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Mixed Norms and Iterated Rearrangements

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Abstract. We prove sharp estimates, and find the optimal range of indices, for the comparison of mixed norms for both functions and their iterated rearrangements.

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1. Introduction

Mixed norm spaces $L^p[L^r]$ were introduced in [4] as an important tool to study generalizations of Sobolev's theorem about the continuity of certain potential operators and the Hausdorff-Young theorem. Previous estimates involving $L^1[L^\infty]$ already appeared in [11, 18], where the authors proved the end-point case $p = 1$ for the Sobolev embedding (a further extension can be also found in [6, 10]). On the other hand, it was shown in [12] that Sobolev embedding can be strengthened using iterated rearrangements (see (4) for the definition). The systematic study of such rearrangements was begun in the work [7] and then was continued in different papers (see [1–3, 14, 15, 20, 22]). In particular, works [2, 3] were devoted to normability properties and embeddings of weighted Lorentz spaces defined in terms of iterated rearrangements. These rearrangements were also used in the study of embeddings of Besov, Lipschitz, and Sobolev type spaces (see [13, 19]). Furthermore, it was shown in [15, 20] that iterated rearrangements, in comparison with usual nonincreasing rearrangements, are better adapted to spaces with dominating mixed smoothness.

As an extension of the results in [6, 10], the works [1, 14] were devoted to estimates of iterated rearrangements in terms of mixed norms $L^1[L^\infty]$.

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It is then a natural question, regarding these estimates, to consider how do the mixed norm spaces $L^p[L^r]$ behave under the iterated rearrangements. We observe that some results in this direction have been already obtained in [7], but they were not optimal, and some proofs were based on false arguments (see Remark 4.11). Contrary to what one would a priori expect, the mixed norm of a function f is not in general determined by its rearrangements, and the other way around. In order to carry on this study, we introduce in Section 2 the preliminary definitions and prove some results of independent interest (like the reverse Minkowski's integral inequality in Proposition 2.2). In Section 3 we establish in Lemma 3.2 one of the principal techniques of this work, namely a discrete version of our main result, Theorem 4.5, which is proved in Section 4. A dual estimate is then considered in Theorem 4.7 and the sharpness of all these positive results is proved in Propositions 4.6, 4.8, 4.9, and 4.10. Section 5 deals with the cross-section estimates of rearrangements of general measurable sets.

2. Definitions and auxiliary propositions

Denote by $S_0(\mathbb{R}^n)$ the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that, the *distribution function*, satisfies that

$$\lambda_f(y) = |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty, \quad \text{for each } y > 0. \tag{1}$$

A *nonincreasing rearrangement* of a function $f \in S_0(\mathbb{R}^n)$ is a nonnegative and nonincreasing function f^* on $\mathbb{R}_+ = (0, +\infty)$ which is equimeasurable with $|f|$, that is, $\lambda_{f^*} = \lambda_f$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} |f(x)|, \quad 0 < t < \infty, \tag{2}$$

where the supremum is taken over all F_σ-sets of measure t (see [9, p. 32]). Moreover, the supremum is attained for some of such sets E. Note also that the function defined by (2) is left continuous on $\mathbb{R}_+$.

Let $0 < p, r < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{p,r}(\mathbb{R}^n)$ if

$$\|f\|_{L^{p,r}} \equiv \|f\|_{p,r} = \left(\frac{r}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} = \left(\int_0^\infty \left(t\lambda_f^{\frac{1}{p}}(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}} < \infty. \tag{3}$$

It is clear that $\|f\|_{p,p} = \|f\|_p$. For a fixed p, the Lorentz spaces $L^{p,r}$ strictly increase as the secondary index r increases; that is, the following strict embedding $L^{p,r} \subset L^{p,s}$, with $r < s$, holds (see [5, Chapter 4], [21, Chapter 5]).

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Denote by $\hat{x}_k$ the $(n-1)$-dimensional vector obtained from the n-tuple x by removal of its kth coordinate. We shall write

$x = (x_k, \hat{x}_k)$. Let $f \in S_0(\mathbb{R}^n)$ and $1 \le k \le n$. We fix $\hat{x}_k \in \mathbb{R}^{n-1}$ and consider the $\hat{x}_k$-section of the function f

$$f_{\hat{x}_k}(x_k) = f(x_k, \hat{x}_k), \quad x_k \in \mathbb{R}.$$

For almost all $\hat{x}_k \in \mathbb{R}^{n-1}$ we have $f_{\hat{x}_k} \in S_0(\mathbb{R})$. We set

$$\mathcal{R}_k f(u, \hat{x}_k) = f^*_{\hat{x}_k}(u), \quad u \in \mathbb{R}_+. \tag{4}$$

Observe that the kth argument of the function $\mathcal{R}_k f$ is equal to u. The function $\mathcal{R}_k f$ is defined almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$; we call it *the rearrangement of f with respect to the kth variable.* It is easy to show that $\mathcal{R}_k f$ is a measurable function equimeasurable with $|f|$. Let $\mathcal{P}_n$ be the collection of all permutations of the set $\{1, \dots, n\}$. For each $\sigma = \{k_1, \dots, k_n\} \in \mathcal{P}_n$ we call the function

$$\mathcal{R}_\sigma f(t) = \mathcal{R}_{k_n} \cdots \mathcal{R}_{k_1} f(t), \quad t \in \mathbb{R}^n_+,$$

the $\mathcal{R}_\sigma$-rearrangement of f. Thus, we obtain $\mathcal{R}_\sigma f$ by "rearranging" f in nonincreasing order successively with respect to the variables $x_{k_1}, \dots, x_{k_n}$. The rearrangement $\mathcal{R}_\sigma f$ is defined on $\mathbb{R}^n_+$. It is nonnegative, nonincreasing in each variable, and equimeasurable with the function $|f|$ (see [7]).

Lemma 2.1. *Let $f_k \in S_0(\mathbb{R}^n)$ $(k \in \mathbb{N})$ and assume that the sequence $\{f_k\}$ converges in measure to a function $f \in S_0(\mathbb{R}^n)$. Then, for each permutation $\sigma \in \mathcal{P}_n$*

$$\lim_{k\to\infty} \mathcal{R}_\sigma f_k(t) = \mathcal{R}_\sigma f(t), \quad \textit{for almost all } t \in \mathbb{R}^n_+.$$

This lemma follows from a similar statement for usual rearrangements (see [16, Chapter 2, §2]).

Our next result is an important tool we will need in Section 5 to prove optimal estimates for sections of measurable sets. It is essentially a reverse Minkowski's integral inequality for the Lorentz spaces $L^{p,r}$ defined in (3), on the range $0 < p \le r \le 1$. We recall that $\|\cdot\|_{p,r}$ is in fact a norm [5, Chapter 4], when $1 \le r \le p < \infty$, and hence Minkowski's integral inequality holds for those indices:

$$\left\| \int_{\mathbb{R}} f(\cdot, y)\, dy \right\|_{p,r} \le \int_{\mathbb{R}} \|f(\cdot, y)\|_{p,r}\, dy.$$

Recall that, if $g \in S_0(\mathbb{R})$ and $t > 0$, then for any set $A \subset \mathbb{R}$, with $|A| = t$,

$$\int_A |g(x)|\, dx \le \int_0^t g^*(s)\, ds,$$

and there exists a set $A_t \subset \mathbb{R}$, with $|A_t| = t$, such that [16, Chapter 2, §2]

$$\int_{A_t} |g(x)|\, dx = \int_0^t g^*(s)\, ds.$$

Then, it follows that

$$\int_{\mathbb{R}\setminus A} |g(x)|\,dx \geq \int_t^\infty g^*(s)\,ds, \tag{5}$$

if $|A| = t$, and

$$\int_{\mathbb{R}\setminus A_t} |g(x)|\,dx = \int_t^\infty g^*(s)\,ds. \tag{6}$$

Proposition 2.2. *Let $0 < p \leq r \leq 1$. Assume that $f \in S_0(\mathbb{R}^2)$ is a nonnegative function. Then,*

$$\left\| \int_{\mathbb{R}} f(\cdot, y)\,dy \right\|_{p,r} \geq \int_{\mathbb{R}} \|f(\cdot, y)\|_{p,r}\,dy. \tag{7}$$

Proof. It is a trivial calculation to show, using that $\|\cdot\|_{\frac{1}{p}}$ is a norm, that the result follows if $0 < p = r \leq 1$:

$$\left\| \int_{\mathbb{R}} f(\cdot, y)\,dy \right\|_{r} \geq \int_{\mathbb{R}} \|f(\cdot, y)\|_{r}\,dy. \tag{8}$$

Now, consider $q = \frac{r}{p} > 1$, and set

$$\Psi(x) = \int_{\mathbb{R}} f(x, y)\,dy.$$

Applying Fubini's theorem, we obtain

$$\|\Psi\|_{p,r} = \left(q \int_0^\infty t^{q-1} \Psi^*(t)^r\,dt \right)^{\frac{1}{r}} = \left(q(q-1) \int_0^\infty t^{q-2} \int_t^\infty \Psi^*(u)^r\,du\,dt \right)^{\frac{1}{r}}.$$

We assume that $\|\Psi\|_{p,r} < \infty$ and thus $\Psi \in S_0(\mathbb{R})$. From (6) it follows that, for any $t > 0$, there exists a measurable set $A_t \subset \mathbb{R}$ such that $|A_t| = t$ and

$$\int_t^\infty \Psi^*(u)^r\,du = \int_{\mathbb{R}\setminus A_t} \Psi(x)^r\,dx = \int_{\mathbb{R}\setminus A_t} \left(\int_{\mathbb{R}} f(x, y)\,dy \right)^r dx.$$

Applying (8), we have $\int_t^\infty \Psi^*(u)^r\,du \geq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}\setminus A_t} f(x, y)^r\,dx \right)^{\frac{1}{r}} dy \right)^r$. Thus,

$$\|\Psi\|_{p,r} \geq \left(q(q-1) \int_0^\infty \left(\int_{\mathbb{R}} \Phi(t, y)\,dy \right)^r dt \right)^{\frac{1}{r}},$$

where

$$\Phi(t, y) = t^{\frac{q-2}{r}} \left(\int_{\mathbb{R}\setminus A_t} f(x, y)^r\,dx \right)^{\frac{1}{r}}.$$

Using again (8), we obtain $\|\Psi\|_{p,r} \geq \big(q(q-1)\big)^{\frac{1}{r}} \int_{\mathbb{R}} \big(\int_0^\infty \Phi(t,y)^r\, dt\big)^{\frac{1}{r}} dy$. Further, by (5) and Fubini's theorem,

$$\begin{aligned} q(q-1)\int_0^\infty \Phi(t,y)^r\, dt &= q(q-1)\int_0^\infty t^{q-2}\int_{\mathbb{R}\setminus A_t} f(x,y)^r\, dx\, dt \\ &\geq q(q-1)\int_0^\infty t^{q-2}\int_t^\infty \mathcal{R}_1 f(s,y)^r\, ds\, dt \\ &= q\int_0^\infty t^{q-1}\mathcal{R}_1 f(t,y)^r\, dt. \end{aligned}$$

Thus

$$\|\Psi\|_{p,r} \geq q^{\frac{1}{r}} \int_{\mathbb{R}} \left(\int_0^\infty t^{q-1}\mathcal{R}_1 f(t,y)^r\, dt\right)^{\frac{1}{r}} dy = \int_{\mathbb{R}} \|f(\cdot,y)\|_{p,r}\, dy. \qquad \square$$

3. Rearrangement inequalities for finite sequences

We are going to prove in this Section the principal tools needed to handle the main results of this work. The idea is to discretize the problem (see the proof of Theorem 4.5) and use Lemmas 3.2 and 3.3.

Lemma 3.1. *Let $a \geq b$ and $c \geq d$ be nonnegative numbers. Then for any $\alpha \in [0,1]$*

$$(a+d)^\alpha + (b+c)^\alpha \geq (a+c)^\alpha + (b+d)^\alpha. \tag{9}$$

Proof. Assume that $a > 0$. Set $u = \frac{b}{a}$, $v = \frac{c}{a}$, and $w = \frac{d}{a}$. Then (9) can be rewritten as $(1+w)^\alpha + (u+v)^\alpha \geq (1+v)^\alpha + (u+w)^\alpha$, or, equivalently,

$$(1+w)^\alpha - (u+w)^\alpha \geq (1+v)^\alpha - (u+v)^\alpha. \tag{10}$$

Let $\varphi(x) = (1+x)^\alpha - (u+x)^\alpha$, $x \geq 0$. Since $0 \leq u \leq 1$ and $0 < \alpha \leq 1$, we have

$$\varphi'(x) = \alpha[(1+x)^{\alpha-1} - (u+x)^{\alpha-1}] \leq 0$$

for all $x > 0$, and thus the function φ decreases on $[0,\infty)$. Since $v \geq w$, this implies inequality (10). $\square$

Lemma 3.2. *Let $A = (a_{jk})$ be a $\mu \times \nu$-matrix of nonnegative numbers. For a fixed column k, let $\{a^*_{jk}\}_{j=1}^\mu$ be the nonincreasing rearrangement of the sequence $\{a_{jk}\}_{j=1}^\mu$. Then for any $\alpha \in (0,1]$*

$$\sum_{j=1}^{\mu}\left(\sum_{k=1}^{\nu} a_{jk}\right)^{\alpha} \geq \sum_{j=1}^{\mu}\left(\sum_{k=1}^{\nu} a^*_{jk}\right)^{\alpha}. \tag{11}$$

Proof. We use the induction with respect to μ to prove that, for any $\mu \in \mathbb{N}$, inequality (11) holds, for all $\nu \in \mathbb{N}$. If $\mu = 1$, then for any $\nu \in \mathbb{N}$ we have equality in (11). Suppose that a number $m \in \mathbb{N}$ has the following property:

(i) *inequality* (11) *is true for* $\mu = m$ *and any* $\nu \in \mathbb{N}$.

Fix m and prove that inequality (11) holds for $\mu = m + 1$ and any $\nu \in \mathbb{N}$. For this, we apply induction with respect to ν. For $\nu = 1$ and any $\mu \in \mathbb{N}$, we have equality in (11). Assume that $n \in \mathbb{N}$ is such that:

(ii) *inequality* (11) *is true for* $\mu = m + 1$ *and* $\nu = n$.

We shall prove that (11) is true for $\mu = m+1$ and $\nu = n+1$; then, by induction, the lemma will be proved.

Set $b_{j1} = a_{j1} + a_{j2}$ $(j = 1, \dots, m+1)$. For definiteness, we may assume that $n \geq 2$ (the case $n = 1$ is simpler). Set $b_{jk} = a_{j,k+1}$ $(k = 2, \dots, n)$. We now obtain the $(m+1) \times n$-matrix $B = (b_{jk})$. We have

$$\sum_{k=1}^{n+1} a_{jk} = \sum_{k=1}^{n} b_{jk}, \quad j = 1, \dots, m+1.$$

Using our assumption (ii), we obtain

$$S \equiv \sum_{j=1}^{m+1} \left(\sum_{k=1}^{n+1} a_{jk} \right)^{\alpha} = \sum_{j=1}^{m+1} \left(\sum_{k=1}^{n} b_{jk} \right)^{\alpha} \geq \sum_{j=1}^{m+1} \left(\sum_{k=1}^{n} b_{jk}^{*} \right)^{\alpha},$$

where $\{b_{jk}^{*}\}_{j=1}^{m+1}$ is the nonincreasing rearrangement of the sequence $\{b_{jk}\}_{j=1}^{m+1}$ (for a fixed k). We have

$$b_{j1}^{*} = b_{\mu_j 1} = a_{\mu_j 1} + a_{\mu_j 2},$$

where $\{\mu_1, \dots, \mu_{m+1}\}$ is some permutation of the set $\{1, \dots, m+1\}$. Denote $\overline{a}_{j1} = a_{\mu_j 1}$, $\overline{a}_{j2} = a_{\mu_j 2}$. Also, we have $b_{jk}^{*} = a_{j,k+1}^{*}$ for $k = 2, \dots, n$. Set

$$s_j = \sum_{k=2}^{n} b_{jk}^{*} = \sum_{k=3}^{n+1} a_{jk}^{*}.$$

Then $s_j \geq s_{j+1}$ $(j = 1, \dots, m)$. We have that

$$S \geq \sum_{j=1}^{m+1} \left(b_{j1}^{*} + s_j \right)^{\alpha} = \sum_{j=1}^{m+1} \left(\overline{a}_{j1} + \overline{a}_{j2} + s_j \right)^{\alpha}. \tag{12}$$

Setting $\overline{a}_{jk} = a_{jk}^{*}$ for $k = 3, \dots, n+1$ and $j = 1, \dots, m+1$, we obtain the $(m+1) \times (n+1)$-matrix $\overline{A} = (\overline{a}_{jk})$. Further, we consider the first column of this matrix. We shall put the greatest element of this column to the first place, carrying out (if necessary) an interchange of elements. Let j' be the least index such that

$$a_{11}^* = \max_{1\le j\le m+1} a_{j1} = \overline{a}_{j'1}.$$

If $j' = 1$, no interchange is needed. Let $j' > 1$, that is, $\overline{a}_{11} < a_{11}^* = \overline{a}_{j'1}$. Since $\overline{a}_{11} + \overline{a}_{12} = b_{11}^* \ge b_{j'1}^* = \overline{a}_{j'1} + \overline{a}_{j'2}$, we have that $\overline{a}_{j'2} < \overline{a}_{12}$. Set

$$a = \overline{a}_{j'1} = a_{11}^*, \quad b = \overline{a}_{11}, \quad c = \overline{a}_{12} + s_1, \quad d = \overline{a}_{j'2} + s_{j'}.$$

Then $a > b$, $c > d$. Hence, applying Lemma 3.1, we obtain

$$(\overline{a}_{j'1} + \overline{a}_{j'2} + s_{j'})^\alpha + (\overline{a}_{11} + \overline{a}_{12} + s_1)^\alpha \ge (a_{11}^* + \overline{a}_{12} + s_1)^\alpha + (\overline{a}_{11} + \overline{a}_{j'2} + s_{j'})^\alpha.$$

It follows that, interchanging the elements $\overline{a}_{11}$ and $\overline{a}_{j'1}$ in the first column of $\overline{A}$, we do not increase the sum at the right-hand side of (12). The elements of the new column obtained by this interchange will be denoted by $\widetilde{a}_{11}, \dots, \widetilde{a}_{m+1,1}$; clearly, $\widetilde{a}_{11} = a_{11}^*$. We have the inequality

$$S \ge (a_{11}^* + \overline{a}_{12} + s_1)^\alpha + \sum_{j=2}^{m+1} (\widetilde{a}_{j1} + \overline{a}_{j2} + s_j)^\alpha. \tag{13}$$

Next, we consider the second column of the matrix $\overline{A}$. As above, our objective is to put the greatest element of this column to the first place (possibly interchanging two elements). Let j'' be the least index such that

$$a_{12}^* = \max_{1\le j\le m+1} a_{j2} = \overline{a}_{j''2}.$$

If $j'' = 1$, we are done. Assume that $j'' > 1$, that is, $\overline{a}_{12} < a_{12}^* = \overline{a}_{j''2}$. Set

$$a = \overline{a}_{j''2} = a_{12}^*, \quad b = \overline{a}_{12}, \quad c = a_{11}^* + s_1, \quad d = \widetilde{a}_{j''1} + s_{j''}.$$

Then $a > b$, $c \ge d$. Applying Lemma 3.1, we obtain

$$\begin{aligned}(\overline{a}_{j''2} + \widetilde{a}_{j''1} + s_{j''})^\alpha + (\overline{a}_{12} + a_{11}^* + s_1)^\alpha &\ge (\overline{a}_{j''2} + a_{11}^* + s_1)^\alpha + (\overline{a}_{12} + \widetilde{a}_{j''1} + s_{j''})^\alpha \\ &= \left(\sum_{k=1}^{n+1} a_{1k}^*\right)^\alpha + (\widetilde{a}_{j''1} + \overline{a}_{12} + s_{j''})^\alpha.\end{aligned}$$

As above, we see that the interchange of the elements $\overline{a}_{12}$ and $\overline{a}_{j''2}$ in the second column of $\overline{A}$ does not increase the sum at the right-hand side of (13). Making this interchange, we denote by $\widetilde{a}_{12}, \dots, \widetilde{a}_{m+1,2}$ the elements of the new column; here $\widetilde{a}_{12} = a_{12}^*$ and $\widetilde{a}_{j''2} = \overline{a}_{12}$. Now, by (13), we have that

$$S \ge \left(\sum_{k=1}^{n+1} a_{1k}^*\right)^\alpha + \sum_{j=2}^{m+1} \left(\widetilde{a}_{j1} + \widetilde{a}_{j2} + \sum_{k=3}^{n+1} a_{jk}^*\right)^\alpha. \tag{14}$$

Denote by C the $m \times (n+1)$-matrix formed by the columns

$$\{\widetilde{a}_{j1}\}_{j=2}^{m+1},\ \{\widetilde{a}_{j2}\}_{j=2}^{m+1},\ \{a^*_{j,3}\}_{j=2}^{m+1}, \dots, \{a^*_{j+1,n+1}\}_{j=2}^{m+1}.$$

It is easy to see that the rearrangements of the first two columns of C are $\{a^*_{j1}\}_{j=2}^{m+1}$ and $\{a^*_{j2}\}_{j=2}^{m+1}$, respectively. Then, applying assumption (i), we have

$$\sum_{j=2}^{m+1}\left(\widetilde{a}_{j1}+\widetilde{a}_{j2}+\sum_{k=3}^{n+1}a_{jk}\right)^{\alpha} \ge \sum_{j=2}^{m+1}\left(\sum_{k=1}^{n+1}a^*_{jk}\right)^{\alpha}.$$

Using this inequality and (14), we obtain $S \ge \sum_{j=1}^{m+1}\left(\sum_{k=1}^{n+1}a^*_{jk}\right)^{\alpha}$. By induction, this completes the proof. □

Let $A = (a_{jk})$ be a $\mu \times \nu$-matrix of nonnegative numbers. For a fixed $1 \le k \le \nu$, denote by $\{\mathcal{R}_1 a_{jk}\}_{j=1}^{\mu}$ the nonincreasing rearrangement of the sequence $\{a_{jk}\}_{j=1}^{\mu}$. Further, for a fixed $1 \le j \le \mu$, let $\{\mathcal{R}_{1,2}a_{jk}\}_{k=1}^{\nu}$ be the nonincreasing rearrangement of the sequence $\{\mathcal{R}_1 a_{jk}\}_{k=1}^{\nu}$.

Similarly we define $\mathcal{R}_2 a_{jk}$ and $\mathcal{R}_{2,1}a_{jk}$.

Lemma 3.3. *Let $H = (h_{jk})$ be a $\mu \times \nu$-matrix of nonnegative numbers. Then for any $\alpha \in (0,1]$*

$$\sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}h_{jk}\right)^{\alpha} \ge \sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}\mathcal{R}_{1,2}h_{jk}\right)^{\alpha} \tag{15}$$

and

$$\sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}h_{jk}\right)^{\alpha} \ge \sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}\mathcal{R}_{2,1}h_{jk}\right)^{\alpha}. \tag{16}$$

Proof. Applying Lemma 3.2 to the transposed matrix $(\mathcal{R}_1 h_{jk})^T$, we obtain

$$\sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}h_{jk}\right)^{\alpha} = \sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}\mathcal{R}_1 h_{jk}\right)^{\alpha} \ge \sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}\mathcal{R}_{1,2}h_{jk}\right)^{\alpha}.$$

Further, by Lemma 3.2 applied to the transposed matrix $(h_{jk})^T$, we have

$$\sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}h_{jk}\right)^{\alpha} \ge \sum_{k=1}^{\nu}\left(\sum_{j=1}^{\mu}\mathcal{R}_2 h_{jk}\right)^{\alpha}. \tag{17}$$

For any fixed k the sequence $\{\mathcal{R}_{2,1}h_{jk}\}_{j=1}^{\mu}$ is the rearrangement of the sequence $\{\mathcal{R}_2 h_{jk}\}_{j=1}^{\mu}$ and therefore

$$\sum_{j=1}^{\mu}\mathcal{R}_2 h_{jk} = \sum_{j=1}^{\mu}\mathcal{R}_{2,1}h_{jk}.$$

Together with (17), this implies (16). □

Remark 3.4. Let $T = (t_{jk})$ be a $\mu \times \nu$-matrix of nonnegative numbers. Then for any $p \geq 1$

$$\sum_{k=1}^{\nu} \left(\max_{1\leq j\leq \mu} t_{jk} \right)^p \geq \sum_{k=1}^{\nu} \left(\max_{1\leq j\leq \mu} \mathcal{R}_{1,2} t_{jk} \right)^p$$

and

$$\sum_{k=1}^{\nu} \left(\max_{1\leq j\leq \mu} t_{jk} \right)^p \geq \sum_{k=1}^{\nu} \left(\max_{1\leq j\leq \mu} \mathcal{R}_{2,1} t_{jk} \right)^p .$$

For the proof, we apply Lemma 3.3 to $h_{j,k} = t_{j,k}^r$, $\alpha = \frac{p}{r}$, with $r > p$, and then let r tend to ∞.

4. Mixed norm spaces $L^p[L^r]$

We begin with the following simple observation. Let φ be a nonnegative step function on $\mathbb{R}$, $\varphi(x) = 0$ for $|x| > H > 0$, and

$$\varphi(x) = a_k \quad \text{for } x \in \left(\frac{(k-\nu-1)H}{\nu}, \frac{(k-\nu)H}{\nu} \right], \quad k = 1, \dots, 2\nu.$$

Then $\varphi^*(t) = 0$ for $t > 2H$ and

$$\varphi^*(t) = a_k^* \quad \text{for } t \in \left(\frac{(k-1)H}{\nu}, \frac{kH}{\nu} \right], \quad k = 1, \dots, 2\nu,$$

where $\{a_k^*\}_{k=1}^{2\nu}$ is the nonincreasing rearrangement of the sequence $\{a_k\}_{k=1}^{2\nu}$.

Further, we introduce the following definition.

Definition 4.1. Let $Q = [-H, H]^2$ $(H > 0)$. Denote by $\mathfrak{S}(Q)$ the set of all nonnegative functions f defined on $\mathbb{R}^2$ such that $f(x, y) = 0$ for all $(x, y) \notin Q$, and there exists $\nu \in \mathbb{N}$ such that f is constant on each of the squares

$$Q_{jk} = \left(\frac{(j-\nu-1)H}{\nu}, \frac{(j-\nu)H}{\nu} \right] \times \left(\frac{(k-\nu-1)H}{\nu}, \frac{(k-\nu)H}{\nu} \right]$$

$(j, k = 1, \dots, 2\nu)$.

Lemma 4.2. *Let f be a measurable function defined on $\mathbb{R}^2$ such that $0 \leq f(x, y) \leq M$ for all $(x, y) \in \mathbb{R}^2$ and f vanishes outside of some square $Q = [-H, H]^2$ $(H > 0)$. Then there exists a sequence $\{f_k\}$ of functions $f_k \in \mathfrak{S}(Q)$ such that $0 \leq f_k(x, y) \leq M$ for all $(x, y) \in \mathbb{R}^2$ and $\{f_k\}$ converges to f in measure.*

This lemma follows from the Luzin C-property (see, e.g., [17, Chapter XII, §1]).

Definition 4.3. Let X and Y be Banach spaces of functions defined on $\mathbb{R}$. Then by $Y[X]$ we denote the mixed norm space of functions f defined on $\mathbb{R}^2$, with the finite norm

$$\|f\|_{Y[X]} = \|\varphi\|_Y, \quad \text{where} \quad \varphi(y) = \|f(\cdot, y)\|_X.$$

Remark 4.4. It is easy to see that $L^p[L^p] = L^p$, isometrically. It can be proved that, for the nondiagonal case $1 \le p \ne r \le \infty$, then $L^p[L^r] \ne L^q$, for all $1 \le q \le \infty$ (see [8] for further results regarding this question).

The following result is one of the main estimates of this work. We will prove in Proposition 4.6 that it is sharp.

Theorem 4.5. *Let* $1 \le p \le r \le \infty$ *and let* $f \in L^p[L^r]$. *Then* $\mathcal{R}_{1,2}f, \mathcal{R}_{2,1}f \in L^p[L^r]$ *and*

$$\|\mathcal{R}_{1,2}f\|_{L^p[L^r]} \le \|f\|_{L^p[L^r]}, \tag{18}$$

$$\|\mathcal{R}_{2,1}f\|_{L^p[L^r]} \le \|f\|_{L^p[L^r]}. \tag{19}$$

Proof. First we assume that $1 \le r < \infty$ and $f \in \mathfrak{S}(Q_N)$, where $Q_N = [-N, N]^2$, $N \in \mathbb{N}$. Let c_{jk} be the constant value that f takes on the square Q_{jk} (see Definition 4.1). Then

$$\|f\|_{L^p[L^r]}^p = \left(\frac{N}{\nu}\right)^2 \sum_{k=1}^{2\nu} \left(\sum_{j=1}^{2\nu} c_{jk}^r\right)^{\frac{p}{r}}$$

and (by the observation above)

$$\|\mathcal{R}_{1,2}f\|_{L^p[L^r]}^p = \left(\frac{N}{\nu}\right)^2 \sum_{k=1}^{2\nu} \left(\sum_{j=1}^{2\nu} (\mathcal{R}_{1,2}c_{jk})^r\right)^{\frac{p}{r}}.$$

Applying inequality (15) with $\alpha = \frac{p}{r}$ and $h_{jk} = c_{jk}^r$, we obtain that inequality (18) holds for any $f \in \mathfrak{S}(Q_N)$.

Similarly,

$$\|\mathcal{R}_{2,1}f\|_{L^p[L^r]}^p = \left(\frac{N}{\nu}\right)^2 \sum_{k=1}^{2\nu} \left(\sum_{j=1}^{2\nu} (\mathcal{R}_{2,1}c_{jk})^r\right)^{\frac{p}{r}},$$

and (16) implies that (19) also holds for any function $f \in \mathfrak{S}(Q_N)$.

Assume now that f is an arbitrary measurable function such that $0 \le f(x,y) \le N$ for some $N \in \mathbb{N}$ and for all $(x,y) \in \mathbb{R}^2$, and f vanishes outside of the square Q_N. Then (18) and (19) follow from the preceding case by the use of Lemma 4.2, Lemma 2.1, and the dominated convergence theorem (see [4]).

Finally, for any nonnegative function $f \in L^p[L^r]$, set

$$g_N(x,y) = \min(N, f(x,y))\chi_{Q_N}(x,y) \quad (N \in \mathbb{N}).$$

Then the sequence $g_N(x,y)$ increases and tends to $f(x,y)$ at every point $(x,y) \in \mathbb{R}^2$. Thus, $\mathcal{R}_{1,2}g_N(s,t)$ increases and tends to $\mathcal{R}_{1,2}f(s,t)$ for any $(s,t) \in \mathbb{R}^2_+$ (see [5, p. 41]). Since for any $N \in \mathbb{N}$ we have

$$\|\mathcal{R}_{1,2}g_N\|_{L^p[L^r]} \le \|g_N\|_{L^p[L^r]},$$

then, applying the monotone convergence theorem, we obtain (18). Similarly we obtain (19).

In the same way, the case $r = \infty$ follows using Remark 3.4. □

We shall show that Theorem 4.5 fails to hold for $p > r$.

Proposition 4.6. *Let $1 \le r < p \le \infty$. Then there exists a function $f \in L^p[L^r]$ such that $\mathcal{R}_{1,2}f(\cdot,t) = \mathcal{R}_{2,1}f(\cdot,t) \notin L^r$, for all $t > 0$.*

Proof. Set

$$\psi(y) = e^y - 1, \quad F(y) = [e^y(y+1)]^{-\frac{1}{r}} \quad (y \ge 0),$$

and

$$f(x,y) = \begin{cases} F(y), & \text{if } y \ge 0,\ 0 \le x \le \psi(y), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|f(\cdot,y)\|_{L^r} = F(y)\psi(y)^{\frac{1}{r}} \le (y+1)^{-\frac{1}{r}} \quad (y \ge 0).$$

Thus, $\|f\|_{L^p[L^r]} < \infty$, for $r < p \le \infty$.

Further, $\mathcal{R}_1 f(s,y) = f(s,y)$. It follows that

$$\mathcal{R}_{1,2}f(s,t) = \mathcal{R}_2 f(s,t) = \mathcal{R}_{2,1}f(s,t).$$

Let $\varphi(x) = \psi^{-1}(x) = \ln(1+x)$, $x \ge 0$. For a fixed $s \ge 0$ we have

$$f(s,y) = \begin{cases} F(y), & \text{if } \varphi(s) \le y < \infty, \\ 0, & \text{if } 0 \le y < \varphi(s). \end{cases}$$

It follows easily from (2) that

$$\mathcal{R}_{1,2}f(s,t) = F(\varphi(s)+t), \quad \text{for all } s,t \ge 0.$$

Hence,

$$\|\mathcal{R}_{1,2}f(\cdot,t)\|^r_{L^r} = \int_0^\infty F(\varphi(s)+t)^r\,ds = \int_0^\infty F(z+t)^r\psi'(z)\,dz = e^{-t}\int_0^\infty \frac{dz}{z+t+1} = \infty,$$

for any $t \ge 0$. □

Now we shall prove a statement which in a sense can be considered as a "dual" to Theorem 4.5. However, its proof is much simpler.

Theorem 4.7. *Let $1 \le r \le p \le \infty$. Assume that $\mathcal{R}_{1,2}f \in L^p[L^r]$. Then $f \in L^p[L^r]$ and*

$$\|f\|_{L^p[L^r]} \le \|\mathcal{R}_{1,2}f\|_{L^p[L^r]}. \tag{20}$$

Similarly, if $\mathcal{R}_{2,1}f \in L^p[L^r]$, then $f \in L^p[L^r]$ and

$$\|f\|_{L^p[L^r]} \le \|\mathcal{R}_{2,1}f\|_{L^p[L^r]}. \tag{21}$$

Proof. First we assume that $p < \infty$. Set

$$\Phi(y) = \int_0^\infty \mathcal{R}_1 f(s,y)^r \, ds.$$

Then

$$\Phi(y) = \int_{\mathbb{R}} |f(x,y)|^r \, dx \tag{22}$$

and

$$\|f\|_{L^p[L^r]}^r = \left(\int_{\mathbb{R}} \Phi(y)^q \, dy \right)^{\frac{1}{q}},$$

where $q = \frac{p}{r}$. By duality, there exists a function $g \in L^{q'}(\mathbb{R})$ with $\|g\|_{q'} = 1$ (as usual, q' denotes the conjugate exponent, $\frac{1}{q} + \frac{1}{q'} = 1$) such that $\|f\|_{L^p[L^r]}^r = \int_{\mathbb{R}} \Phi(y) g(y) \, dy = \int_{\mathbb{R}} \left(\int_0^\infty \mathcal{R}_1 f(s,y)^r \, ds \right) g(y) \, dy = \int_0^\infty \left(\int_{\mathbb{R}} \mathcal{R}_1 f(s,y)^r g(y) \, dy \right) ds$. Applying to the interior integral the Hardy-Littlewood inequality, we obtain

$$\|f\|_{L^p[L^r]}^r \le \int_0^\infty \int_0^\infty \mathcal{R}_{1,2} f(s,t)^r g^*(t) \, dt \, ds = \int_0^\infty g^*(t) \left(\int_0^\infty \mathcal{R}_{1,2} f(s,t)^r ds \right) dt.$$

Finally, applying to the latter integral Hölder's inequality, we get

$$\|f\|_{L^p[L^r]}^r \le \left(\int_0^\infty \left(\int_0^\infty \mathcal{R}_{1,2} f(s,t)^r ds \right)^q dt \right)^{\frac{1}{q}} = \|\mathcal{R}_{1,2} f\|_{L^p[L^r]}^r.$$

Let now $p = \infty$. Using (22), we have that

$$\|f\|_{L^\infty[L^r]}^r = \operatorname{ess\,sup}_{y \in \mathbb{R}} \Phi(y) \le \int_0^\infty \operatorname{ess\,sup}_{y \in \mathbb{R}} \mathcal{R}_1 f(s,y)^r \, ds.$$

Further, $\operatorname{ess\,sup}_{y \in \mathbb{R}} \mathcal{R}_1 f(s,y) = \operatorname{ess\,sup}_{t>0} \mathcal{R}_{1,2} f(s,t) = \mathcal{R}_{1,2} f(s, 0+)$, for any $s > 0$. On the other hand, by the monotone convergence theorem,

$$\|\mathcal{R}_{1,2} f\|_{L^\infty[L^r]}^r = \lim_{t \to 0+} \int_0^\infty \mathcal{R}_{1,2} f(s,t)^r \, ds = \int_0^\infty \mathcal{R}_{1,2} f(s, 0+)^r \, ds.$$

Thus, we obtain (20) for $p = \infty$.

We now consider the proof of (21). By a simple change of notation, we can assume, without loss of generality, that $r = 1$. If $1 < p < \infty$, given f, we can find a suitable nonnegative function $h \in L^{p'}$, with $\|h\|_{p'} = 1$, such that

$$\begin{aligned}
\|f\|_{L^p[L^1]} &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,dx \right)^p dy \right)^{\frac{1}{p}} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,dx \right) h(y)\,dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| h(y)\,dy \right) dx \\
&\le \int_{\mathbb{R}} \left(\int_0^\infty \mathcal{R}_2 f(x,t) h^*(t)\,dt \right) dx \\
&= \int_0^\infty \left(\int_{\mathbb{R}} \mathcal{R}_2 f(x,t)\,dx \right) h^*(t)\,dt \\
&= \int_0^\infty \left(\int_0^\infty \mathcal{R}_{2,1} f(s,t)\,ds \right) h^*(t)\,dt \\
&\le \left(\int_0^\infty \left(\int_0^\infty \mathcal{R}_{2,1} f(s,t)\,ds \right)^p dt \right)^{\frac{1}{p}} \|h\|_{p'} = \|\mathcal{R}_{2,1} f\|_{L^p[L^1]}.
\end{aligned}$$

If $p = \infty$,

$$\begin{aligned}
\|f\|_{L^\infty[L^1]} &= \operatorname{ess\,sup}_{y\in\mathbb{R}} \int_{\mathbb{R}} |f(x,y)|\,dx \\
&\le \int_{\mathbb{R}} \operatorname{ess\,sup}_{y\in\mathbb{R}} |f(x,y)|\,dx \\
&= \int_{\mathbb{R}} \mathcal{R}_2 f(x,0+)\,dx \\
&= \int_0^\infty \mathcal{R}_{2,1} f(s,0+)\,ds \\
&= \lim_{t\to 0+} \int_0^\infty \mathcal{R}_{2,1} f(s,t)\,ds \\
&= \operatorname{ess\,sup}_{t>0} \int_0^\infty \mathcal{R}_{2,1} f(s,t)\,ds = \|\mathcal{R}_{2,1} f\|_{L^\infty[L^1]}. \qquad \square
\end{aligned}$$

We now show that Theorem 4.7 does not hold for $1 \le p < r < \infty$ and $\mathcal{R}_{1,2}$, although, contrary to what we have proved in Proposition 4.6, the end-point case $1 \le p < r = \infty$ is also true (see Proposition 4.9).

Proposition 4.8. *Let $1 \le p < r < \infty$. Then there exists a function $f \in S_0(\mathbb{R}^2)$ such that $\mathcal{R}_{1,2} f \in L^p[L^r]$ but $f \notin L^p[L^r]$.*

Proof. Set
$$\psi(y)=e^y-1,\quad F(y)=e^{-\frac{y}{r}}(y+1)^{-\frac{1}{p}}\quad (y\ge 0),$$
and
$$f(x,y)=\begin{cases}F(y), & \text{if } y\ge 0,\ 0\le x\le \psi(y),\\ 0, & \text{otherwise.}\end{cases}$$
Then $\|f(\cdot,y)\|_{L^r}=F(y)\psi(y)^{\frac{1}{r}}\ge \frac{1}{2(y+1)^{\frac{1}{p}}}$ $(y\ge 1)$ and thus $\|f\|_{L^p[L^r]}=\infty$.

As in Proposition 4.6, we have
$$\mathcal{R}_{1,2}f(s,t)=F(\varphi(s)+t),\quad \text{for all } s,t\ge 0,$$
where $\varphi(x)=\psi^{-1}(x)=\ln(1+x)$. Thus,
$$\|\mathcal{R}_{1,2}f(\cdot,t)\|_{L^r}^r=\int_0^\infty F(\varphi(s)+t)^r ds=\int_0^\infty F(z+t)^r\psi'(z)\,dz=e^{-t}\int_0^\infty \frac{dz}{(z+t+1)^{\frac{r}{p}}}\le\frac{pe^{-t}}{r-p}.$$
Thus, it follows that $\|\mathcal{R}_{1,2}f\|_{L^p[L^r]}<\infty$. □

Proposition 4.9. *For any $p\in[1,\infty]$ and any function $f\in S_0(\mathbb{R}^2)$,*
$$\|f\|_{L^p[L^\infty]}=\|\mathcal{R}_{1,2}f\|_{L^p[L^\infty]}. \tag{23}$$

Proof. For any $y\in\mathbb{R}$
$$\lim_{s\to 0+}\mathcal{R}_1f(s,y)=\mathcal{R}_1f(0+,y)=\|\mathcal{R}_1f(\cdot,y)\|_\infty=\|f(\cdot,y)\|_\infty,$$
and the convergence is monotone. Thus,
$$\lim_{s\to 0+}\int_{\mathbb{R}}\mathcal{R}_1f(s,y)^p\,dy=\int_{\mathbb{R}}\|f(\cdot,y)\|_\infty^p\,dy=\|f\|_{L^p[L^\infty]}^p.$$
On the other hand, for any $s>0$, $\int_{\mathbb{R}}\mathcal{R}_1f(s,y)^p\,dy=\int_0^\infty\mathcal{R}_{1,2}f(s,t)^p\,dt$, and by the monotone convergence theorem
$$\lim_{s\to 0+}\int_0^\infty\mathcal{R}_{1,2}f(s,t)^p dt=\int_0^\infty\mathcal{R}_{1,2}f(0+,t)^p dt=\int_0^\infty\|\mathcal{R}_{1,2}f(\cdot,t)\|_\infty^p\,dt=\|\mathcal{R}_{1,2}f\|_{L^p[L^\infty]}^p.$$
These equalities imply (23). □

Finally, we see that Theorem 4.7 does not hold either for $\mathcal{R}_{2,1}$ in the whole range $1\le p<r\le\infty$ (including $r=\infty$):

Proposition 4.10. *Let $1\le p<r\le\infty$. Then there exists a function $f\in S_0(\mathbb{R}^2)$ such that $\mathcal{R}_{2,1}f\in L^p[L^r]$ but $f\notin L^p[L^r]$.*

Proof. Assume first that $1 \le p < r < \infty$. Define

$$f(x,y) = \begin{cases} n^{-\frac{1}{p}}, & \text{if } n-1 < x, y \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Then

$$\mathcal{R}_{2,1}f(s,t) = \begin{cases} n^{-\frac{1}{p}}, & \text{if } n-1 < s \le n, \text{ and } 0 < t \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and hence,

$$\|\mathcal{R}_{2,1}f\|_{L^p[L^r]} = \left(\int_0^1 \left(\int_0^\infty \left(\mathcal{R}_{2,1}f(s,t) \right)^r ds \right)^{\frac{p}{r}} dt \right)^{\frac{1}{p}} = \left(\sum_{n=1}^\infty \frac{1}{n^{\frac{r}{p}}} \right)^{\frac{1}{r}} < \infty.$$

On the other hand,

$$\|f\|_{L^p[L^r]} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(f(x,y) \right)^r dx \right)^{\frac{p}{r}} dy \right)^{\frac{1}{p}} = \left(\sum_{n=1}^\infty \frac{1}{n} \right)^{\frac{1}{p}} = \infty.$$

The case $1 \le p < r = \infty$ is handled similarly with the same function in (24). □

Remark 4.11. A more general inequality than (20) was proved, in [7], related to mixed Lorentz spaces (see [7, Theorem 4.5.I]). Nevertheless, Theorem 4.7 does not follow from this result since the constant in the corresponding inequality in [7] blows up as $p \to r+$. Further, it was stated in [7] that inequality of the type (18) (with a constant greater than 1) can be obtained by duality arguments. However, the duality relation given in [7, Theorem 3.12.II] is false. In the case $X = L^1, Y = L^\infty$ this relation can be formulated as follows:
Inclusion

$$\mathcal{R}_{1,2}f \in L^\infty[L^1] \tag{25}$$

holds if and only if

$$\sup_g \iint_{\mathbb{R}^2} |f(x,y)g(x,y)|dxdy < \infty, \tag{26}$$

where the supremum is taken over all g *such that* $\|\mathcal{R}_{1,2}g\|_{L^1[L^\infty]} \le 1$.

We shall show that this statement is false. Set

$$f(x,y) = \begin{cases} \frac{1}{y}, & \text{if } 0 < y \le 1, \ 0 < x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f\|_{L^\infty[L^1]} = 1$. Further, $\mathcal{R}_{1,2}f(s,t) = \mathcal{R}_2 f(s,t) = \mathcal{R}_{2,1}f(s,t)$. We have

$$\mathcal{R}_{1,2}f(s,t) = \begin{cases} (s+t)^{-1}, & \text{if } 0 < t \le 1, \ 0 \le s \le 1-t, \\ 0, & \text{otherwise.} \end{cases}$$

Thus
$$\int_0^1 \mathcal{R}_{1,2}f(s,t)\,ds = \int_0^{1-t} \frac{ds}{s+t} = \ln\frac{1}{t}.$$
It follows that $\|\mathcal{R}_{1,2}f\|_{L^\infty[L^1]} = \infty$.

Assume that $g \in S_0(\mathbb{R}^2)$ and $\|\mathcal{R}_{1,2}g\|_{L^1[L^\infty]} \leq 1$. Then
$$\begin{aligned}\iint_{\mathbb{R}^2} |f(x,y)g(x,y)|dxdy &= \int_0^1 \frac{1}{y}\int_0^y |g(x,y)|dxdy \\ &\leq \int_0^1 \frac{1}{y}\int_0^y \mathcal{R}_1 g(s,y)dsdy \\ &\leq \int_0^1 \operatorname{ess\,sup}_{s>0} \mathcal{R}_1 g(s,y)dy \\ &= \int_0^1 \mathcal{R}_1 g(0+,y)dy \\ &= \|g\|_{L^1[L^\infty]} \\ &\leq 1.\end{aligned}$$
Thus, we see that (26) does not imply (25).

5. Mixed norms and rearrangements for measurable sets

We now study the relationship between both iterated rearrangements, $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,1}$, on characteristic functions of measurable sets $E \subset \mathbb{R}^2$. It is interesting to observe that, in this situation, the estimates on mixed normed spaces can be described in terms of Lorentz "norms" with main indices below one. For this purpose, we will work on rearrangement invariant spaces X on $\mathbb{R}$ (see [5, 16]). In this setting, we recall the definition of the *fundamental function* of X:
$$\varphi_X(t) = \|\chi_E\|_X,$$
where $|E| = t$. Given a set $E \subset \mathbb{R}^2$, we also define the following functions (representing the linear measures of sections):
$$\Psi_E^1(x) = \int_{\mathbb{R}} \chi_E(x,y)\,dy \quad \text{and} \quad \Psi_E^2(y) = \int_{\mathbb{R}} \chi_E(x,y)\,dx.$$

Lemma 5.1. *Let X be a rearrangement invariant space on $\mathbb{R}$ and $E \subset \mathbb{R}^2$. Then,*
$$\|\Psi_E^1\|_X \leq \int_{\mathbb{R}} \varphi_X(\Psi_E^2(y))\,dy. \tag{27}$$
In particular, if $1 \leq p \leq r < \infty$, then
$$\left\| \int_{\mathbb{R}} \chi_E(\cdot,y)\,dy \right\|_{L^{\frac{r}{p},1}} \leq \left\| \int_{\mathbb{R}} \chi_E(x,\cdot)\,dx \right\|_{L^{\frac{p}{r}}}^{\frac{p}{r}}. \tag{28}$$

Proof. This is an easy consequence of the Minkowski's integral inequality. In fact,

$$\|\Psi_E^1\|_X = \left\| \int_{\mathbb{R}} \chi_E(\cdot, y)\, dy \right\|_X \leq \int_{\mathbb{R}} \|\chi_E(\cdot, y)\|_X\, dy = \int_{\mathbb{R}} \varphi_X(\Psi_E^2(y))\, dy.$$

Now, if we set $q = \frac{r}{p} \geq 1$, with $X = L^{q,1}$ and taking into account that the fundamental function of $L^{q,1}$ satisfies that $\varphi_{L^{q,1}}(t) = t^{\frac{1}{q}}$, using (27) we get,

$$\left\| \int_{\mathbb{R}} \chi_E(\cdot, y)\, dy \right\|_{L^{\frac{r}{p},1}} = \|\Psi_E^1\|_{L^{q,1}} \leq \int_{\mathbb{R}} \Psi_E^2(y)^{\frac{1}{q}}\, dy = \left\| \int_{\mathbb{R}} \chi_E(x, \cdot)\, dx \right\|_{L^{\frac{p}{r}}}^{\frac{p}{r}}. \qquad \square$$

Theorem 5.2. *Let $E \subset \mathbb{R}^2$ be a measurable subset.*

(i) *If $1 \leq p \leq r \leq \infty$, then*

$$\|\mathcal{R}_{2,1}\chi_E\|_{L^p[L^r]} \leq \|\mathcal{R}_{1,2}\chi_E\|_{L^p[L^r]}.$$

(ii) *If $1 \leq r \leq p \leq \infty$, then*

$$\|\mathcal{R}_{1,2}\chi_E\|_{L^p[L^r]} \leq \|\mathcal{R}_{2,1}\chi_E\|_{L^p[L^r]}. \tag{29}$$

Proof. A first step to prove these results is the following: If $\varphi_1(x) = \Psi_E^1(x)$ and $\varphi_2(y) = \Psi_E^2(y)$, then

$$\mathcal{R}_{1,2}\chi_E(s,t) = \chi_{(0,\varphi_2^*(t))}(s) \quad \text{and} \quad \mathcal{R}_{2,1}\chi_E(s,t) = \chi_{(0,\varphi_1^*(s))}(t).$$

In fact, this follows readily from the definition of the iterated rearrangements and the identities:

$$\mathcal{R}_1\chi_E(s,y) = \chi_{(0,\varphi_2(y))}(s) \quad \text{and} \quad \mathcal{R}_2\chi_E(x,t) = \chi_{(0,\varphi_1(x))}(t).$$

Hence, for $1 \leq p \leq r < \infty$,

$$\int_0^\infty \mathcal{R}_{1,2}\chi_E(s,t)^r\, ds = \int_0^\infty \chi_{(0,\varphi_2^*(t))}(s)\, ds = \varphi_2^*(t),$$

and

$$\|\mathcal{R}_{1,2}\chi_E\|_{L^p[L^r]} = \left(\int_0^\infty \varphi_2^*(t)^{\frac{p}{r}}\, dt \right)^{\frac{1}{p}} = \left\| \int_{\mathbb{R}} \chi_E(x, \cdot)\, dx \right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}}. \tag{30}$$

Similarly, recalling the definition of the distribution function given in (1),

$$\int_0^\infty \mathcal{R}_{2,1}\chi_E(s,t)^r\, ds = \int_0^\infty \chi_{(0,\varphi_1^*(s))}(t)\, ds = \left|\{s > 0 : \varphi_1^*(s) > t\}\right| = \lambda_{\varphi_1}(t),$$

and, using (3), we obtain

$$\|\mathcal{R}_{2,1}\chi_E\|_{L^p[L^r]} = \left(\int_0^\infty \lambda_{\varphi_1}(t)^{\frac{p}{r}}\,dt\right)^{\frac{1}{p}} = \left\|\int_{\mathbb{R}} \chi_E(\cdot,y)\,dy\right\|_{L^{\frac{r}{p},1}}^{\frac{1}{p}}. \tag{31}$$

Thus, from (30), (31), and (28) we finally get (i), if $r<\infty$. When $r=\infty$, we only need to combine Proposition 4.9 with (19) in Theorem 4.5.

To prove (ii), when $p<\infty$, we use (30), (31), and the reverse Minkowski's integral inequality (7), for the pair of indices $0<q=\frac{r}{p}\le 1$ and the Lorentz space $L^{q,1}$:

$$\begin{aligned}\|\mathcal{R}_{2,1}\chi_E\|_{L^p[L^r]} &= \left\|\int_{\mathbb{R}} \chi_E(\cdot,y)\,dy\right\|_{L^{q,1}}^{\frac{1}{p}}\\ &\ge \left(\int_{\mathbb{R}} \|\chi_E(\cdot,y)\|_{L^{q,1}}\,dy\right)^{\frac{1}{p}}\\ &= \left(\int_0^\infty \varphi_2^*(t)^{\frac{1}{q}}\,dt\right)^{\frac{1}{p}}\\ &= \|\mathcal{R}_{1,2}\chi_E\|_{L^p[L^r]}.\end{aligned}$$

As before, the case $p=\infty$ requires an extra argument (however this situation is much simpler because we are dealing with the exterior norm). Indeed, we first assume that E is a bounded set; i.e., for some $H>0$,

$$E\subset[-H,H]^2. \tag{32}$$

Then, the functions $g_1=\mathcal{R}_{1,2}\chi_E$ and $g_2=\mathcal{R}_{2,1}\chi_E$ vanish outside the square $[0,2H]^2$. Using (29), for $1\le r<q<\infty$, which we have just proved, we have that

$$\left(\int_0^{2H}\left(\int_0^{2H} g_1(s,t)^r ds\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} \le \left(\int_0^{2H}\left(\int_0^{2H} g_2(s,t)^r ds\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}.$$

Denote

$$\psi_j(t)=\left(\int_0^{2H} g_j(s,t)^r ds\right)^{\frac{1}{r}},\quad j=1,2.$$

Then

$$\left(\int_0^{2H}\psi_1(t)^q dt\right)^{\frac{1}{q}} \le \left(\int_0^{2H}\psi_2(t)^q dt\right)^{\frac{1}{q}}.$$

If we now let $q\to\infty$, then we obtain $\|\psi_1\|_\infty\le\|\psi_2\|_\infty$, as we wanted to show. The general case, without assuming (32), follows by the monotone convergence theorem (see [4, p. 302]). □

Remark 5.3. It is easy to see that Theorem 5.2 is sharp. In fact, let $0 < \gamma < 1$, $\beta_n = \sum_{k=1}^{n} \frac{1}{k^\gamma}$, $Q_n = [\beta_n, \beta_{n+1}] \times [\beta_n, \beta_{n+1}]$, and $E = \bigcup_{n=1}^{\infty} Q_n$. Define $f_\gamma = \chi_E$. Then

$$\|\mathcal{R}_{1,2} f_\gamma\|^p_{L^p[L^r]} = \sum_{n=1}^{\infty} \frac{1}{n^{\gamma(1+\frac{p}{r})}}$$

and

$$\|\mathcal{R}_{2,1} f_\gamma\|^p_{L^p[L^r]} = \sum_{n=1}^{\infty} \left(\frac{1}{n^\gamma} - \frac{1}{(n+1)^\gamma} \right) \beta_n^{\frac{p}{r}} \approx \sum_{n=1}^{\infty} \frac{1}{n^\lambda},$$

where $\lambda = \gamma + 1 + (\gamma - 1)\frac{p}{r}$.

If $1 \le r < p < \infty$, set $\gamma = \frac{p}{r+p}$. Then,

$$\|\mathcal{R}_{1,2} f_\gamma\|^p_{L^p[L^r]} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p}{r}}} < \infty \quad \text{and} \quad \|\mathcal{R}_{2,1} f_\gamma\|^p_{L^p[L^r]} \approx \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Conversely, if $1 \le p < r < \infty$, set $\gamma = \frac{r}{r+p}$. Then,

$$\|\mathcal{R}_{1,2} f_\gamma\|^p_{L^p[L^r]} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{and} \quad \|\mathcal{R}_{2,1} f_\gamma\|^p_{L^p[L^r]} \approx \sum_{n=1}^{\infty} \frac{1}{n^{2-\frac{p}{r}}} < \infty.$$

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