tp_{a-1}(x) + (1-t)p_a(x)} \leq 1$$

and

$$p_{a-1}(A(0,m)y) = \frac{p_{a-1}(x)}{tp_{a-1}(x) + (1-t)p_a(x)} > 1.$$

Hence, $a = \tau(A(0,m)y)$. On the other hand, since

$$x = [tp_{a-1}(x) + (1-t)p_a(x)]A(0,m)y, \tag{10}$$

we have

$$\|x\| = te^{\rho_{a-1}-\rho_m} + (1-t)e^{\rho_a-\rho_m} = [tp_{a-1}(x) + (1-t)p_a(x)]\|A(0,m)y\|$$

or, equivalently,

$$tp_{a-1}(x) + (1-t)p_a(x) = \frac{te^{\rho_{a-1}-\rho_m} + (1-t)e^{\rho_a-\rho_m}}{\|A(0,m)y\|}.$$

This leads to

$$x = \frac{te^{\rho_{a-1}-\rho_m} + (1-t)e^{\rho_a-\rho_m}}{\|A(0,m)y\|}A(0,m)y,$$

where $a = \tau(A(0,m)y)$. In order to express x in terms of m and y, it remains to write down explicitly $t = t(m,x)$ in terms of m and y. For this we note that letting

$$z = tp_{a-1}(x) + (1-t)p_a(x),$$

it follows from (10) that

$$p_a(x) = zp_a(A(0,m)y) \quad \text{and} \quad p_{a-1}(x) = zp_{a-1}(A(0,m)y).$$

Multiplying the first identity by $1-t$ and the second by t we obtain

$$z = z[tp_{a-1}(A(0,m)y) + (1-t)p_a(A(0,m)y)]$$

and thus, $tp_{a-1}(A(0,m)y) + (1-t)p_a(A(0,m)y) = 1$. Therefore,

$$t = \frac{1 - p_a(A(0,m)y)}{p_{a-1}(A(0,m)y) - p_a(A(0,m)y)}.$$

We showed that for each $y \in P_mE$, any $x \in E$ satisfying $f_m(x) = y$ is of the form

$$x = \frac{ue^{\rho_{c-1}-\rho_m} + (1-u)e^{\rho_c-\rho_m}}{\|A(0,m)y\|}A(0,m)y, \tag{11}$$

where $c = \tau(A(0,m)y)$ and

$$u = \frac{1 - p_c(A(0,m)y)}{p_{c-1}(A(0,m)y) - p_c(A(0,m)y)}.$$

It remains to prove that any such x indeed satisfies $f_m(x) = y$. We first observe that $u \in [0,1)$. Now let

$$r = ue^{\rho_{c-1}-\rho_m} + (1-u)e^{\rho_c-\rho_m}. \tag{12}$$

By (11) we have $\|x\| = r$ and since $u \in [0,1)$, we obtain

$$e^{\rho_{c-1}-\rho_m} < \|x\| \le e^{\rho_c-\rho_m}.$$

Hence, $c = a$ and, by (4), we have $u = t$. Moreover, by (11),

$$p_a(x) = p_c(x) = \frac{rp_c(A(0,m)y)}{\|A(0,m)y\|} \quad \text{and} \quad p_{a-1}(x) = p_{c-1}(x) = \frac{rp_{c-1}(A(0,m)y)}{\|A(0,m)y\|}.$$

Writing $y_m = A(0,m)y$, we obtain

$$\begin{aligned} &up_{c-1}(x) + (1-u)p_c(x) \\ &= \frac{r}{\|A(0,m)y\|}\left[\frac{1-p_c(y_m)}{p_{c-1}(y_m)-p_c(y_m)}p_{c-1}(y_m) + \frac{p_{c-1}(y_m)-1}{p_{c-1}(y_m)-p_c(y_m)}p_c(y_m)\right] \\ &= \frac{r}{\|A(0,m)y\|}. \end{aligned} \tag{13}$$

On the other hand, it follows from (11) and (12) that $A(m,0)x = \frac{ry}{\|A(0,m)y\|}$ and thus, by (13),

$$f_m(x) = \frac{A(m,0)x}{tp_{a-1}(x) + (1-t)p_a(x)} = \frac{1}{up_{c-1}(x) + (1-u)p_c(x)} \cdot \frac{ry}{\|A(0,m)y\|} = y.$$

This shows that the map f_m is onto. A similar argument applies to g_m and so the map h_m is onto.

Step 4: Existence of an increasing map L_1. Without loss of generality, we assume that $\nu \le 1$ and $\mu \ge 1$. Take $x \in PE \setminus \{0\}$ with $\|x\| \le 1$. By (3), we have $m > a-1$ and thus $m \ge a$. Write $\kappa = \frac{1}{\nu}$. Then

$$\begin{aligned} &\|A(m,0)x\|^\kappa \\ &= t\|A_P(m,0)x\|^\kappa + (1-t)\|A_P(m,0)x\|^\kappa \\ &\le t\big[\|A_P(m,a-1)\| \cdot \|A_P(a-1,0)x\|\big]^\kappa + (1-t)\big[\|A_P(m,a)\| \cdot \|A_P(a,0)x\|\big]^\kappa \\ &\le t\big[Ne^{-\nu(\rho_m-\rho_{a-1})}p_{a-1}\big]^\kappa + (1-t)\big[Ne^{-\nu(\rho_m-\rho_a)}p_a\big]^\kappa \\ &= N^\kappa te^{\rho_{a-1}-\rho_m}p_{a-1}^\kappa + N^\kappa(1-t)e^{\rho_a-\rho_m}p_a^\kappa, \end{aligned}$$

where $p_a = p_a(x)$ and $p_{a-1} = p_{a-1}(x)$. Using the convexity of the function $x \mapsto x^\kappa$ (we are assuming that $0 < \nu \le 1$), we obtain

$$[tp_{a-1} + (1-t)p_a]^\kappa \ge tp_{a-1}^\kappa + (1-t)p_a^\kappa$$

and hence,

$$\|f_m(x)\|^\kappa \le N^\kappa \frac{te^{\rho_{a-1}-\rho_m}p_{a-1}^\kappa + (1-t)e^{\rho_a-\rho_m}p_a^\kappa}{tp_{a-1}^\kappa + (1-t)p_a^\kappa} = N^\kappa \frac{tud + (1-t)cv}{td + (1-t)c},$$

where

$$u = e^{\rho_{a-1}-\rho_m} < v = e^{\rho_a-\rho_m} \quad \text{and} \quad c = p_a^\kappa < d = p_{a-1}^\kappa.$$

It follows from (4) that $\frac{tud+(1-t)cv}{td+(1-t)c} \leq tu + (1-t)v = \|x\|$ and hence,

$$\|f_m(x)\| \leq N\|x\|^\nu \quad \text{whenever} \quad \|x\| \leq 1. \tag{14}$$

Now take $x \in PE$ with $\|x\| > 1$. By (3), we have $a > m$ and thus $a-1 \geq m$. Write $\lambda = \frac{1}{\mu}$. Using the concavity of the function $x \mapsto x^\lambda$ (we are assuming that $\mu \geq 1$), we obtain

$$\begin{aligned}
&\|A_P(m,0)x\|^\lambda \\
&= t\|A_P(m,0)x\|^\lambda + (1-t)\|A_P(m,0)x\|^\lambda \\
&\leq t\big[\|A(m,a-1)\| \cdot \|A_P(a-1,0)x\|\big]^\lambda + (1-t)\big[\|A(m,a)\| \cdot \|A_P(a,0)x\|\big]^\lambda \\
&\leq t\big[Ne^{\mu(\rho_{a-1}-\rho_m)}p_{a-1}\big]^\lambda + (1-t)\big[Ne^{\mu(\rho_a-\rho_m)}p_a\big]^\lambda \\
&= tN^\lambda e^{\rho_{a-1}-\rho_m}p_{a-1}^\lambda + (1-t)N^\lambda e^{\rho_a-\rho_m}p_a^\lambda
\end{aligned}$$

and hence,

$$\|f_m(x)\|^\lambda \leq N^\lambda \frac{te^{\rho_{a-1}-\rho_m}p_{a-1}^\lambda + (1-t)e^{\rho_a-\rho_m}p_a^\lambda}{[tp_{a-1}+(1-t)p_a]^\lambda} = N^\lambda \frac{tuc^\lambda + (1-t)vd^\lambda}{[tc+(1-t)d]^\lambda},$$

where

$$u = e^{\rho_{a-1}-\rho_m} < v = e^{\rho_a-\rho_m} \quad \text{and} \quad d = p_a < c = p_{a-1}. \tag{15}$$

We will show that

$$tuc^\lambda + (1-t)vd^\lambda \leq [tu + (1-t)v] \cdot [tc + (1-t)d]^\lambda. \tag{16}$$

For this we define $f\colon [0,1] \to \mathbb{R}$ by

$$f(t) = [tc + (1-t)d]^\lambda.$$

Since $\lambda \leq 1$, the function is concave and so its derivative f' is decreasing and positive. Note that inequality (16) is equivalent to

$$(1-t)v[f(t) - f(0)] - tu[f(1) - f(t)] \geq 0. \tag{17}$$

By the mean value theorem, there exist $\xi \in (0,t)$ and $\eta \in (t,1)$ such that

$$f(t) - f(0) = tf'(\xi) \quad \text{and} \quad f(1) - f(t) = (1-t)f'(\eta).$$

Hence, taking these ξ and η, inequality (17) is equivalent to

$$t(1-t)vf'(\xi) - t(1-t)uf'(\eta) \geq 0.$$

Therefore, it is sufficient to prove that $vf'(\bar{\xi}) - uf'(\bar{\eta}) > 0$ for any $\bar{\xi} > \bar{\eta}$. In view of (15) we have indeed $vf'(\bar{\xi}) - uf'(\bar{\eta}) > u(f'(\bar{\xi}) - f'(\bar{\eta})) > 0$, which yields inequality (17). Hence, (16) holds and so

$$\|f_m(x)\| \le N\|x\|^\mu \quad \text{whenever} \quad \|x\| > 1. \tag{18}$$

Finally, let

$$M_1(\theta) = N \max\{\theta^\nu, \theta^\mu\}.$$

Note that M_1 is increasing and it follows from (14) and (18) that

$$\|f_m(x)\| \le M_1(\|x\|).$$

A similar argument applies to the maps g_m to produce an increasing function M_2 such that

$$\|g_m(x)\| \le M_2(\|x\|)$$

for $x \in QE$. We define an increasing map L_1 by

$$L_1(\|x\|) = M_1(\|Px\|) + M_2(\|Qx\|).$$

Clearly $L_1(0) = 0$ and $\|h_m(x)\| \le L_1(\|x\|)$ for $x \in E$.

Step 5: Existence of an increasing map L_2. We continue to assume that $\nu \le 1$ and $\mu \ge 1$. Take $x \in PE$ with $\|x\| > 1$. Then $a - 1 \ge m$ and

$$\begin{aligned}
&\|A(m,0)x\| \\
&= t\|A_P(m,0)x\| + (1-t)\|A_P(m,0)x\| \\
&\ge t\|A_P(a-1,m)\|^{-1} \cdot \|A_P(a-1,0)x\| + (1-t)\|A_P(a,m)\|^{-1} \cdot \|A_P(a,0)x\| \\
&\ge tN'e^{-\nu(\rho_{m-1}-\rho_a)}p_{a-1} + (1-t)N'e^{-\nu(\rho_{m-1}-\rho_{a+1})}p_a \\
&\ge N'te^{\nu(\rho_a-\rho_m)}p_{a-1} + N'(1-t)e^{\nu(\rho_a-\rho_m)}p_a \\
&= N'e^{\nu(\rho_a-\rho_m)}(tp_{a-1} + (1-t)p_a),
\end{aligned}$$

where $N' = \frac{1-e^{-\nu c}}{2N^2}$, $p_a = p_a(x)$ and $p_{a-1} = p_{a-1}(x)$. Therefore,

$$\|f_m(x)\| \ge N'e^{\nu(\rho_a-\rho_m)} \frac{tp_{a-1} + (1-t)p_a}{tp_{a-1} + (1-t)p_a} = N'e^{\rho_a-\rho_m} \ge N'\|x\|^\nu.$$

Letting $y = f_m(x)$, we obtain

$$\|f_m^{-1}(y)\| \le (N')^{-\kappa}\|y\|^\kappa \quad \text{whenever} \quad \|x\| > 1. \tag{19}$$

Now take $x \in PE \setminus \{0\}$ with $\|x\| \leq 1$. Then $m \geq a$ and

$$\begin{aligned}
&\|A(m,0)x\| \\
&= t\|A_P(m,0)x\| + (1-t)\|A_P(m,0)x\| \\
&\geq t\|A_P(a-1,m)\|^{-1} \cdot \|A_P(a-1,0)x\| + (1-t)\|A_P(a,m)\|^{-1} \cdot \|A_P(a,0)x\| \\
&\geq tN'e^{-\mu(\rho_{m-1}-\rho_a)}p_{a-1} + (1-t)N'e^{-\mu(\rho_{m-1}-\rho_{a+1})}p_a \\
&\geq N'te^{\mu(\rho_a-\rho_m)}p_{a-1} + N'(1-t)e^{\mu(\rho_a-\rho_m)}p_a \\
&= N'e^{\mu(\rho_a-\rho_m)}(tp_{a-1} + (1-t)p_a).
\end{aligned}$$

Therefore,

$$\|f_m(x)\| \geq N'e^{\mu(\rho_a-\rho_m)}\frac{tp_{a-1} + (1-t)p_a}{tp_{a-1} + (1-t)p_a} = N'e^{\mu(\rho_a-\rho_m)} \geq N'\|x\|^{\mu}.$$

Letting $y = f_m(x)$, we obtain

$$\|f_m^{-1}(y)\| \leq (N')^{-\lambda}\|y\|^{\lambda} \quad \text{whenever} \quad \|x\| \leq 1. \tag{20}$$

Finally, let

$$M_3(\theta) = \max\bigl\{(N')^{-\kappa}\theta^{\nu}, (N')^{-\lambda}\theta^{\mu}\bigr\}.$$

Note that M_3 is increasing and it follows from (19) and (20) that

$$\|f_m^{-1}(y)\| \leq M_3(\|y\|).$$

A similar argument applies to the maps g_m to produce an increasing function M_4 such that

$$\|g_m^{-1}(y)\| \leq M_4(\|y\|)$$

for $y \in QE$. We define an increasing map L_2 by

$$L_2(\|y\|) = M_3(\|Py\|) + M_4(\|Qy\|).$$

Clearly $L_2(0) = 0$ and $\|h_m^{-1}(y)\| \leq L_2(\|y\|)$ for $y \in E$. This completes the proof of Theorem 2.1.

3. Equivalence of growth rates

In this section we introduce a notion of equivalence between sequences in $\mathcal{X}$ and we show that it characterizes completely the notion of a topological equivalence between evolution families.

A sequence $(x_m)_{m\in\mathbb{Z}}$ is said to be *almost increasing* if there exists a constant $\delta \geq 0$ such that

$$x_m - x_n \geq -\delta \quad \text{for} \quad m \geq n.$$

Notice that any sequence that is increasing on $\mathbb{Z} \setminus [a, b]$ for some $a < b$ is also almost increasing.

We define a binary relation $\succ$ in the set $\mathcal{X}$ as follows: $(x_m)_{m\in\mathbb{Z}} \succ (y_m)_{m\in\mathbb{Z}}$ if there exists $a > 0$ such that the sequence $z_m = x_m - a y_m$ is almost increasing, that is, if there exist $a, b \geq 0$ such that

$$x_m - x_n \geq a(y_m - y_n) - b \quad \text{for} \quad m \geq n. \tag{21}$$

Proposition 3.1. *$(x_m)_{m\in\mathbb{Z}} \succ (y_m)_{m\in\mathbb{Z}}$ if and only if there exists a constant $\alpha > 0$ such that for any $\theta \geq 1$ we have*

$$x_m - x_n \leq \theta \quad \Rightarrow \quad y_m - y_n \leq \alpha\theta \tag{22}$$

for each $m \geq n$.

Proof. Assume that $(x_m)_{m\in\mathbb{Z}} \succ (y_m)_{m\in\mathbb{Z}}$ and $x_m - x_n \leq \theta$. It follows from (21) that

$$y_m - y_n \leq \frac{1}{a}(x_m - x_n) + \frac{b}{a} < \frac{1}{a}\theta + \frac{b}{a}\theta = \left(\frac{1}{a} + \frac{b}{a}\right)\theta = \alpha\theta.$$

Now assume that (22) holds. Take $m > n$ and $\theta = q(x_m - x_n) \geq 1$ for some $q \in \mathbb{N}$. Then $\theta \geq qc$ and so,

$$y_m - y_n \leq \alpha\theta = q\alpha(x_m - x_n).$$

This yields that $x_m - x_n \geq \frac{1}{a}(y_m - y_n)$ with $a = q\alpha$. $\square$

We introduce an equivalence relation $\sim$ on the set $\mathcal{X}$ by declaring that $(x_m)_{m\in\mathbb{Z}} \sim (y_m)_{m\in\mathbb{Z}}$ if

$$(x_m)_{m\in\mathbb{Z}} \succ (y_m)_{m\in\mathbb{Z}} \quad \text{and} \quad (y_m)_{m\in\mathbb{Z}} \succ (x_m)_{m\in\mathbb{Z}}.$$

The following result relates in an optimal manner the notion of equivalence between growth rates and the existence of a topological equivalence. Let $x = (x_m)_{m\in\mathbb{Z}}$ and $y = (y_m)_{m\in\mathbb{Z}}$ be growth rates.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be evolution families. If $\mathcal{A}$ admits an x-strong exponential dichotomy with projections P_1 and Q_1 at the origin and $\mathcal{B}$ admits a y-strong exponential dichotomy with projections P_2 and Q_2, then $\mathcal{A}$ and $\mathcal{B}$ are topologically equivalent if and only if:*

1. *the subspaces P_1E and P_2E are homeomorphic;*
2. *the subspaces Q_1E and Q_2E are homeomorphic;*
3. *$x \sim y$.*

Proof. We start with an auxiliary result.

Lemma 3.3. *$x \sim y$ if and only if the evolution families $e^{x_m - x_n}\mathrm{Id}$ and $e^{y_m - y_n}\mathrm{Id}$ are topologically equivalent.*

Proof. Assume first that $x \sim y$. Given $\theta \in \mathbb{R}$ and $m \in \mathbb{Z}$, there exists a unique $a = a(m, |\theta|) \in \mathbb{Z}$ such that

$$e^{x_m - x_a} \le |\theta| < e^{x_m - x_{a-1}}.$$

One can easily verify that $a(m, |\theta| e^{x_m}) = a(0, |\theta|)$. Moreover, let $t = t(m, |\theta|) \in [0, 1)$ be the unique real number for which

$$\log|\theta| = t(x_m - x_{a-1}) + (1 - t)(x_m - x_a). \tag{23}$$

One can also show that $t(m, |\theta| e^{x_m}) = t(0, |\theta|)$.

Similarly, given $\eta \in \mathbb{R}$ and $m \in \mathbb{Z}$, there exists a unique $b = b(m, |\eta|) \in \mathbb{Z}$ such that

$$e^{y_m - y_b} \le |\eta| < e^{y_m - y_{b-1}}.$$

One can easily verify that $b(m, |\eta| e^{y_m}) = b(0, |\eta|)$. Moreover, let $s = s(m, |\eta|) \in [0, 1)$ be the unique real number for which

$$\log|\eta| = s(y_m - y_{a-1}) + (1 - s)(y_m - y_a). \tag{24}$$

One can also show that $s(m, |\eta| e^{y_m}) = s(0, |\eta|) = s_0$.

Now let

$$\varepsilon(\theta) = \begin{cases} 1, & \theta > 0, \\ 0, & \theta = 0, \\ -1, & \theta < 0. \end{cases}$$

For each $m \in \mathbb{Z}$ we consider the function $h_m : \mathbb{R} \to \mathbb{R}$ defined by

$$h_m(\theta) = \varepsilon(\theta) e^{t(y_m - y_{a-1}) + (1-t)(y_m - y_a)}$$

with t as in (23), and the function $g_m : \mathbb{R} \to \mathbb{R}$ defined by

$$g_m(\eta) = \varepsilon(\eta) e^{s(x_m - x_{b-1}) + (1-s)(x_m - x_b)}$$

with s as in (24). It is easy to check that h_m and g_m are continuous and that

$$h_m \circ g_m = g_m \circ h_m = \mathrm{Id}.$$

Moreover, writing $a_0 = a(0, |\theta|)$ and $t_0 = t(0, |\theta|)$, we obtain

$$\begin{aligned} h_m(e^{x_m}\theta) &= \varepsilon(e^{x_m}\theta) e^{t_0(x_m - x_{a_0 - 1}) + (1 - t_0)(x_m - x_a)} \\ &= \varepsilon(\theta) e^{y_m} e^{-t_0 y_{a_0 - 1} - (1 - t_0) y_{a_0}} \\ &= e^{y_m} h_0(\theta). \end{aligned}$$

Therefore, $h_m(e^{x_m-x_n}\theta)=e^{y_m}h_0(e^{-x_n}\theta)=e^{y_m}e^{-y_n}h_n(\theta)=e^{y_m-y_n}h_n(\theta)$.

If $m\ge a$, then

$$\begin{aligned}|h_m(\theta)|&=e^{t(y_m-y_{a-1})+(1-t)(y_m-y_a)}\\&\le e^{\frac{1}{a_1}[t(x_m-x_{a-1})+tb_1+(1-t)(x_m-x_a)+(1-t)b_1]}\\&=e^{\frac{b_1}{a_1}}|\theta|^{\frac{1}{a_1}}=L_{1,1}(|\theta|)\end{aligned}$$

for some $a_1,b_1>0$. If $m<a$, then

$$\begin{aligned}|h_m(\theta)|&=e^{t(y_m-y_{a-1})+(1-t)(y_m-y_a)}\\&\le e^{ta_2(y_m-y_{a-1})+tb_2+(1-t)a_2(y_m-y_a)+(1-t)b_2}\\&=e^{b_2}|\theta|^{a_2}=L_{1,2}(|\theta|)\end{aligned}$$

for some $a_2,b_2>0$. If $m\ge b$, then

$$\begin{aligned}|h_m^{-1}(\eta)|&=e^{s(x_m-x_{b-1})+(1-s)(x_m-x_b)}\\&\le e^{\frac{1}{a_2}[s(y_m-y_{b-1})+sb_2+(1-s)(y_m-y_b)+(1-s)b_2]}\\&=e^{\frac{b_2}{a_2}}|\eta|^{\frac{1}{a_2}}=L_{2,1}(|\eta|).\end{aligned}$$

Finally, if $m<b$, then

$$\begin{aligned}|h_m^{-1}(\eta)|&=e^{s(x_m-x_{b-1})+(1-s)(x_m-x_b)}\\&\le e^{sa_1(y_m-y_{b-1})+sb_1+(1-s)a_1(y_m-y_b)+(1-s)b_1}\\&=e^{b_1}|\eta|^{a_1}=L_{2,2}(|\eta|).\end{aligned}$$

Therefore, one can take

$$L(\theta)=\max\bigl\{L_{1,1}(\theta),L_{1,2}(\theta),L_{2,1}(\theta),L_{2,2}(\theta)\bigr\}.$$

Now let $h_m\colon\mathbb{R}\to\mathbb{R}$ be the functions in the notion of a topological equivalence, which thus satisfy

$$h_m(e^{x_m-x_n}\theta)=e^{y_m-y_n}h_n(\theta).$$

We have $|h_m(\theta)|=e^{y_m-y_n}|h_n(e^{x_n-x_m}\theta)|$ and so,

$$|\theta|\le e^{y_m-y_n}|h_n(e^{x_n-x_m}(h_m^{-1}(\theta)))|\le e^{y_m-y_n}L\bigl(e^{x_n-x_m}L(|\theta|)\bigr).$$

This yields the inequality

$$e^{y_n-y_m}|\theta|\le L(e^{x_n-x_m}L(|\theta|)).$$

For $|\theta| = 1$ we have

$$e^{y_n - y_m} \le L(e^{x_n - x_m} L(1)).$$

If $\theta \ge 1$ and $x_n - x_m \le \theta$, then

$$e^{y_n - y_m} \le L(e^{\theta} L(1)) \le a e^{b\theta} < e^{\theta(\log a + b)}$$

for some constants $a \ge 1$ and $b > 0$ (it follows from the proof of Theorem 2.1 that one can take $L(\theta) = a\theta^b$ for some $a \ge 1$ and $b > 0$). Therefore,

$$y_n - y_m < \theta(\log a + b) = c\theta$$

for $n \ge m$. By Proposition 3.1 we conclude that $y \succ x$. One can show in a similar manner that $x \succ y$. $\square$

We proceed with the proof of the theorem. Assume that $\mathcal{A}$ and $\mathcal{B}$ are topologically equivalent. We consider the evolution families

$$\begin{aligned} A'(m,n) &= e^{x_n - x_m} P_1 + e^{x_m - x_n} Q_1, \\ B'(m,n) &= e^{y_n - y_m} P_2 + e^{y_m - y_n} Q_2, \\ C(m,n) &= e^{x_n - x_m} P_2 + e^{x_m - x_n} Q_2. \end{aligned}$$

Using the symbol $\sim$ to denote topological equivalence, it follows from Theorem 2.1 that $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$. In particular, $\mathcal{A}' \sim \mathcal{B}'$ and so there exist maps h_m as in the notion of topological equivalence such that

$$h_m(e^{x_n - x_m} P_1 z + e^{x_m - x_n} Q_1 z) = (e^{y_n - y_m} P_2 + e^{y_m - y_n} Q_2) h_n(z) \tag{25}$$

for $m, n \in \mathbb{Z}$ and $z \in E$. Replacing z by $P_1 z$ we obtain

$$h_m(e^{x_n - x_m} P_1 z) = e^{y_n - y_m} P_2 h_n(P_1 z) + e^{y_m - y_n} Q_2 h_n(P_1 z).$$

Since

$$\lim_{m \to \infty} h_m(e^{x_n - x_m} P_1 z) = 0 \quad \text{and} \quad \lim_{m \to \infty} e^{y_n - y_m} P_2 h_n(P_1 z) = 0$$

(using in the first identity the second property in the notion of a topological equivalence), we obtain

$$\lim_{m \to \infty} e^{y_m - y_n} Q_2 h_n(P_1 z) = 0$$

and so $Q_2 h_n(P_1 z) = 0$. Therefore, $P_2 h_n(P_1 z) = h_n(P_1 z)$, which yields that

$$h_n(P_1 E) \subset P_2 E. \tag{26}$$

Now we rewrite identity (25) in the form

$$h_m^{-1}(e^{y_n - y_m} P_2 z + e^{y_m - y_n} Q_2 z) = e^{x_n - x_m} P_1 h_n^{-1}(z) + e^{x_m - x_n} Q_1 h_n^{-1}(z).$$

Replacing z by $P_2 z$ we obtain

$$h_m^{-1}(e^{y_n-y_m}P_2z) = e^{x_n-x_m}P_1h_n^{-1}(P_2z) + e^{x_m-x_n}Q_1h_n^{-1}(P_2z).$$

Since

$$\lim_{m\to\infty} h_m^{-1}(e^{y_n-y_m}P_2z) = 0 \quad \text{and} \quad \lim_{m\to\infty} e^{x_n-x_m}P_1h_n^{-1}(P_2z) = 0,$$

we obtain $Q_1h_n^{-1}(P_2z) = 0$. Therefore, $P_1h_n^{-1}(P_2z) = h_n^{-1}(P_2z)$, which yields that

$$h_n^{-1}(P_2E) \subset P_1E. \tag{27}$$

It follows from (26) and (27) that the spaces P_1E and P_2E are homeomorphic. Using similar arguments, one can show that the spaces Q_1E and Q_2E are also homeomorphic. This implies that $\mathcal{A}' \sim \mathcal{C}$ and so $\mathcal{B}' \sim \mathcal{C}$. Hence, it follows from Lemma 3.3 that $x \sim y$.

Now we assume that the three conditions in the theorem are satisfied. Then the unit spheres $S(P_1E)$ and $S(P_2E)$ are homeomorphic, and the same happens to the unit spheres $S(Q_1E)$ and $S(Q_2E)$, via homeomorphisms say

$$f\colon S(P_1E) \to S(P_2E) \quad \text{and} \quad g\colon S(Q_1E) \to S(Q_2E).$$

We define maps $F\colon P_1E \to P_1E$ and $G\colon Q_1E \to Q_2E$ by

$$F(z) = \begin{cases} \|z\| f\big(\frac{z}{\|z\|}\big), & z \in P_1E \setminus \{0\}, \\ 0, & z = 0 \end{cases}$$

and

$$G(z) = \begin{cases} \|z\| g\big(\frac{z}{\|z\|}\big), & z \in Q_1E \setminus \{0\}, \\ 0, & z = 0. \end{cases}$$

One can easily verify that F and G are homeomorphisms, with inverses

$$F^{-1}(z) = \begin{cases} \|z\| f^{-1}\big(\frac{z}{\|z\|}\big), & z \in P_2E \setminus \{0\}, \\ 0, & z = 0, \end{cases}$$

and

$$G^{-1}(z) = \begin{cases} \|z\| g^{-1}\big(\frac{z}{\|z\|}\big), & z \in Q_2E \setminus \{0\}, \\ 0, & z = 0. \end{cases}$$

Then $H = F \oplus G$ is a homeomorphism of E, with inverse $H^{-1} = F^{-1} \oplus G^{-1}$. We have

$$\begin{aligned} H(A'(m,n)z) &= F(e^{x_n-x_m}P_1z) + G(e^{x_m-x_n}Q_1z) \\ &= e^{x_n-x_m}F(P_1z) + e^{x_m-x_n}G(Q_1z) \\ &= (e^{x_n-x_m}P_2 + e^{x_m-x_n}Q_2)H(z) \\ &= C(m,n)H(z) \end{aligned}$$

and so $\mathcal{A}' \sim \mathcal{C}$. On the other hand, by Theorem 2.1, we have $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$, while Lemma 3.3 implies that $\mathcal{B}' \sim \mathcal{C}$. Together with $\mathcal{A}' \sim \mathcal{C}$ this yields that $\mathcal{A} \sim \mathcal{B}$. $\square$

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