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Sharp Estimates and Existence for Anisotropic Elliptic Problems with General Growth in the Gradient

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Abstract. In this paper, we prove sharp estimates and existence results for anisotropic nonlinear elliptic problems with lower order terms depending on the gradient. Our prototype is:

$$\begin{cases} -\mathcal{Q}_p u = [H(Du)]^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, $0 < p-1 < q \leq p < N$, and $\mathcal{Q}_p$ is the anisotropic operator

$$\mathcal{Q}_p u = \operatorname{div}\left([H(Du)]^{p-1} H_\xi(Du)\right),$$

where H is a suitable norm of $\mathbb{R}^N$. Moreover, f belongs to a suitable Marcinkiewicz space.

Keywords. Nonlinear elliptic problems with gradient dependent terms, anisotropic Laplacian, convex symmetrization, a priori estimates

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1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and $1 < p < N$. Consider a convex, 1-homogeneous function $H\colon \mathbb{R}^N \to [0, +\infty[$ in $C^1(\mathbb{R}^N \setminus \{0\})$. The aim of this paper is to obtain sharp a priori estimates and existence results for elliptic Dirichlet problems modeled on the following:

$$\begin{cases} -\mathcal{Q}_p u = [H(Du)]^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $p-1<q\le p$, and Q_p is the anisotropic operator

$$\mathcal{Q}_p u = \operatorname{div}\left([H(Du)]^{p-1}H_\xi(Du)\right).$$

Moreover, we assume that f belongs to the Marcinkiewicz space $M^{\frac{N}{\gamma}}(\Omega)$, with $\gamma = \frac{q}{q-(p-1)}$. In order to consider a datum f which is (at least) in L^1, we will suppose that $\frac{N}{N-1}(p-1) < q \le p$. In general, $\mathcal{Q}_p$ is highly nonlinear, and it extends some well-known classes of operators. In particular, for $H(\xi) = (\sum_k |\xi_k|^r)^{\frac{1}{r}}$, $r>1$, $\mathcal{Q}_p$ becomes

$$\mathcal{Q}_p v = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left(\sum_{k=1}^{N}\left|\frac{\partial v}{\partial x_k}\right|^r\right)^{\frac{p-r}{r}}\left|\frac{\partial v}{\partial x_i}\right|^{r-2}\frac{\partial v}{\partial x_i}\right).$$

Note that for $r=2$, it coincides with the usual p-Laplace operator, while for $r=p$ it is the so-called pseudo-p-Laplace operator. This kind of operators has attracted an increasing interest in recent years. We refer, for example, to [2,15,20] ($p=2$) and [6,14,17] ($1<p<+\infty$).

In the Euclidean setting, that is when $H(\xi) = (\sum_i \xi_i^2)^{\frac{1}{2}}$, problem (1.1) reduces to

$$\begin{cases} -\Delta_p u = |Du|^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Δ_p is the well-known p-Laplace operator.

Problem (1.2) has been widely studied in literature. In general, for equations with q-growth in the gradient, existence results can be given under suitable sign conditions on the gradient-dependent term (see for example [9] and the references therein). On the other hand, if $f \in L^r(\Omega)$, in order to obtain an existence result for (1.2) it is necessary to impose a smallness assumption on the L^r norm of f. For example, if $f\in L^r$, $r>\frac{N}{p}$, and $\|f\|_r$ is small enough, then a bounded solution exists (see for instance [27,29]). As regards the case of unbounded solutions, depending on the summability of f, several results are known. For example, in [23], the case of $q=p$ and $f\in L^{N/p}$ is considered, and a sharp condition (in a suitable sense) on $\|f\|_{N/p}$ is given. For the general case $p-1<q\le p$, with different summability assumptions of f, we refer the reader to [1,3,12–14,18,21,24–26,32].

In this paper we deal with a problem whose prototype is (1.1), for a general norm H (see Section 2 for the precise assumptions), and looking for solutions in $W_0^{1,q}(\Omega)$ not necessarily bounded. More precisely, under a suitable smallness hypothesis on $\|f\|_{M^{N/\gamma}}$, $\gamma = \left(\frac{q}{p-1}\right)'$, we obtain some sharp a priori estimates, comparing the solutions to suitable approximating problems of (1.1), with the solutions to the anisotropic radially symmetric problem

$$\begin{cases} -\mathcal{Q}_p u = [H(Du)]^q + \dfrac{\lambda}{H^o(x)^\gamma} & \text{in } \Omega^\star, \\ u = 0 & \text{on } \partial\Omega^\star. \end{cases} \tag{1.3}$$

Here H^o is the polar function of H, $\Omega^\star$ is the sublevel set of H^o with the same Lebesgue measure of Ω and $\lambda = \kappa_N^{\frac{\gamma}{N}} \|f\|_{M^{\frac{N}{\gamma}}}$, with $\kappa_N = |\{x\colon H^o(x) < 1\}|$ (see Section 2 for the precise definitions). The comparison result is obtained by means of symmetrization techniques. Taking into account the structure of the equation, we use a suitable notion of symmetrization, known as convex symmetrization (see [2], and Section 2 for the definition). In this order of ideas, to obtain uniform bounds on the solutions of approximating problems it is sufficient to study the anisotropic radial problem (1.3). Hence, a key role is played by an existence and uniqueness result for a special class of positive solutions of (1.3) whose level sets are homothetic to H^o. This kind of solutions u are exactly the ones that allow to perform a change of variable $V = \varphi(u)$, such that V solves

$$\begin{cases} -\mathcal{Q}_p V = \dfrac{\lambda}{H^o(x)^\gamma} \left(\dfrac{V+1}{\gamma - 1}\right)^{\gamma-1} & \text{in } \Omega^\star, \\ V = 0 & \text{on } \partial\Omega^\star. \end{cases} \tag{1.4}$$

The solutions to (1.4) can be explicitly written, and then also the solutions to (1.3).

We refer the reader to Theorem 2.3 and Theorem 2.4 in Section 2.2, where the main results of the paper are stated.

The structure of the paper is the following. In Section 2, we recall the notation and the main assumptions used throughout all the paper, and we state the main results. In Section 3, we study the anisotropic radial problem (1.3). Finally, in Section 4 we prove the quoted comparison result and a priori estimates for the approximating problems. Finally, we give the proof of the main results.

2. Notation, preliminaries and main results

Let $N \geq 2$, and $H : \mathbb{R}^N \to [0, +\infty[$ be a $C^1(\mathbb{R}^N \setminus \{0\})$ function such that

$$H(t\xi) = |t| H(\xi), \quad \forall \xi \in \mathbb{R}^N, \ \forall t \in \mathbb{R}, \tag{2.1}$$

and such that any level set $\{\xi \in \mathbb{R}^n\colon H(\xi) \leq t\}$, with $t > 0$ is strictly convex. Moreover, suppose that there exist two positive constants $c_1 \leq c_2$ such that

$$c_1|\xi| \leq H(\xi) \leq c_2|\xi|, \quad \forall \xi \in \mathbb{R}^N. \tag{2.2}$$

Remark 2.1. We stress that the homogeneity of H and the convexity of its level sets imply the convexity of H. Indeed, by (2.1), it is sufficient to show that, for any $\xi_1, \xi_2 \in \mathbb{R}^n \setminus \{0\}$,

$$H(\xi_1 + \xi_2) \le H(\xi_1) + H(\xi_2). \tag{2.3}$$

By the convexity of the level sets, we have

$$\begin{aligned} &H\left(\frac{\xi_1}{H(\xi_1)+H(\xi_2)} + \frac{\xi_2}{H(\xi_1)+H(\xi_2)}\right) \\ &= H\left(\frac{H(\xi_1)}{H(\xi_1)+H(\xi_2)}\frac{\xi_1}{H(\xi_1)} + \frac{H(\xi_2)}{H(\xi_1)+H(\xi_2)}\frac{\xi_2}{H(\xi_2)}\right) \\ &\le 1, \end{aligned}$$

and by (2.1) we get (2.3).

We define the polar function $H^o\colon \mathbb{R}^N \to [0,+\infty[$ of H as

$$H^o(v) = \sup_{\xi\neq 0} \frac{\xi \cdot v}{H(\xi)}.$$

It is easy to verify that also H^o is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$H(v) = \sup_{\xi\neq 0} \frac{\xi \cdot v}{H^o(\xi)}.$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^N \colon H^o(\xi) < 1\}.$$

is the so-called Wulff shape centered at the origin. We put $\kappa_N = |\mathcal{W}|$, and denote $\mathcal{W}_r = r\mathcal{W}$.

In the following, we often make use of some well-known properties of H and H^o :

$$\begin{aligned} H(\xi) &= DH(\xi)\cdot\xi, && \forall \xi \in \mathbb{R}^N \setminus \{0\}, \\ H^o(\xi) &= DH^o(\xi)\cdot\xi, && \forall \xi \in \mathbb{R}^N \setminus \{0\}, \\ H(DH^o(\xi)) &= H^o(DH(\xi)) = 1, && \forall \xi \in \mathbb{R}^N \setminus \{0\}, \\ H^o(\xi)DH(DH^o(\xi)) &= H(\xi)DH^o(DH(\xi)) = \xi, && \forall \xi \in \mathbb{R}^N \setminus \{0\}. \end{aligned}$$

Let Ω be an open subset of $\mathbb{R}^N$. The total variation of a function $u \in BV(\Omega)$ with respect to H is (see [4]):

$$\int_\Omega |Du|_H = \sup\left\{\int_\Omega u \operatorname{div}\sigma dx \colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ H^o(\sigma) \le 1\right\}.$$

This yields the following definition of anisotropic perimeter of $F \subset \mathbb{R}^N$ in Ω:

$$P_H(F;\Omega) = \int_\Omega |D\chi_F|_H = \sup\left\{ \int_F \operatorname{div}\sigma dx \colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ H^o(\sigma) \le 1 \right\}.$$

The following co-area formula for the anisotropic perimeter

$$\int_{\{u>t\}} H(Du)dx = \int_\Omega P_H(\{u>s\},\Omega)\, ds, \quad \forall u \in BV(\Omega) \tag{2.4}$$

holds, moreover

$$P_H(F;\Omega) = \int_{\Omega\cap\partial^* F} H(\nu_F) d\mathcal{H}^{N-1}$$

where $\mathcal{H}^{N-1}$ is the $(N-1)-$dimensional Hausdorff measure in $\mathcal{R}^N$, $\partial^* F$ is the reduced boundary of F and ν_F is the outer normal to F (see [4]).

The anisotropic perimeter of a set F is finite if and only if the usual Euclidean perimeter

$$P(F;\Omega) = \sup\left\{ \int_F \operatorname{div}\sigma dx \colon \sigma \in C_0^1(\Omega;\mathbb{R}^N),\ |\sigma| \le 1 \right\}$$

is finite. Indeed, by properties (2.1) and (2.2) we have that

$$\frac{1}{c_2}|\xi| \le H^o(\xi) \le \frac{1}{c_1}|\xi|, \tag{2.5}$$

and then $c_1 P(E;\Omega) \le P_H(E;\Omega) \le c_2 P(E;\Omega)$. A fundamental inequality for the anisotropic perimeter is the isoperimetric inequality

$$P_H(E;\mathbb{R}^N) \ge N\kappa_N^{\frac{1}{N}} |E|^{1-\frac{1}{N}}, \tag{2.6}$$

which holds for any measurable subset E of $\mathbb{R}^N$ (see for example [2] for a proof). See also [16] for some questions related to an anisotropic relative isoperimetric inequality).

We recall that if $u \in W^{1,1}(\Omega)$, then (see [4])

$$\int_\Omega |Du|_H = \int_\Omega H(Du)dx.$$

2.1. Rearrangements and convex symmetrization. We recall some basic definition on rearrangements. Let Ω be an bounded open set of $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}$ be a measurable function, and denote with $|\Omega|$ the Lebesgue measure of Ω.

The distribution function of u is the map $\mu_u : \mathbb{R} \to [0,\infty[$ defined by

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|.$$

Such function is decreasing and right continuous.

The decreasing rearrangement of u is the map $u^*: [0,\infty[\to \mathbb{R}$ defined by

$$u^*(s) := \sup\{t \in \mathbb{R} : \mu_u(t) > s\}.$$

The function u^* is the generalized inverse of μ_u.

Following [2], the convex symmetrization of u is the function $u^\star(x)$, $x \in \Omega^\star$ defined by:

$$u^\star(x) = u^*(\kappa_N H^o(x)^N),$$

where $\Omega^\star$ is a set homothetic to the Wulff shape centered at the origin having the same measure of Ω, that is, $\Omega^\star = \mathcal{W}_R$, with $R = \left(\frac{|\Omega|}{\kappa_N}\right)^{\frac{1}{N}}$.

We will say that any $w(x)$, $x \in \Omega^\star$ is an anisotropic radial function if for any $x \in \Omega^\star$, $w(x) = \tilde{w}(H^o(x))$, for some function $\tilde{w}(r)$, $r \in [0,R]$. For the sake of brevity, we will refer to such functions as radial functions. For example, $u^\star$ is radial.

The following results will be useful in the sequel. First, a basic tool will be the Hardy inequality, stated below.

Proposition 2.2. *For any $u \in W^{1,\gamma}(\mathbb{R}^N)$, $1 < \gamma < N$,*

$$\int_{\mathbb{R}^N} H(Du)^\gamma dx \geq \Lambda_\gamma \int_{\mathbb{R}^N} \frac{|u|^\gamma}{H^o(x)^\gamma} dx, \tag{2.7}$$

and the constant $\Lambda_\gamma = \left(\frac{N-\gamma}{\gamma}\right)^\gamma$ is optimal, and it is not achieved.

If $H(x) = |x|$, (2.7) is the classical Hardy inequality. For a general H, (2.7) has been proved in [33].

Finally, we recall the definition of Marcinkiewicz spaces. We say that a measurable function $u\colon \Omega \to \mathbb{R}$ belongs to $M^r(\Omega)$, $r > 1$, if there exists a constant C such that

$$\mu_u(t) \leq C t^{-r}, \quad \forall t > 0,$$

or, equivalently, $u^*(s) \leq C\sigma^{-\frac{1}{r}}$, for all $\sigma \in]0, |\Omega|]$. Then, we denote

$$\|u\|_{M^r(\Omega)} = \sup_{\sigma \in]0,|\Omega|[} u^*(\sigma)\sigma^{\frac{1}{r}}.$$

2.2. Statement of the problem and main results. Our aim is to prove a priori estimates and existence results for problems of the type

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = b(x,u,Du) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory functions verifying

$$a(x,s,\xi)\cdot\xi \geq H(\xi)^p, \tag{2.9}$$

and

$$|a(x,s,\xi)| \leq \alpha(|\xi|^{p-1}+|s|^{p-1}+k(x)), \tag{2.10}$$

for a.e. $x\in\Omega$, for any $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $\alpha>0$, $k\in L_+^{p'}(\Omega)$, and $1<p<N$. Moreover,

$$(a(x,s,\xi)-a(x,s,\xi'))\cdot(\xi-\xi')>0, \tag{2.11}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}, \xi\neq\xi'\in\mathbb{R}^N$. As regards the lower order terms, we suppose that $b\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory functions such that

$$|b(x,s,\xi)|\leq H(\xi)^q \tag{2.12}$$

for a.e. $x\in\Omega$, for any $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, for $p-1<q\leq p$.

Finally, we take f such that

$$f^\star(x)\leq\frac{\lambda}{H^o(x)^\gamma},\ x\in\Omega^\star,\quad \text{with } \gamma=\left(\frac{q}{p-1}\right)'=\frac{q}{q-(p-1)}. \tag{2.13}$$

We observe that such hypothesis implies that f belongs to the Marcinkiewicz space $M^{\frac{N}{\gamma}}(\Omega)$. It is worth to recall that $M^{\frac{N}{\gamma}}(\Omega)\subset L^s(\Omega)$ for any $s<\frac{N}{\gamma}$, but $M^{\frac{N}{\gamma}}(\Omega)\supset L^{\frac{N}{\gamma}}(\Omega)$.

Assume first that

$$p\geq q>p-1+\frac{p}{N}. \tag{2.14}$$

Then (2.13) implies that $f\in L^{(p^*)'}(\Omega)$, where $p^*=\frac{Np}{N-p}$ is the Sobolev conjugate of p. Indeed in this case, $p\leq\gamma<\frac{Np-N+p}{p}$, that is $\frac{N}{\gamma}>(p^*)'$. Hence, if (2.14) holds, we say that $u\in W_0^{1,p}(\Omega)$ is a weak solution to (2.8) if

$$\int_\Omega a(x,u,Du)\cdot D\varphi\,dx=\int_\Omega[b(x,u,Du)+f]\varphi\,dx, \tag{2.15}$$

for any $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Second, suppose that

$$p-1+\frac{p}{N}\geq q>\frac{N}{N-1}(p-1). \tag{2.16}$$

Then (2.13) gives that $f\in L^s(\Omega)$, with $1<s<\frac{N}{\gamma}$. Hence, if (2.16) holds, we say that u is a distributional solution to (2.8) if $u\in W_0^{1,q}(\Omega)$ and (2.15) is satisfied for any $\varphi\in C_0^\infty(\Omega)$.

Finally, let us observe that if $q=\frac{N}{N-1}(p-1)$, then $\frac{N}{\gamma}=1$, and f is not in $L^1(\Omega)$.

The main results of our paper will be the following.

Theorem 2.3. *Suppose that the assumptions* (2.9),(2.10), (2.11) *and* (2.12) *hold. Moreover, let* $f \in M^{\frac{N}{\gamma}}(\Omega)$ *such that*

$$f^\star(x) \le \frac{\lambda}{H^o(x)^\gamma}, \quad x \in \Omega^\star, \text{ for some } 0 \le \lambda < c_\gamma \Lambda_\gamma, \tag{2.17}$$

with $c_\gamma = (\gamma-1)^{\gamma-1}$, *and* $\Lambda_\gamma = \left(\frac{N-\gamma}{\gamma}\right)^\gamma$. *Then,*

(a) *if* $p \ge q > p-1+\frac{p}{N}$, *then problem* (2.8) *admits a weak solution* $u \in W_0^{1,p}(\Omega) \cap L^s(\Omega)$, *with* $s < +\infty$ *if* $p = q$, *or* $s < \frac{N[q-(p-1)]}{p-q}$ *otherwise;*

(b) *if* $p-1+\frac{p}{N} \ge q > \frac{N}{N-1}(p-1)$, *then problem* (2.8) *admits a distributional solution* $u \in W_0^{1,r}(\Omega)$, *with* $r < N[q-(p-1)]$.

Theorem 2.4. *Under the assumptions of Theorem* 2.3, *the obtained solution* u *verifies*

$$u^*(s) \le v^*(s), \quad s \in]0, |\Omega|],$$

where $v \in W_0^{1,s}(\Omega^\star)$, $s < N(q-(p-1))$ *is the radial solution to problem*

$$\begin{cases} -\mathcal{Q}_p v = H(Dv)^q + \dfrac{\lambda}{H^o(x)^\gamma} & \text{in } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases}$$

given in Theorem 3.4 (*see Section* 3).

Remark 2.5. We explicitly observe that requiring the condition (2.17) on f in the above Theorems is equivalent to assume that

$$\|f\|_{M^{\frac{N}{\gamma}}} < \kappa_N^{\frac{\gamma}{N}} (\gamma-1)^{\gamma-1} \left(\frac{N-\gamma}{\gamma}\right)^\gamma .$$

3. The radial case

We first study the problem

$$\begin{cases} -\mathcal{Q}_p v = H(Dv)^q + \dfrac{\lambda}{H^o(x)^\gamma} & \text{in } \mathcal{W}_R, \\ v = 0 & \text{on } \partial\mathcal{W}_R, \end{cases} \tag{3.1}$$

where $\lambda \ge 0$, $p-1 < q \le p$, $\mathcal{W}_R$ is the Wulff shape centered at the origin and radius R, and $\gamma = \left(\frac{q}{p-1}\right)' = \frac{q}{q-(p-1)}$.

In order to prove an existence and uniqueness result for problem (3.1), we first study the following problem:

$$\begin{cases} -\mathcal{Q}_\gamma V = \dfrac{\lambda}{c_\gamma H^o(x)^\gamma}(V+1)^{\gamma-1} & \text{in } \mathcal{W}_R, \\ V = 0 & \text{on } \partial\mathcal{W}_R, \end{cases} \tag{3.2}$$

with $c_\gamma = (\gamma-1)^{\gamma-1}$.

Remark 3.1. If we look for radial solutions $V(r) = V(H^o(x))$ of (3.2), these solves the equation

$$-|V'|^{\gamma-2}\left((\gamma-1)V'' + \frac{N-1}{r}V'\right) = \frac{\lambda}{c_\gamma}\frac{(V+1)^{\gamma-1}}{r^\gamma} \quad \text{in }]0,R[, \tag{3.3}$$

which follows from the equation in (3.2), plugging in the function $V(r) = V(H^o(x))$ and using the properties of H. It is a straightforward computation to show that $\Phi(r) = \left(\frac{R}{r}\right)^\beta - 1$ solves (3.3) if and only if β is such that

$$-(\gamma-1)\,\beta^\gamma + (N-\gamma)\,\beta^{\gamma-1} = \frac{\lambda}{c_\gamma}. \tag{3.4}$$

For $0 \le \lambda < c_\gamma \Lambda_\gamma$, this equation has exactly two different solutions, but there exists a unique solution β such that

$$\beta \in \left[0, \frac{N-\gamma}{\gamma}\right[\quad \text{and} \quad \Phi(x) = \left(\frac{R}{H^o(x)}\right)^\beta - 1 \in W_0^{1,\gamma}(\mathcal{W}_R) \tag{3.5}$$

(see Figure 1).

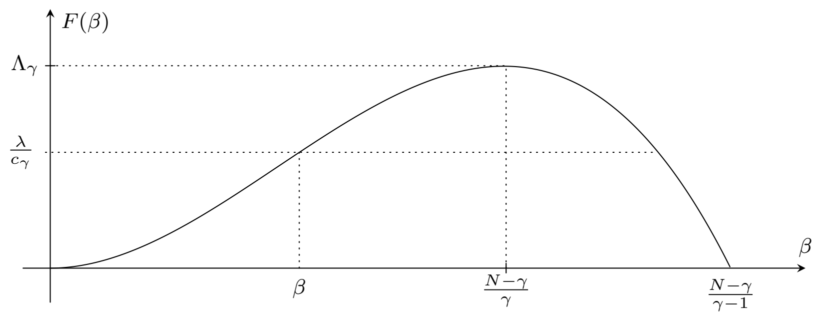

Figure 1: $F(\beta) = -(\gamma-1)\beta^\gamma + (N-\gamma)\beta^{\gamma-1}$. For any $\lambda \in [0, c_\gamma\Lambda_\gamma[$, there exists a unique $\beta \ge 0$ such that $F(\beta) = \frac{\lambda}{c_\gamma}$ and $r^{-\beta} \in W^{1,\gamma}(\mathcal{W}_R)$.

The following result holds:

Theorem 3.2. *Let $1 < \gamma < N$, and*

$$0 \le \lambda < c_\gamma \Lambda_\gamma,$$

where $\Lambda_\gamma = \left(\frac{N-\gamma}{\gamma}\right)^\gamma$ is the best constant of the Hardy inequality (2.7)*. Then, if $\lambda > 0$, the problem* (3.2) *admits a unique positive solution $\Phi \in W_0^{1,\gamma}(\mathcal{W}_R)$, in the sense that*

$$\int_{\mathcal{W}_R} H(D\Phi)^{\gamma-1} H_\xi(D\Phi) \cdot D\varphi \, dx = \frac{\lambda}{c_\gamma} \int_{\mathcal{W}_R} \frac{1}{H^o(x)^\gamma} (\Phi+1)^{\gamma-1} \varphi \, dx, \tag{3.6}$$

$\varphi \in W_0^{1,\gamma}(\mathcal{W}_R)$, where Φ is given in (3.5)*. Moreover, if $\lambda = 0$ the unique solution in $W_0^{1,\gamma}(\Omega)$ to* (3.2) *in the sense of* (3.6) *is $\Phi \equiv 0$.*

Proof. By Remark 3.1, we have to prove only the uniqueness issue. We first assume that $0 < \lambda < c_\gamma \lambda_\gamma$. Reasoning as in [6], we prove that there are no other positive solutions in $W_0^{1,\gamma}(\Omega)$ of (3.2). As a matter of fact, the positive solutions to (3.2) are stationary points of the functional

$$F(\psi) = \frac{1}{\gamma}\int_{\mathcal{W}_R}\left[H(D\psi)^\gamma - \frac{\lambda}{c_\gamma H^o(x)^\gamma}[(|\psi|+1)^\gamma - 1]\,\mathrm{sign}\,\psi\right]dx, \tag{3.7}$$

$\psi \in W_0^{1,\gamma}(\mathcal{W}_R)$. The functional $F(\psi)$ is even. Moreover, it is strictly convex in the variable ψ^γ. Indeed, if $U, V > 0$, $U, V \in W_0^{1,\gamma}(\Omega)$, then the function

$$\phi = \left(\frac{U^\gamma + V^\gamma}{2}\right)^{\frac{1}{\gamma}}$$

is an admissible test function for F in (3.7). Computing $D\phi$, by the homogeneity of H it follows that

$$H(D\phi) = \phi\, H\left(\frac{1}{2}\frac{U^\gamma}{\phi^\gamma}\frac{DU}{U} + \frac{1}{2}\frac{V^\gamma}{\phi^\gamma}\frac{DV}{V}\right).$$

Let $s(x) = \frac{Z^\gamma}{2\phi^\gamma}$. Observing that $0 < s < 1$, by convexity and homogenity of H we have that

$$\begin{aligned} H(D\phi)^\gamma &= \phi^\gamma H\left(s(x)\frac{DU}{U} + (1-s(x))\frac{DV}{V}\right)^\gamma \\ &\le \phi^\gamma\left(s(x)H\left(\frac{DU}{U}\right)^\gamma + (1-s(x))H\left(\frac{DV}{V}\right)^\gamma\right) \\ &= \frac{U^\gamma}{2}H\left(\frac{DU}{U}\right)^\gamma + \frac{V^\gamma}{2}H\left(\frac{DV}{V}\right)^\gamma \\ &= \frac{1}{2}\left[H(DU)^\gamma + H(DV)^\gamma\right]. \end{aligned}$$

On the other hand, the function $g(t) = \left(t^{\frac{1}{\gamma}} + 1\right)^\gamma$, $t \ge 0$ is strictly concave, and then $F(\psi)$ is stricly convex in ψ^γ. Finally, F admits only the positive critical point Φ.

The theorem is completely proved if we show that, when $\lambda = 0$, $\Phi = 0$ is the unique solution in $W_0^{1,\gamma}$. This follows observing that, in this case, the functional F becomes

$$F(\psi) = \frac{1}{\gamma}\int_{\mathcal{W}_R}[H(D\psi)]^\gamma dx,$$

which is strictly convex, since $H^\gamma(\xi)$ is strictly convex in ξ. □

Remark 3.3. It is worth noting that the argument of Theorem 3.2 can be used, for example, also in order to obtain uniqueness for problems of the type

$$\begin{cases} -\mathcal{Q}_\gamma v = b(x)|v|^{\gamma-2}v + f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

with Ω bounded open set of $\mathbb{R}^N$, b such that

$$b(x) \in L\left(\frac{N}{\gamma}, \infty\right), \quad \text{with } (b^+)^\star(x) \le \frac{\lambda}{H^o(x)^\gamma} \text{ in } \Omega^\star, \quad 0 < \lambda < \Lambda_\gamma, \tag{3.9}$$

and $f \in L((\gamma^*)', \gamma')$, $f \ge 0$, $f \not\equiv 0$ in Ω. Under this assumptions, problem (3.8) admits at most a (positive) weak solution. Indeed, if v is a solution to (3.8), using the Polya-Szegö inequality in the anisotropic case (see [2]), and the Hardy-Littlewood inequality we get that

$$\int_{\Omega^\star} H(D(v^-)^\star)^\gamma dx \le \int_\Omega H(Dv^-)^\gamma dx \le \int_\Omega b^+(v^-)^\gamma dx \le \int_{\Omega^\star} (b^+)^\star[(v^-)^\star]^\gamma dx,$$

Recalling the assumptions on b in (3.9), the Hardy inequality assures that $v^- \equiv 0$. Actually, by the maximum principle v must be positive in Ω. Hence we can proceed similarly as in the proof of Theorem 3.2 obtaining the uniqueness of the solution (see also [6, 15, 19]).

Theorem 3.4. *Let* $p \ge q > (p-1)\frac{N}{N-1}$, $\Lambda_\gamma = \left(\frac{N-\gamma}{\gamma}\right)^\gamma$, $c_\gamma = (\gamma-1)^{\gamma-1}$, $\gamma = \left(\frac{q}{p-1}\right)'$ *and*

$$0 \le \lambda < c_\gamma \Lambda_\gamma.$$

Then, if $\lambda > 0$ *there exists a unique positive, radially decreasing, distributional solution* $v(x) = v(r)$ *of* (3.1) *in* $W_0^{1,s}(\mathcal{W}_R)$, *with* $s < N(q-(p-1)) = \tilde{s}$, *such that, defining*

$$V(x) = \exp\left[\frac{1}{\gamma-1}\int_{H^o(x)}^R (-v'(\tau))^{q-(p-1)} d\tau\right] - 1, \tag{3.10}$$

it holds that

$$V \in W_0^{1,\gamma}(\mathcal{W}_R),\ (V+1)^{\gamma-1} \in W^{1,\delta}(\mathcal{W}_R), \quad \text{for some } \delta > \tilde{\delta} = \left(\frac{\tilde{s}}{p-1}\right)'. \tag{3.11}$$

Moreover, if $q < p$,

$$v(r) = \theta\left[r^{-\frac{p-q}{q-(p-1)}} - R^{-\frac{p-q}{q-(p-1)}}\right],$$

with $\theta = [(\gamma-1)\beta]^{\frac{1}{q-(p-1)}} \frac{q-(p-1)}{p-q}$, *while, for* $q = p$,

$$v(r) = (p-1)\beta \log \frac{R}{r},$$

where β *is the solution to* (3.4) *given in* (3.5). *Finally, if* $\lambda = 0$ *and* $(p-1)\frac{N}{N-1} < q \le p$, *the unique radially decreasing solution* v *such that* (3.10),(3.11) *holds is* $v = 0$.

Proof. Using the notation of Theorem 3.2, being $0 < \lambda < c_\gamma \Lambda_\gamma$, we can consider $\Phi = \left(\frac{R}{r}\right)^\beta - 1$, $0 \le \beta < \frac{N-\gamma}{\gamma}$ as the unique positive solution in $W_0^{1,\gamma}(\mathcal{W}_R)$ of (3.2). We reason as in [21], performing the change of variable

$$v(r) = (\gamma-1)^{\frac{1}{q-(p-1)}} \int_r^R \left(\frac{-\Phi'(s)}{\Phi(s)+1}\right)^{\frac{1}{q-(p-1)}} ds = \theta\left[r^{-\frac{p-q}{q-(p-1)}} - R^{-\frac{p-q}{q-(p-1)}}\right],$$

with $\theta = [(\gamma-1)\beta]^{\frac{1}{q-(p-1)}} \frac{q-(p-1)}{p-q}$. A direct computation shows that, being $q > \frac{N}{N-1}(p-1)$, v belongs to $W_0^{1,s}(\mathcal{W}_R)$, for all $s < \tilde{s} = N(q-(p-1))$ and it is a solution to (3.1). Moreover, the function $V(x)$ defined in (3.10) coincides with $\Phi(x)$, and, being $0 \le \beta < \frac{N-\gamma}{\gamma}$, there exists $\delta > \tilde{\delta}$ such that (3.11) holds.

On the contrary, let us suppose that $v(x) \in W_0^{1,s}(\mathcal{W}_R)$ for any $s < N(q-(p-1))$, with v is a radially decreasing, and solves (3.1). Moreover, suppose that the function V defined in (3.10) verifies (3.11).

Following the method contained in [22, Proposition 1.8], we show that $V(x)$ is a solution to (3.2), in the sense of (3.6). Being $V \in W_0^{1,\gamma}(\mathcal{W}_R)$, by a density argument and the Hardy inequality (2.7) it is sufficient to show that

$$-\mathcal{Q}_\gamma(V) = \frac{\lambda}{c_\gamma H^o(x)^\gamma}(V+1)^{\gamma-1} \quad \text{in } \mathcal{D}'(\mathcal{W}_R). \tag{3.12}$$

Being $v \in W_0^{1,s}(\mathcal{W}_R)$ for any $s < \tilde{s}$, the integral $\int_{\mathcal{W}_R} H^{p-1}(Dv) H_\xi(Dv) \cdot D\phi \, dx$ is finite as $\phi \in W_0^{1,\delta}(\mathcal{W}_R)$, with $\delta > \tilde{\delta}$ given in (3.11). This ensures that the operator $T := -\mathcal{Q}_p v$ belongs to $W^{-1,\delta'}$. Hence, by (3.11) the following product $(V+1)^{\gamma-1}T$ is well defined in $\mathcal{D}'$:

$$\begin{aligned}
&\langle (V+1)^{\gamma-1}T, \varphi\rangle := \langle T, (V+1)^{\gamma-1}\varphi\rangle \\
&= \int_{\mathcal{W}_R} H(Dv)^{p-1} H_\xi(Dv) \cdot D\left[(V+1)^{\gamma-1}\varphi\right] dx \\
&= \int_{\mathcal{W}_R} H(Dv)^{p-1} H_\xi(Dv) \cdot \left[(V+1)^{\gamma-1} D\varphi + \varphi\,(\gamma-1)(V+1)^{\gamma-1} H(Dv)^{q-p} Dv\right] dx, \\
&\forall \varphi \in C_0^\infty(\mathcal{W}_R).
\end{aligned}$$

We obtain that

$$\begin{aligned}&(V+1)^{\gamma-1}T\\&\quad=-\operatorname{div}\left[(V+1)^{\gamma-1}H(Dv)^{p-1}H_\xi(Dv)\right]+(V+1)^{\gamma-1}H(Dv)^q\quad\text{in }\mathcal{D}'.\end{aligned}\tag{3.13}$$

Being v a solution to (3.1), $-\mathcal{Q}_p(v)=H(Dv)^q+\lambda H^o(x)^{-\gamma}\in L^1$. Furthermore, $[H(Dv)^q+\lambda H^o(x)^{-\gamma}](V+1)^{\gamma-1}\in L^1$. Indeed, recalling (3.11), we have that $(V+1)^{\gamma-1}H(Dv)^q\le C\frac{|DV|^\gamma}{V+1}\in L^1$, and $H^o(x)^{-\gamma}(V+1)^{\gamma-1}\varphi\in L^1$ by the Hardy inequality. Hence, we can use the result of Brezis and Browder [10], obtaining that, as $\varphi\in\mathcal{C}_0^\infty(\mathcal{W}_R)$,

$$\begin{aligned}&\int_{\mathcal{W}_R}H^{p-1}(Dv)H_\xi(Dv)\cdot D\left[(V+1)^{\gamma-1}\varphi\right]dx\\&\quad=\int_{\mathcal{W}_R}\left[(V+1)^{\gamma-1}\left(H(Dv)^q+\frac{\lambda}{H^0(x)^\gamma}\right)\right]\varphi\,dx,\end{aligned}$$

that is

$$(V+1)^{\gamma-1}T=(V+1)^{\gamma-1}\left(H(Dv)^q+\frac{\lambda}{H^0(x)^\gamma}\right)\quad\text{in }\mathcal{D}'(\mathcal{W}_R).$$

On the other hand, it is easy to see that

$$-\mathcal{Q}_\gamma(V)=-\frac{1}{c_\gamma}\operatorname{div}\left[(V+1)^{\gamma-1}H(Dv)^{p-1}H_\xi(Dv)\right]\quad\text{in }\mathcal{D}'(\mathcal{W}_R).\tag{3.14}$$

Putting (3.13)–(3.14) together, we get that $V\in W_0^{1,\gamma}(\mathcal{W}_R)$ satisfies (3.12). Then $V(x)=\Phi(x)$ by Theorem 3.2, and this concludes the proof. □

Remark 3.5. We explicitly observe that problem (3.1) admits at least two nonnegative solutions in $W_0^{1,s}(\mathcal{W}_R)$, for all $s<\tilde{s}$. For example, if $\lambda=0$, $R=1$ and $\frac{N}{N-1}<q<p$, the problem

$$-\mathcal{Q}_p(u)=[H(Du)]^q,\quad u\in W_0^{1,q}(\mathcal{W}_1)$$

admits the radially decreasing solutions $u_1=0$ and

$$u_2(x)=K\left(\frac{1}{H^o(x)^{\frac{p-q}{q-(p-1)}}}-1\right),\quad\text{with }K=\frac{q-(p-1)}{p-q}\left(\frac{(N-1)q-(p-1)N}{q-(p-1)}\right)^{\frac{1}{q-(p-1)}}.$$

As a matter of fact, $u_2\in W_0^{1,s}(\mathcal{W}_1)$, $s<\tilde{s}$ but, making the change of variable (3.10), the function

$$V(x)=\exp\left[\frac{q-(p-1)}{p-1}\int_r^1(-u_2'(\tau))^{q-(p-1)}d\tau\right]-1=\left(\frac{1}{r}\right)^{\frac{(N-1)q-(p-1)N}{q-(p-1)}}-1,\quad r=H^o(x)$$

does not verify (3.11).

For the uniqueness issue of problem (2.8), we refer the reader to [5] and the references therein.

4. A priori estimates and proof of Theorems 2.3 and 2.4

The key role in order to prove Theorem 2.3 is played by some a priori estimates, given in Theorem 4.1 and in Proposition 4.2 below, for the approximating problems

$$\begin{cases} -\operatorname{div}(a(x,u_\varepsilon,Du_\varepsilon)) = b_\varepsilon(x,u_\varepsilon,Du_\varepsilon) + T_{\frac{1}{\varepsilon}}(f(x)) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\varepsilon > 0$,

$$b_\varepsilon(x,s,\xi) = \frac{b(x,s,\xi)}{1+\varepsilon|b(x,s,\xi)|}, \quad \text{for a.e. } x\in\Omega,\ \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N,$$

and $T_t(s) = \min\{s, \max\{-s,t\}\}$, $t>0$ is the standard truncature function. Since $|b_\varepsilon| \le \frac{1}{\varepsilon}$ and $f_\varepsilon \in L^\infty(\Omega)$, the assumptions (2.9)–(2.11) allows to apply the classical results contained in [28]. Then there exists a weak solution $u_\varepsilon \in W_0^{1,p}(\Omega)$. Moreover, $u_\varepsilon \in L^\infty(\Omega)$.

The theorem below is in the spirit of the comparison results contained in [2,21,31].

Theorem 4.1. *Let $u_\varepsilon \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be a weak solution to (4.1), under the assumptions (2.9)–(2.12), with $f\in M^{\frac{N}{\gamma}}(\Omega)$ such that*

$$f^\star(x) \le \frac{\lambda}{H^o(x)^\gamma}, \quad x\in\Omega^\star, \quad \textit{for some } 0\le\lambda< c_\gamma\Lambda_\gamma,$$

with $c_\gamma = (\gamma-1)^{\gamma-1}$, and $\Lambda_\gamma = \left(\frac{N-\gamma}{\gamma}\right)^\gamma$. Then

$$u_\varepsilon^*(s) \le v^*(s), \quad s\in]0,|\Omega|]. \tag{4.2}$$

where $v\in W_0^{1,s}(\Omega^\star)$, for all $s<N(q-(p-1))$, is the solution to problem

$$\begin{cases} -\mathcal{Q}_p v = H(Dv)^q + \dfrac{\lambda}{H^o(x)^\gamma} & \textit{in } \Omega^\star, \\ v = 0 & \textit{on } \partial\Omega^\star, \end{cases}$$

given by Theorem 3.4.

Proof. The first step consists in proving the following differential inequality:

$$\begin{aligned} &(-u_\varepsilon^*(s))'(N\kappa_N^{\frac{1}{N}}s^{1-\frac{1}{N}})^{\frac{p}{p-1}} \\ &\le \left[\int_0^s \lambda\left(\frac{\kappa_N}{\varrho}\right)^{\frac{\gamma}{N}} \left\{\exp\left(\int_\varrho^s \frac{1}{(N\kappa_N^{\frac{1}{N}})^{p-q}} \frac{(-(u_\varepsilon^*)'(\tau))^{q-(p-1)}}{\tau^{(1-\frac{1}{N})(p-q)}}d\tau\right)\right\}d\varrho\right] \\ &\quad \text{a.e. in}]0,|\Omega|[. \end{aligned} \tag{4.3}$$

Given $t, h > 0$, we take $\varphi = (T_{t+h}(u_\varepsilon) - T_t(u_\varepsilon)) \operatorname{sign} u_\varepsilon$ as test function for (2.8). Hence we get

$$-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx \leq \int_{|u_\varepsilon|>t} H(Du_\varepsilon)^q dx + \int_{|u_\varepsilon|>t} \frac{\lambda}{H^o(x)^\gamma}. \tag{4.4}$$

The Hölder inequality gives that

$$\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^q dx \leq \int_t^{+\infty} \left[\left(-\frac{d}{d\tau}\int_{|u_\varepsilon|>\tau} H(Du_\varepsilon)^p dx\right)^{\frac{q}{p}} (-\mu'_{u_\varepsilon}(\tau))^{1-\frac{q}{p}}\right] d\tau.$$

Hence, the Hölder inequality, the coarea formula (2.4) and the isoperimetric inequality (2.6) give that

$$\left(-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx\right)^{\frac{p-q}{p}} \geq (N\kappa_N^{\frac{1}{N}} \mu_{u_\varepsilon}(t)^{1-\frac{1}{N}})^{p-q}(-\mu'_{u_\varepsilon}(t))^{-\frac{(p-1)(p-q)}{p}}.$$

Using the above inequalities and the Hardy-Littlewood inequality in (4.4), we obtain that

$$\begin{aligned}
&-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx \\
&\leq \int_0^{\mu_{u_\varepsilon}(t)} \lambda \left(\frac{\kappa_N}{\varrho}\right)^{\frac{\gamma}{N}} d\varrho + \frac{1}{\left(N\kappa_N^{\frac{1}{N}}\right)^{p-q}} \int_t^{+\infty} \left(-\frac{d}{d\tau}\int_{|u_\varepsilon|>\tau} H(Du_\varepsilon)^p dx\right)\left(\frac{-\mu'_{u_\varepsilon}(\tau)}{(\mu_{u_\varepsilon}(\tau))^{1-\frac{1}{N}}}\right)^{p-q} d\tau.
\end{aligned}$$

The Gronwall Lemma guarantees that

$$\begin{aligned}
&-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx \\
&\leq \int_0^{\mu_{u_\varepsilon}(t)} \lambda \left(\frac{\kappa_N}{\varrho}\right)^{\frac{\gamma}{N}} d\varrho + \int_t^{+\infty} \frac{1}{(N\kappa_N^{\frac{1}{N}})^{p-q}} \left(\frac{-\mu'_{u_\varepsilon}(\tau)}{(\mu_{u_\varepsilon}(\tau))^{1-\frac{1}{N}}}\right)^{p-q} \\
&\quad \times \left(\int_0^{\mu_{u_\varepsilon}(\tau)} \lambda \left(\frac{\kappa_N}{\varrho}\right)^{\frac{\gamma}{N}} d\varrho\right) \exp\left\{\int_t^\tau \frac{1}{(N\kappa_N^{\frac{1}{N}})^{p-q}} \left(\frac{-\mu'_{u_\varepsilon}(r)}{(\mu_{u_\varepsilon}(r))^{1-\frac{1}{N}}}\right)^{p-q} dr\right\} d\tau.
\end{aligned}$$

As matter of fact, reasoning as in [11, 30] it is possible to prove that

$$\begin{aligned}
\int_t^\tau \left(\frac{-\mu'_{u_\varepsilon}(r)}{(\mu_{u_\varepsilon}(r))^{1-\frac{1}{N}}}\right)^{p-q} dr &= \frac{1}{N^{q-(p-1)}\kappa_N^{\frac{N-p+q}{N}}} \int_{\tau>u_\varepsilon^\star(x)>t} \frac{H(Du_\varepsilon^\star)^{q-(p-1)}}{H^o(x)^{N-1}} dx \\
&= \int_{\mu_{u_\varepsilon}(\tau)}^{\mu_{u_\varepsilon}(t)} \frac{(-(u_\varepsilon^*)'(r))^{q-(p-1)}}{r^{(1-\frac{1}{N})(p-q)}} dr.
\end{aligned}$$

Then we can proceed similarly than [21], and get (4.3).

Now we observe that the solution v obtained in Theorem 3.4 verifies (4.3), where the inequality is replaced by an equality. Hence, from now on, recalling that the function $V(x)$ defined in (3.10) verifies (3.11), we can follow line by line the proof of [21, Theorem 4.1], in order to get that

$$(-u_\varepsilon^*(s))' \leq (-v^*(s))', \quad \text{for a.e. } s \in]0, |\Omega|],$$

and this gives the quoted comparison (4.2). □

From the proof of the above Theorem, we easily get estimates of the solutions in Lebesgue and Sobolev spaces.

Proposition 4.2. *Under the assumptions of Theorem* 4.1*, the following uniform estimates hold:*

(1) *if* $p \geq q > \frac{N}{N-1}(p-1)$,

$$\|u_\varepsilon\|_s \leq C,$$

for all $s < +\infty$ *if* $p = q$, *or* $s < \frac{N[q-(p-1)]}{p-q}$ *otherwise*;

(2) *if* $p \geq q > p - 1 + \frac{p}{N}$, *then*

$$\|Du_\varepsilon\|_p \leq C;$$

(3) *if* $p - 1 + \frac{p}{N} \geq q > \frac{N}{N-1}(p-1)$, *then*

$$\|DT_k(u_\varepsilon)\|_p \leq C, \quad \|Du_\varepsilon\|_r \leq C,$$

for any $k > 0$ *and all* $r < N[q - (p-1)]$.

In any case, C *denotes a constant independent on* ε.

Proof. Using (4.2) and the equimeasurability of the rearrangements, we have that

$$\|u\|_s \leq \|v\|_s,$$

and the explicit expression of v, given by Theorem 3.4, allows to obtain immediately the estimate in (1).

In order to get the gradient estimates in (2) and (3), we recall the proof of Theorem 4.1, and integrate by parts in (4.3). It follows that

$$\begin{aligned}
&-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx \\
&\leq \lambda \int_t^{+\infty} \left(\frac{\kappa_N}{\mu_{u_\varepsilon}(\tau)}\right)^{\frac{\gamma}{N}} (-\mu'_{u_\varepsilon}(\tau)) \exp\left\{\frac{1}{\left(N\kappa_N^{\frac{1}{N}}\right)^{p-q}} \int_t^\tau \left(\frac{-\mu'_{u_\varepsilon}(r)}{\mu_{u_\varepsilon}(r)^{1-\frac{1}{N}}}\right)^{p-q} dr\right\} d\tau \qquad (4.5)\\
&\leq \lambda \kappa_N^{\frac{\gamma}{N}} \int_0^{\mu_{u_\varepsilon}(t)} \varrho^{-\frac{\gamma}{N}} \exp\left\{\int_{\left(\frac{\varrho}{\kappa_N}\right)^{\frac{1}{N}}}^{\left(\frac{\mu_{u_\varepsilon}(t)}{\kappa_N}\right)^{\frac{1}{N}}} (-v'(r))^{q-(p-1)} dr\right\} d\varrho.
\end{aligned}$$

Last inequality follows by a change of variable and (4.2), recalling also that $v(r) = v^*(\kappa_N r^N)$.

Hence, substituting the explicit expression of v, after some computation we get that

$$-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx \le C\mu_{u_\varepsilon}^{1-\frac{\gamma}{N}}(t), \tag{4.6}$$

where $C = C(N, \kappa_N, \gamma, \beta, \lambda) \ge 0$.

Now, suppose that $p \ge q > p - 1 - \frac{p}{N}$. Integrating (4.6), we get:

$$\int_\Omega H(Du_\varepsilon)^p dx \le C\int_0^{|\Omega|} s^{1-\frac{\gamma}{N}}(-u^*(s))'ds \le C\int_0^{|\Omega|} s^{-\frac{p}{N[q-(p-1)]}}ds,$$

and the right-hand side is finite if and only if $q > p - 1 + \frac{p}{N}$. This proves (2).

Consider now the condition in (3), $\frac{N}{N-1}(p-1) < q \le p - 1 - \frac{p}{N}$. We have that

$$-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx = \frac{d}{dt}\int_{|u_\varepsilon|\le t} H(Du_\varepsilon)^p dx \quad \text{a.e. in } [0, +\infty[.$$

Hence we can integrate (4.6) between 0 and k and reason as before, obtaining that

$$\int_\Omega |DT_k(u_\varepsilon)|^p \le Ck^{N-\frac{p}{q-(p-1)}}.$$

Moreover, if $r < p$, using the Hölder inequality we get

$$-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^r dx \le \left(-\frac{d}{dt}\int_{|u_\varepsilon|>t} H(Du_\varepsilon)^p dx\right)^{\frac{r}{p}} [-\mu'_{u_\varepsilon}(t)]^{1-\frac{r}{p}}. \tag{4.7}$$

Using (4.5) and proceeding as before, we can integrate both terms of (4.7), obtaining that

$$\int_\Omega H(Du_\varepsilon)^r dx \le C\int_0^{|\Omega|} s^{-\frac{r}{N[q-(p-1)]}}ds,$$

which is finite if and only if $r < N(q - (p-1))$. □

Proof of Theorems 2.3 *and* 2.4. The estimates of Proposition 4.2 allow to obtain that the approximating sequence u_ε converges, up to a subsequence, to a function u which solves problem (2.8). The proof, whose main difficulties relies in the nonlinearity of the operator, can be obtained by compactness arguments which allow to get the strong convergence of the gradients of the approximating solutions. We skip the details and refer the reader, for example, to [7, 8, 23, 25] and the results quoted therein.

The convergence of u_ε implies that $u_\varepsilon^* \to u^*$ in some Lebesgue space, and then u_ε^* converges pointwise (up to a subsequence) to u^* in $]0, |\Omega|]$. Passing to the limit in (4.2), we are done. □

Remark 4.3. We stress that the bounds (2.2) and (2.5) on H and H^o, and the conditions (2.9), (2.12) and (2.13) give that

$$a(x,s,\xi)\cdot\xi \geq c_1^p|\xi|^p, \quad |b(x,s,\xi)| \leq c_2^q\,|\xi|^q, \quad \text{and} \quad f^\star(x) \leq \frac{\lambda c_2^p}{|x|^p}.$$

Hence, under the above growth conditions, the classical Schwarz symmetrization technique can be applied to problem (2.8). In this way, it is possible to obtain results analogous to those of Theorems 4.1 and Proposition 4.2, and hence to those of Theorems 2.3, 2.4 (in the spirit of the existence results, for example, of [3, 21, 24]), but requiring a stronger assumption on the smallness of $\lambda > 0$. This justifies the use of the more general convex symmetrization (see also [2] and [14, Remark 3.4]).

Remark 4.4. As regards the optimality of the smallness assumption on f, we refer the reader to [3, Section 3]. In such a paper the authors give some examples in the Euclidean radial case where, if $\lambda > 0$ and (2.17) is not satisfied, then in a suitable sense, there are no solutions.

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