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Sobolev Embedding Theorem for Irregular Domains and Discontinuity of $p \to p^*(p,n)$

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Abstract. For a domain $\Omega \subset \mathbb{R}^n$ we denote

$$q_\Omega(p) := \sup\left\{r \in [1,\infty];\ \text{for all } f : \Omega \to \mathbb{R} : (f \in W^{1,p}(\Omega) \Rightarrow f \in L^r(\Omega))\right\}.$$

Let $p_0 \in [2,\infty)$. We construct a domain $\Omega \subset \mathbb{R}^2$ such that $q_\Omega(p)$ is discontinuous at p_0.

Keywords. Sobolev space, Sobolev embedding

Mathematics Subject Classification (2010). 46E35

1. Introduction

We study the Sobolev embedding theorem on irregular domains with non-Lipschitz boundary. The Sobolev embedding theorem on a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary claims

$$f \in W^{1,p}(\Omega),\ p \neq n \ \Rightarrow\ f \in L^{p^*(p)}(\Omega),\ \text{where}$$

$$p^*(p) = \begin{cases} \frac{np}{n-p}, & \text{for } 1 \leq p < n, \\ \infty, & \text{for } n < p < \infty. \end{cases} \tag{1}$$

Inspired by this theorem, we can define the optimal embedding exponent for a domain $\Omega \subset \mathbb{R}^n$ as

$$q_\Omega(p) := \sup\left\{r \in [1,\infty];\ \text{for all } f : \Omega \to \mathbb{R} : (f \in W^{1,p}(\Omega) \Rightarrow f \in L^r(\Omega))\right\}. \tag{2}$$

There are a lot of results in the field of characterization of $q_\Omega(p)$ for various classes of domains. For a Lipschitz domain Ω the function $p^*(p) = q_\Omega(p)$ is continuous and even smooth, (see (1)), this was proven by Sobolev in 1938 [12]. Later, embeddings ware examined on some more problematic classes of domains

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by V. G. Maz'ya [9,10], O. V. Besov and V. P. Il'in [3], T. Kilpeläinen and J. Malý [5], D. A. Labutin [6,7], B. V. Trushin [13,14] and others. For further results and motivation we recommend the introduction by O. V. Besov [2].

For any domain Ω, it holds that $p \leq q_\Omega(p) \leq p^*(p)$. The "nicer" the domain Ω is, the greater the function $q_\Omega(p)$ is. The greatest possible values of the embedding exponent are $q_\Omega(p) = p^*(p)$. Even considering domains, which are irregular in some sense, the exponent $q_\Omega(p)$ has always been continuous and in most cases even smooth. We construct a domain Ω such that the function of the optimal embedding $q_\Omega(p)$ is continuous up to some point, jumps at this point and then it is continuous again. The point of discontinuity $p_0 \in [n, \infty)$ and the size of the jump can be chosen as desired.

Our work is inspired by the construction of a domain in [4], but our proof is completely different. The original article shows the construction of such a domain only in case $p_0 = n = 2$ and the proof is based on change of variables. We prove the same result by chaining Poincaré inequalities and we generalize the construction for the point of discontinuity anywhere in $[n, \infty)$. This result can be generalized to any dimension too, but for simplicity we show the calculations only in case $n = 2$.

An explicit example of a domain with the point of discontinuity under the point of dimension, i.e. $p_0 \in (1, n)$ would be of interest.

Our proof will be as follows. We choose a domain Ω and verify a given embedding. Then we continue the proof that the embedding is optimal by counterexamples.

1.1. Construction of $\Omega_{\alpha,\beta}$ and the embedding. Firstly, we construct a domain $\Omega_{\alpha,\beta} \subset \mathbb{R}^2$ for parameters $\alpha \geq 1, \beta > \alpha$. The point of discontinuity of $q_{\Omega_{\alpha,\beta}}(p)$ is $p_0 = \alpha + 1$, parameter β determinates the size of the jump $\lim_{t\to p_0+} q_{\Omega_{\alpha,\beta}}(t) - \lim_{t\to p_0-} q_{\Omega_{\alpha,\beta}}(t)$.

Let us denote by T_i the family of domains in $\mathbb{R}^2$

$$T_i := \left\{ [x_1, x_2] \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x_1 \in \left(-2^{-i^2} i^{-1}, (-2^{-i^2} + 2^{-i}) i^{-1} \right), \\ x_2 \in \left(2^{-i+1}, 2^{-i+1} + (x_1 + 2^{-i^2} i^{-1})^\alpha 2^{-i(\beta-\alpha)} i^{-1+\alpha} \right) \end{array} \right\} \tag{3}$$

The shape of T_i is the subgraph of $y(x) = x^\alpha$ function on some right neighbourhood of 0. By S we denote $S := (-4, 0) \times (-2, 2)$. Now we define

$$\Omega_{\alpha,\beta} := \bigcup_{i\in\mathbb{N}} T_i \cup S.$$

We define $q_{\Omega_{\alpha,\beta}}(p) : [1, \beta + 1) \to [1, \infty)$ by

$$q_{\Omega_{\alpha,\beta}}(p) := \begin{cases} p & \text{for } 1 \leq p < \alpha + 1, \\ \frac{(\beta+1)p}{\beta+1-p} & \text{for } \alpha + 1 \leq p < \beta + 1. \end{cases}$$

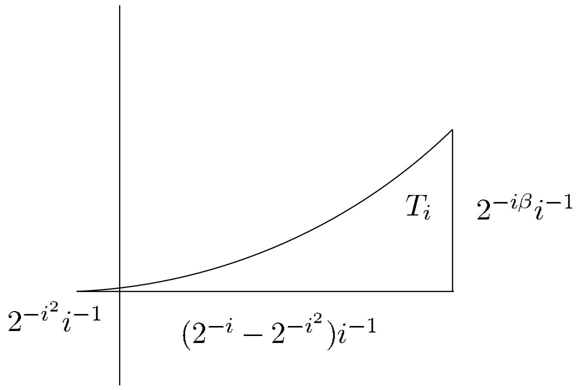

Figure 1: The domain T_i

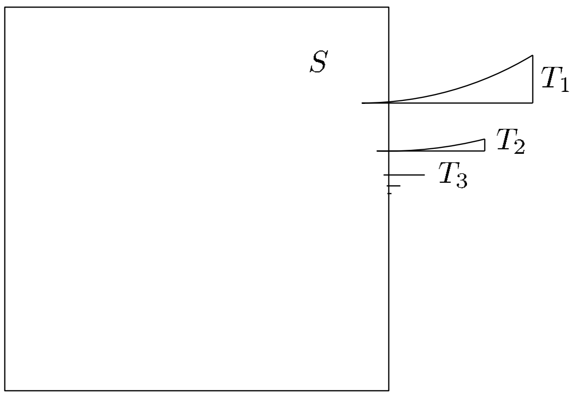

Figure 2: The domain $\Omega_{\alpha,\beta}$

The function $q_{\Omega_{\alpha,\beta}}(p)$ has a jump at $p_0 = \alpha + 1$ of size

$$\lim_{t\to p_0+} q_{\Omega_{\alpha,\beta}}(t) - \lim_{t\to p_0-} q_{\Omega_{\alpha,\beta}}(t) = \frac{(\alpha+1)^2}{\beta-\alpha}.$$

Theorem 1.1 (Optimal Sobolev embedding Theorem for $\Omega_{\alpha,\beta}$)**.** *Let* $\alpha \geq 1$, $\beta > \alpha$ *and* $1 \leq p < 1+\beta$, $p \neq \alpha + 1$. *Then*

$$W^{1,p}(\Omega_{\alpha,\beta}) \subset L^{q_{\Omega_{\alpha,\beta}}(p)}(\Omega_{\alpha,\beta}).$$

Moreover, for every $q(p) > q_{\Omega_{\alpha,\beta}}(p)$ *there exists a function* $g : \Omega_{\alpha,\beta} \to \mathbb{R}$ *satisfying*

$$g \in W^{1,p}(\Omega_{\alpha,\beta}) \quad \textit{and} \quad g \notin L^{q(p)}(\Omega_{\alpha,\beta}).$$

We prove the first part of Theorem 1.1 in Section 3. The optimality part of Theorem 1.1 is proven in Section 4.

Remark 1.2 (Embedding for $p=\alpha+1$)**.** We do not formulate previous Theorem for case $p=\alpha+1$. However, if we use supremum definition (2) instead of $q(p)=\max\{r\colon r>1, W^{1,p}\subset L^r\}$, then the Theorem would be valid even in case $p=\alpha+1$ and it can be proven by the same means described in Section 3.

We decided to exclude the case $p=\alpha+1$, so we show that the maximum and the supremum definition of $q(p)$ are equivalent for all cases $p\neq\alpha+1$. The discontinuity of $q(p)$ is clear irrespective of precise value at this point and of the choice of the maximum or the supremum. We do not answer the question if $L^{q_{\Omega_{\alpha,\beta}}(\alpha+1)}\subset W^{1,\alpha+1}$ holds, we suppose that the answer is no.

2. Preliminaries

For simplicity we use the notation $\Omega=\Omega_{\alpha,\beta}$ and $q(p)=q_{\Omega_{\alpha,\beta}}(p)$. By C we denote a generic positive constant whose exact value may change at each occurrence. We write for example $C(a,b,c)$ if C may depend on parameters a,b and c.

We use standard notation for weak derivatives and Lebesgue and Sobolev spaces. We denote the Sobolev norm $\|f\|_{W^{1,p}(\Omega)}$ for the function $f\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}$, $p\in[1,\infty]$ as

$$\|f\|_{W^{1,p}(\Omega)}=\begin{cases}\left(\|f\|^p_{L^p(\Omega)}+\sum_{i=1}^n\|D_if\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & \text{for } p\in[1,\infty)\\ \max\{\|f\|_{L^p(\Omega)},\|D_1f\|_{L^p(\Omega)},\dots,\|D_nf\|_{L^p(\Omega)}\} & \text{for } p=\infty.\end{cases}\tag{4}$$

We denote the Sobolev space $W^{1,p}(\Omega)$ as the set of all functions with finite norm $\|f\|_{W^{1,p}(\Omega)}$.

We use the notation $a_i\simeq b_i$, if there exists a constant $K>0$ such that

$$\frac{1}{K}<\frac{a_i}{b_i}<K\quad\text{for every } i\in\mathbb{N}.$$

We denote the integral average by

$$f_A:=\fint_A f=\frac{1}{|A|}\int_A f(x)\,\mathrm{d}x.$$

The following Poincaré-type inequality will be essential.

Lemma 2.1. *Let $b: B(0,r)\subset\mathbb{R}^n\to\mathbb{R}^n$ be a bi-Lipschitz mapping with a bi-Lipschitz constant $L>0$, and $A=b(B(0,r))$. Let $1\le p\le\infty$, $p\neq n$ and $1\le m\le p^*(p)$. Then there exists a constant $C(n,p,m,L)$ such that for $f\in W^{1,p}(A)$ we have*

$$|A|^{-\frac{1}{m}}\|f-f_A\|_{L^m(A)}\le C(n,p,m,L)r|A|^{-\frac{1}{p}}\|Df\|_{L^p(A)}.$$

We use the convention $|A|^{-\frac{1}{\infty}} = 1$.

Let $p = n$ *and* $1 \leq m < \infty$. *Then there exists a constant* $C(n, m, L)$, *such that for* $f \in W^{1,p}(A)$ *it holds*

$$|A|^{-\frac{1}{m}} \|f - f_A\|_{L^m(A)} \leq C(n, m, L) r |A|^{-\frac{1}{n}} \|Df\|_{L^n(A)}.$$

Proof. In case b is the identity mapping and $p = q$ we get the classical result. The more difficult case $1 \leq q \leq p^*(p)$ follows from [8] as Theorem 12.23 and Exercise 12.24 and by applying Hölder's inequality. The general case where b is not the identity follows from a simple change of variables. □

3. The proof of Sobolev embedding Theorem for $\Omega_{\alpha,\beta}$

In this section we prove the embedding part of Theorem 1.1 for the case $\alpha \geq 1$. We give the details for $\alpha > 1$ and the case $\alpha = 1$ is only sketched.

Let us suppose that $\alpha > 1$. Then for every $i \in \mathbb{N}$ we define the covering of $T_i \setminus S$ by domains, which are bi-Lipschitz equivalent to balls. The proof of $W^{1,p} \subset L^{q(p)}$ for $p < \alpha + 1$ is elementary and follows from (4), as every function in $W^{1,p}$ belongs to L^p. Further we suppose that $\beta + 1 > p > \alpha + 1$.

3.1. Covering of T_i. We define $k_\alpha = \frac{3}{2}\left(\frac{2}{\alpha-1}\right)^{\frac{1}{\alpha-1}}$,

$$s_{i,j} := k_\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j^{-\frac{1}{\alpha-1}}, \quad r_{i,j} := \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j^{-\frac{\alpha}{\alpha-1}}. \tag{5}$$

For fixed $i \in \mathbb{N}$ we define the sequence of domains $Q_{i,j}$, $j \in \mathbb{N}$

$$Q_{i,j} := \left\{ [x_1, x_2] \in T_i \;\middle|\; \begin{aligned} x_1 + 2^{-i^2} i^{-1} \in \big(s_{i,j} - r_{i,j}, s_{i,j} + r_{i,j}\big) \\ \cap \big(-2^{-i^2} i^{-1}, (-2^{-i^2} + 2^{-i}) i^{-1}\big) \end{aligned} \right\}. \tag{6}$$

Lemma 3.1 (Covering lemma). *Let* $i \in \mathbb{N}$, T_i *be given by* (3) *and the sequence of domains* $Q_{i,j}$ *by* (6). *Then*

(i) $Q_{i,j}$ *are bi-Lipschitz equivalent to balls with radius* $r_{i,j}$ *with the same bi-Lipschitz constant* L *independent of* i *and* j.

(ii) *For fixed* j_0 *there exists only a finite number of domains* $Q_{i,j}$ *with non-empty intersection with* Q_{i,j_0}. *This number is bounded by some constant* $C(\alpha, \beta)$.

(iii) *For fixed* j_0 *let* $A_{i,j_0} := Q_{i,j_0} \cap Q_{i,j_0+1}$. *There exists some positive constant* $C(\alpha, \beta)$ *such that* $C(\alpha, \beta) < \frac{|A_{i,j_0}|}{|Q_{i,j_0}|} < \frac{|A_{i,j_0}|}{|Q_{i,j_0+1}|}$.

(iv) *There exists a smallest index* $j_{i,\infty}$ *satisfying* $Q_{i,j_{i,\infty}} \subset S$, *and there exists a biggest index* $j_{i,0}$ *satisfying* $s_{i,j_{i,0}} + r_{i,j_{i,0}} \geq (-2^{-i^2} + 2^{-i}) i^{-1} =$ *"height of* T_i*". Estimated values are*

$$j_{i,\infty} \simeq 2^{i(\beta-\alpha)+i^2(\alpha-1)}, \quad j_{i,0} \simeq 2^{i(\beta-1)}. \tag{7}$$

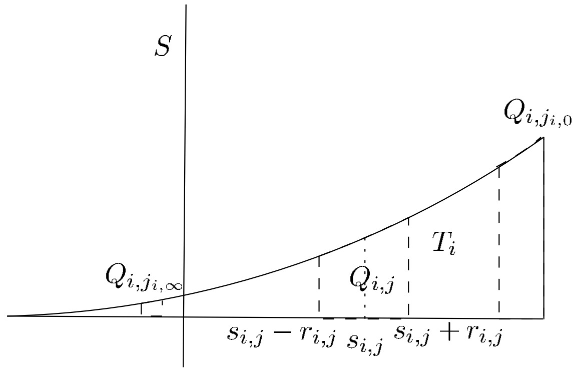

Figure 3: The covering of T_i

The proof is rather technical but straightforward and can be done by basic calculus, therefore we only outline it.

Sketch of the proof of Lemma 3.1. We define two bi-Lipschitz mappings:

$$\begin{aligned}
&b_{1,i,j} : B(0, r_{i,j}) \to (-r_{i,j}, r_{i,j}) \times (0, r_{i,j}),\\
&b_{2,i,j} : (-r_{i,j}, r_{i,j}) \times (0, r_{i,j}) \to Q_{i,j},\\
b_{2,i,j}(x_1, x_2) := &\left(x_1 + s_{i,j}, 2^{-i+1} + \frac{x_2}{r_{i,j}} (x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha} 2^{-i(\beta-\alpha)} i^{-1+\alpha} \right).
\end{aligned}$$

The mapping $b_{1,i,j}$ maps a ball to half a square and has bi-Lipschitz constant L_1 independent of i and j, its exact formula can be found easily. Let us consider Jacobi matrices of both $b_{2,i,j}$ and $b_{2,i,j}^{-1}$

$$D_{b_{2,i,j}}(x_1, x_2) = \begin{pmatrix} 1 & \frac{x_2}{r_{i,j}} \alpha (x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha-1} 2^{-i(\beta-\alpha)} i^{-1+\alpha} \\ 0 & r_{i,j}^{-1} (x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha} 2^{-i(\beta-\alpha)} i^{-1+\alpha} \end{pmatrix}$$

$$D_{b_{2,i,j}^{-1}}(b_{2,i,j}(x_1, x_2)) = \begin{pmatrix} 1 & -x_2 \alpha (x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{-1} \\ 0 & r_{i,j} (x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{-\alpha} 2^{i(\beta-\alpha)} i^{1-\alpha} \end{pmatrix}.$$

By a direct computation it is not difficult to check that all partial derivatives are bounded by a constant, i.e. the second mapping $b_{2,i,j}$ has bi-Lipschitz constant $L_{2,i,j}$ dependent on i and j, and it can be estimated by an same L_2 common for all i and j. The key observation is, that $L_{2,i,j}$ is a monotone sequence in both i and j. We have found a bi-Lipschitz mappings $b_{2,i,j} \circ b_{1,i,j} : B(0, r_{i,j}) \to Q_{i,j}$ with constant $L = L_1 L_2$ and the first part is proven.

The second part can be proven by verifying that the statement $\lim_{j\to\infty}(s_{i,j} - s_{i,j+1} - r_{i,j}) = 0$ holds true for every $i \in \mathbb{N}$.

To prove the third part we define $P_{i,j} \subset A_{i,j}$,

$$P_{i,j} := (s_{i,j+1}, s_{i,j}) \times (2^{-i+1}, 2^{-i+1} + r_{i,j+1}).$$

We estimate $\frac{|P_{i,j}|}{|Q_{i,j}|}$ and we easily find $C(\alpha,\beta)$ such that $C(\alpha,\beta) < \frac{|P_{i,j}|}{|Q_{i,j}|}$.

The fourth part is important for further calculations. We estimate the indices $j_{i,0}$ and $j_{i,\infty}$ by the definition of $r_{i,j}$ (5). We observe that $\operatorname{diam}(Q_{i,j_{i,\infty}}) \simeq$ "height of $T_i \setminus S$ on the left edge" and $\operatorname{diam}(Q_{i,j_{i,\infty}}) \simeq r_{i,j_{i,\infty}}$ for $j_{i,\infty}$ and that $\operatorname{diam}(Q_{i,j_{i,0}}) \simeq$ "height of $T_i \setminus S$ on the right edge" and $\operatorname{diam}(Q_{i,j_{i,0}}) \simeq r_{i,j_{i,0}}$ for $j_{i,0}$. By this observation we have

$$\begin{aligned} 2^{-i\beta} i^{-1} &\simeq r_{i,j_{i,0}} \simeq \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j_{i,0}^{-\frac{\alpha}{\alpha-1}}, \\ (2^{-i^2} i^{-1})^\alpha 2^{-i(\beta-\alpha)} i^{-1+\alpha} &\simeq r_{i,j_{i,\infty}} \simeq \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j_{i,\infty}^{-\frac{\alpha}{\alpha-1}} \end{aligned} \tag{8}$$

which implies (7). □

3.2. Proof of Theorem 1.1 for $\beta+1 > p > \alpha+1$, $\alpha \ge 1$. We denote $q = q(p) = q_{\Omega_{\alpha,\beta}}(p)$. We estimate the power of the norm

$$\|f\|^q_{L^q(\Omega)} \le \|f\|^q_{L^q(S)} + \sum_{i\in\mathbb{N}} \|f\|^q_{L^q(T_i\setminus S)} \le \|f\|^q_{L^q(S)} + \sum_{i\in\mathbb{N}} \sum_{j=j_{i,0}}^{j_{i,\infty}} \|f\|^q_{L^q(Q_{i,j})}.$$

The part $\|f\|^q_{L^q(S)}$ is bounded for any $q \in [1,\infty)$ thanks to Sobolev embedding theorem for Lipschitz domains $W^{1,p}(S) \subset L^\infty(S), p > n = 2$. Therefore we have $\|f\|^q_{L^q(S)} \le C$ and also $|f_{Q_{i,j_{i\infty}}}| < C$. We estimate

$$\begin{aligned} &\|f\|^q_{L^q(\Omega)} \\ &\le \|f\|^q_{L^q(S)} + \sum_{i=1}^\infty \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(|f(x) - f_{Q_{i,j}}| + \sum_{k=j+1}^{j_{i,\infty}} |f_{Q_{i,k}} - f_{Q_{i,k-1}}| + C \Big)^q \, \mathrm{d}x \\ &\le C + C \sum_{i=1}^\infty \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \sum_{k=j+1}^{j_{i,\infty}} \Big(|f_{Q_{i,k}} - f_{Q_{i,k-1}}| \Big)^q \, \mathrm{d}x \\ &\quad + C \sum_{i=1}^\infty \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(|f(x) - f_{Q_{i,j}}| \Big)^q \, \mathrm{d}x. \end{aligned} \tag{9}$$

By (3), Lemma 2.1 for $m = \infty$, Lemma 3.1(ii) and $r_{i,j} \le 1$ we have

$$\begin{aligned} \sum_{i=1}^\infty \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(|f(x) - f_{Q_{i,j}}| \Big)^q \, \mathrm{d}x &\le \sum_{i=1}^\infty \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(\|f(x) - f_{Q_{i,j}}\|_{L^\infty(Q_{i,j})} \Big)^q \, \mathrm{d}x \\ &\le \sum_{i=1}^\infty \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(C r_{i,j}^{\frac{p-2}{p}} \|Df\|_{W^{1,p}(Q_{i,j})} \Big)^q \, \mathrm{d}x \\ &\le C \sum_{i=1}^\infty |T_i| \|Df\|^q_{W^{1,p}(\Omega)} \\ &\le C \end{aligned} \tag{10}$$

By Lemma 2.1 and Lemma 3.1(iii) we have the following estimate

$$\begin{aligned}|f_{Q_{i,j}}-f_{Q_{i,j-1}}| &\leq \left(\fint_{A_{i,j-1}}|f_{Q_{i,j}}-f(y)|\,\mathrm{d}y+\fint_{A_{i,j-1}}|f_{Q_{i,j-1}}-f(y)|\,\mathrm{d}y\right)\\ &\leq C\left(\fint_{Q_{i,j}}|f_{Q_{i,j-1}}-f(y)|\,\mathrm{d}y+\fint_{Q_{i,j-1}}|f_{Q_{i,j-1}}-f(y)|\,\mathrm{d}y\right)\\ &\leq C\left(r_{i,j}^{\frac{p-2}{p}}\left(\int_{Q_{i,j}}|Df(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}+r_{i,j-1}^{\frac{p-2}{p}}\left(\int_{Q_{i,j-1}}|Df(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\right).\end{aligned}$$

By this estimate and the Hölder inequality for sums and Lemma 3.1(ii) we get

$$\begin{aligned}&\sum_{j=j_{i,0}}^{j_{i,\infty}}\int_{Q_{i,j}}\left(\sum_{k=j+1}^{j_{i,\infty}}|f_{Q_{i,k}}-f_{Q_{i,k-1}}|\right)^q\,\mathrm{d}x\\ &\leq C\sum_{j=j_{i,0}}^{j_{i,\infty}}\int_{Q_{i,j}}\left(\sum_{k=j}^{j_{i,\infty}}r_{i,k}^{\frac{p-2}{p}}\left(\int_{Q_{i,k}}|Df(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\right)^q\,\mathrm{d}x\\ &\leq C|T_i|\left(\sum_{k=j_{i,0}}^{j_{i,\infty}}r_{i,k}^{\frac{p-2}{p}}\left(\int_{Q_{i,k}}|Df(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\right)^q \qquad (11)\\ &\leq C|T_i|\left(\sum_{k=j_{i,0}}^{j_{i,\infty}}r_{i,k}^{\frac{(p-2)p}{p(p-1)}}\right)^{\frac{q(p-1)}{p}}\left(\sum_{k=j_{i,0}}^{j_{i,\infty}}\left(\int_{Q_{i,k}}|Df(y)|^p\,\mathrm{d}y\right)^{p\frac{1}{p}}\right)^{\frac{q}{p}}\\ &\leq C|T_i|\left(\sum_{k=j_{i,0}}^{j_{i,\infty}}r_{i,k}^{\frac{p-2}{p-1}}\right)^{\frac{q(p-1)}{p}}.\end{aligned}$$

From (5), (9), (10), (11) and (3) we have

$$\begin{aligned}\|f\|_{L^q(\Omega)}^q &\leq C+C\sum_{i=1}^{\infty}i^{-2}2^{-i(\beta+1)}\left(\sum_{j=j_{i,0}}^{j_{i,\infty}}r_{i,j}^{\frac{p-2}{p-1}}\right)^{\frac{q(p-1)}{p}}\\ &\leq C+C\sum_{i=1}^{\infty}i^{-\frac{qp-2q+2p}{p}}2^{i\left(\frac{(\beta-\alpha)q(p-2)}{(\alpha-1)p}-(\beta+1)\right)}\left(\sum_{j=j_{i,0}}^{j_{i,\infty}}j^{-\frac{\alpha(p-2)}{(\alpha-1)(p-1)}}\right)^{\frac{q(p-1)}{p}}.\end{aligned}$$

We estimate the sum over j and by $p>\alpha+1$ and (7) we get

$$\sum_{j=j_{i,0}}^{j_{i,\infty}}j^{-\frac{\alpha(p-2)}{(\alpha-1)(p-1)}}\leq C2^{i\frac{(\beta-1)(\alpha+1-p)}{(\alpha-1)(p-1)}}.$$

Finally we put the estimates together and we get

$$\begin{aligned}\|f\|^q_{L^q(\Omega)} &\le C + C\sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}} 2^{i\left(\frac{(\beta-\alpha)q(p-2)}{(\alpha-1)p}-(\beta+1)\right)} \left(2^{i\frac{(\beta-1)(\alpha+1-p)}{(\alpha-1)(p-1)}}\right)^{\frac{q(p-1)}{p}} \\ &\le C + C\sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}} 2^{i\left(-(\beta+1)+q\frac{\beta+1-p}{p}\right)}.\end{aligned}$$

The proof is complete, because the sum is finite if $q \le \frac{(\beta+1)p}{\beta+1-p}$.

Let us consider the case $\alpha = 1$. We have to change the definition (5) of $s_{i,j}$ and $r_{i,j}$ and the definition (6) of $Q_{i,j}$ as follows,

$$r_{i,j} := r_{i,0}(1+2^{-i(\beta-1)-1})^j, \text{ for } r_{i,0} = 2^{-i^2-i(\beta-1)-1} i^{-1} \quad \text{and} \quad s_{i,j} := \sum_{k=0}^{j-1} r_{i,j}.$$

We define $Q_{i,j}$ as trapezoids which have the average of basis equal to their width. We denote half of its width by $r_{i,j}$, that is

$$Q_{i,j} = T_i \cap \{x \in \mathbb{R}^2 : x_2 \in (s_{i,j} - r_{i,j}, s_{i,j} + r_{i,j})\}.$$

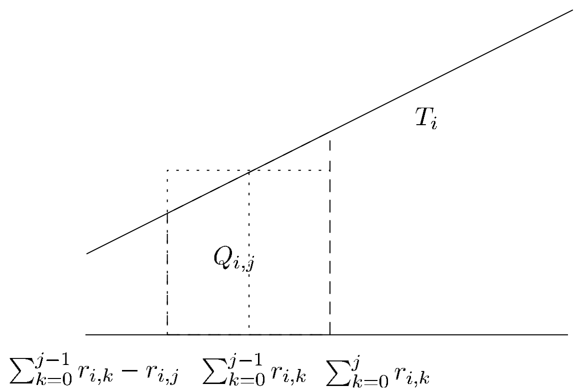

Figure 4: The domain $Q_{i,j}$

Let us denote, that the sequences $r_{i,j}$ and $s_{i,j}$ are strictly decreasing with respect to index j in case $\alpha > 1$, but these sequences are strictly increasing in case $\alpha = 1$.

The Lemma 3.1 holds and is proven in the same way as for $\alpha > 1$, only the indices of border $Q_{i,j}$ are $j_{i,\infty} = -1$ and analogously to (8)

$$2^{-i\beta} i^{-1} \simeq r_{i,j_{i,0}} = (1 + 2^{-i(\beta-1)-1})^{j_{i,0}} 2^{-i^2-i(\beta-1)-1} i^{-1}$$

we get

$$j_{i,0} \simeq \frac{\ln(2)(i^2 - i)}{\ln(1 + 2^{-i(\beta-1)-1})}.$$

The idea of chaining the Poincaré inequality is analogous, and after an easy modification we get our result. We can copy all arguments and calculations from (9), (10), (11), then we use (3) for $\alpha = 1$, the new definition of $r_{i,j}$ and the previous estimates for $j_{i,0}, j_{i,\infty}$ and we get

$$
\begin{aligned}
\|f\|^q_{L^q(\Omega)} &\leq C + C\sum_{i=1}^{\infty} i^{-2} 2^{-i(\beta+1)} \left(\sum_{j=j_{i,\infty}}^{j_{i,0}} r_{i,j}^{\frac{p-2}{p-1}} \right)^{\frac{q(p-1)}{p}} \\
&\leq C + C\sum_{i=1}^{\infty} i^{-\frac{2p+q(p-2)}{p}} 2^{-i(1+\beta)+(-i^2-i(\beta-1))\frac{q(p-2)}{p}} \\
&\quad \times \left(\frac{(1+2^{-i(\beta-1)-1})^{\frac{p-2}{p-1}(\frac{\ln(2)(i^2-i)}{\ln(1+2^{-i(\beta-1)-1})}+2)} - 1}{(1+2^{-i(\beta-1)-1})^{\frac{p-2}{p-1}} - 1} \right)^{\frac{q(p-1)}{p}},
\end{aligned}
$$

where the final term comes from the sum of geometric series. The right hand side can be estimated and after an easy calculation we have

$$
\|f\|^q_{L^q(\Omega)} \leq C + C\sum_{i=1}^{\infty} i^{-\frac{2p+q(p-2)}{p}} 2^{i\frac{q(\beta+1-p)-(\beta+1)p}{p}}.
$$

The right hand side is finite if $q \leq \frac{p(\beta+1)}{\beta+1-p}$ and the proof is complete.

The complete proof for $\alpha = 1$ with all details can be found in [11].

4. Proof of the optimality of $q(p)$ for $\Omega_{\alpha,\beta}$

We fix $q > q(p)$. We define a function g by the choice of proper functions $g_i : T_i \to \mathbb{R}$ and the sequence d_i of positive numbers. We define

$$
g(x_1, x_2) = \begin{cases} 0 & \text{for } (x_1, x_2) \in S, \\ d_i g_i(x_1, x_2) & \text{for } (x_1, x_2) \in T_i \setminus S, \quad \text{for all } i \in \mathbb{N}. \end{cases}
$$

Clearly

$$
\|g\|^p_{W^{1,p}(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{W^{1,p}(T_i)} \text{ and } \|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)}. \tag{12}
$$

The choice of g_i and d_i depends on p and $\alpha + 1$, so we split the proof into two parts.

4.1. The case $p < \alpha + 1$. Let us consider $p \in [1, \alpha+1)$. We define

$$\begin{aligned} g_i(x_1,x_2) &:= \Big(x_1 + 2^{-i^2} i^{-1}\Big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha} \text{ for } (x_1,x_2) \in T_i \setminus S, \\ d_i &:= (2^{-i^2} i^{-1})^{\alpha} 2^{\frac{i(\beta+1)}{q}} i^{\frac{2}{q}}. \end{aligned} \tag{13}$$

For fixed $i \in \mathbb{N}$ we estimate the norm in the space $L^q(T_i)$. By (3) the height of T_i for $x_1 \in \big(-i^{-1}2^{-i^2}, i^{-1}(2^{-i} - 2^{-i^2})\big)$ is

$$l(x_1) = \Big(x_1 + 2^{-i^2} i^{-1}\Big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1} \tag{14}$$

and we get

$$\begin{aligned} &\|g_i\|^q_{L^q(T_i)} \\ &= \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \Big|\big(x_1 + 2^{-i^2} i^{-1}\big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha}\Big|^q l(x_1)\,\mathrm{d}x_1 \\ &= \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \Big|\big(x_1 + 2^{-i^2} i^{-1}\big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha}\Big|^q \big(x_1 + 2^{-i^2} i^{-1}\big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1}\,\mathrm{d}x_1. \end{aligned} \tag{15}$$

We want to estimate the size of the integral. The integrand is positive and concave, so we can estimate its value by its maximum. More precisely, we replace the integrand by constant function $(2^{-i^2} i^{-1})^{-\alpha q} \big(2^{-i} i^{-1}\big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1}$, hence

$$\begin{aligned} \|g_i\|^q_{L^q(T_i)} &\simeq \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} (2^{-i^2} i^{-1})^{-\alpha q} \big(2^{-i} i^{-1}\big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1}\,\mathrm{d}x_1 \\ &\simeq i^{\alpha q - 2} 2^{i^2 \alpha q - i - i\beta}. \end{aligned} \tag{16}$$

By (13) we get

$$\|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)} \geq C \sum_{i=1}^{\infty} i^0 2^0 = \infty.$$

We need to prove the convergence of $\|g\|^p_{W^{1,p}(\Omega_\alpha)}$. First of all we estimate

$$\|g_i\|^p_{W^{1,p}(T_i)} \leq 2\max\{\|g_i\|^p_{L^p(T_i)}, \|Dg_i\|^p_{L^p(T_i)}\}.$$

The estimate of the norm of g_i in $L^p(T_i)$ is analogous to (16), by replacing p by q we get $\|g_i\|^p_{L^p(T_i)} \simeq i^{\alpha p - 2} 2^{i^2 \alpha p - i - i\beta}$. We use $q > q(p) = p$ and we estimate the norm of g in $L^p(\Omega)$

$$\|g\|^p_{L^p(\Omega)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{L^p(T_i)} \leq C \sum_{i=1}^{\infty} i^{-2} (2^{-i-i\beta})^{\frac{q-p}{q}} < \infty.$$

We express the norm of g_i by a derivative $\|Dg_i(x_1,x_2)\|^p_{L^p(T_i)} = \int_{T_i}\left|\frac{\partial g_i(x_1,x_2)}{\partial x_1}\right|^p \mathrm{d}x$. The estimate is similar to (15). The proof splits in two cases. Firstly, we consider $p>1$ and we get

$$\begin{aligned}\int_{T_i}\left|\frac{\partial g_i}{\partial x_1}\right|^p \mathrm{d}x &\le C\int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left(x_1+2^{-i^2}i^{-1}\right)^{\alpha+p(-\alpha-1)}2^{i(-\beta+\alpha)}i^{\alpha-1}\,\mathrm{d}x_1\\ &\le C\,2^{i(-\beta+\alpha)}i^{\alpha-1}\Big[\left(x_1+2^{-i^2}i^{-1}\right)^{(1-p)(\alpha+1)}\Big]_0^{i^{-1}(2^{-i}-2^{-i^2})}\\ &\le C\,2^{i(-\beta+\alpha)}i^{\alpha-1}(2^{-i^2}i^{-1})^{(1-p)(\alpha+1)}.\end{aligned}\tag{17}$$

It follows that

$$\|Dg\|^p_{L^p(\Omega)}=\sum_{i=1}^{\infty}d_i^p\|Dg_i\|^p_{L^p(T_i)}\le\sum_{i=1}^{\infty}Ci^{2\frac{p-q}{q}+p(\alpha+1)}2^{i(-\beta+\alpha+\frac{(\beta+1)p}{q})}2^{i^2(p-\alpha-1)}<\infty.$$

The proof of the finiteness of the norm in case $p=1$ is similar, except the estimate in (17) involves $\int(x_1+C)^{-1}\,\mathrm{d}x_1=\log|x_1+C|$. It is easy to finish the proof in this case, too.

4.2. The case $p>\alpha+1$. We define

$$\begin{aligned}g_i(x_1,x_2)&:=\left(x_1+2^{-i^2}i^{-1}\right)^{\alpha}-(2^{-i^2}i^{-1})^{\alpha}\text{ for }(x_1,x_2)\in T_i\setminus S,\\ d_i&:=2^{i\left(\frac{\beta+1}{q}+\alpha\right)}i^{\alpha+\frac{2}{q}}.\end{aligned}$$

We use (12), (14) and we estimate the norms of g_i as in the previous case. Analogously to (15) and (16) we have

$$\begin{aligned}\|g_i\|^q_{L^q(T_i)}&=\int_0^{\frac{2^{-i}-2^{-i^2}}{i}}\left|\left(x_1+2^{-i^2}i^{-1}\right)^{\alpha}-(2^{-i^2}i^{-1})^{\alpha}\right|^q l(x_1)\,\mathrm{d}x_1\\ &\simeq\int_0^{\frac{2^{-i}-2^{-i^2}}{i}}\left(x_1+2^{-i^2}i^{-1}\right)^{(q+1)\alpha}2^{i\alpha-i\beta}i^{\alpha-1}\,\mathrm{d}x_1\\ &\simeq i^{-\alpha q-2}2^{-i(q\alpha+1+\beta)}.\end{aligned}\tag{18}$$

We estimate $\|g\|^q_{L^q(\Omega)}$ by

$$\|g\|^q_{L^q(\Omega)}=\sum_{i=1}^{\infty}d_i^q\|g_i\|^q_{L^q(T_i)}\ge C\sum_{i=1}^{\infty}i^0 2^0=\infty.$$

Now we need to prove the convergence of the norms of g and Dg in $L^p(\Omega)$. Analogously to (18), by replacing the position of q by p we get

$\|g_i\|^p_{L^p(T_i)} \simeq i^{-\alpha p-2}2^{-i(p\alpha+1+\beta)}$. We use $q > q(p) > p$ and we estimate the norm of g in $L^p(\Omega)$

$$\|g\|^p_{L^p(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{L^p(T_i)} \le C \sum_{i=1}^{\infty} (i^{-2}2^{-i-i\beta})^{\frac{q-p}{q}} < \infty.$$

Let us express the norm of g_i by a derivative and we estimate

$$\begin{aligned}\int_{T_i} \left|\frac{\partial g_i}{\partial x_1}\right|^p \mathrm{d}x &\le C \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left(x_1 + 2^{-i^2} i^{-1}\right)^{(\alpha-1)p+\alpha} 2^{i(-\beta+\alpha)} i^{\alpha-1}\,\mathrm{d}x_1 \\ &\le C\, 2^{i(-\beta+\alpha)} i^{\alpha-1} \Big[\left(x_1 + 2^{-i^2} i^{-1}\right)^{\alpha p - p + \alpha + 1}\Big]_0^{\frac{2^{-i}-2^{-i^2}}{i}} \\ &\le C\, i^{p(-\alpha+1)-2} 2^{i(-\alpha p + p - \beta - 1)}.\end{aligned}$$

It follows that

$$\|Dg\|^p_{L^p(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|Dg_i\|^p_{L^p(T_i)} \le \sum_{i=1}^{\infty} C i^{p-2\frac{q-p}{q}} 2^{i\left(p+(\beta+1)\frac{p-q}{q}\right)} < \infty,$$

where the finiteness follows from $q > q(p) = \frac{(\beta+1)p}{\beta+1-p}$.

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References

[1] Adams, R. A. and Fournier, J. J. F., *Sobolev Spaces* (Second edition). Pure Appl. Math. (Amst.) 140. Amsterdam: Elsevier 2003.

[2] Besov, O. V., Sobolev's Embedding theorem for anisotropically irregural domains. *Eurasian Math. J.* 2 (2011)(1), 32 – 51.

[3] Besov, O. V., Il'in, V. P. and Nikolskii, S. M., *Integral Representations of Functions and Imbedding Theorems. Vols. I, II.* New York: Wiley 1978, 1979.

[4] Gol'dshtein, V. and Gurov, L., Applications of change of variables operators for exact embeddings theorems. *Integral Equ. Oper. Theory* 19 (1994)(1), 1 – 24.

[5] Kilpeläinen, T. and Malý, J., Sobolev inequalities on sets with irregular boundaries. *Z. Anal. Anwend.* 19 (2000)(2), 369 – 380.

[6] Labutin, D. A., Embedding of Sobolev spaces on Hölder domains. *Proc. Steklov Inst. Math.* 227 (1999), 163 – 172.

[7] Labutin, D. A., Sharpness of Sobolev inequalities for a class of irregural domains. *Proc. Steklov Inst. Math.* 232 (2001), 211 – 215.

[8] Leoni, G., *First Course in Sobolev Spaces.* Grad. Stud. Math. 105. Providence (RI): Amer. Math. Soc. 2009.

[9] Maz'ya, V. G., *Sobolev Spaces.* Berlin: Springer 1985.

[10] Maz'ya, V. G. and Poborchi, S. V., *Differentiable Functions on Bad Domains.* River Edge (NJ): World Scientific 1997.

[11] Roskovec, T., Sobolev embedding theorem on domains without Lipschitz boundary (in Czech). Master Thesis. Fac. Math. Physics, Charles Univ. in Prague 2012 (avialable at http://www.karlin.mff.cuni.cz/∼roskovec/mas.pdf).

[12] Sobolev, S. L., *Some Applications of Functional Analysis in Mathematical Physics.* Providence (RI): Amer. Math. Soc. 1991.

[13] Trushin, B. V., Sobolev embedding theorems for a class of anisotropic irregular domains. *Proc. Steklov Inst. Math.* 260 (2008), 287 – 309.

[14] Trushin, B. V., Continuity of embeddings of weighted Sobolev spaces in Lebesgue spaces on anisotropically irregular domains. *Proc. Steklov Inst. Math.* 269 (2010), 265 – 283.

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