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An Existence Result for Fractional Kirchhoff-Type Equations

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Dedicated to Anna and Sandro

Abstract. The aim of this paper is to study a class of nonlocal fractional Laplacian equations of Kirchhoff-type. More precisely, by using an appropriate analytical context on fractional Sobolev spaces, we establish the existence of one non-trivial weak solution for nonlocal fractional problems exploiting suitable variational methods.

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1. Introduction

This paper is devoted to the following nonlocal problem

$$\begin{cases} -(a+b\|u\|_{X_0}^2)\mathcal{L}_K u = f(x,u) & \text{in } \Omega \\ \qquad\qquad\qquad\quad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{P_K^f}$$

Here and in the sequel Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$, where $2s < n < 4s$ and $s \in (0,1)$, with continuous boundary $\partial\Omega$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function verifying the conditions stated in the sequel. Moreover, a, b denote two positive real constants and

$$\|u\|_{X_0}^2 := \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y)\, dxdy.$$

Finally, $\mathcal{L}_K$ is a nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))\, K(y)\, dy, \quad (x \in \mathbb{R}^n)$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a kernel function with the properties that:

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(k_1) *$\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) := \min\{|x|^2, 1\}$;*

(k_2) *there exists $\theta > 0$ such that*

$$K(x) \geq \theta |x|^{-(n+2s)},$$

for any $x \in \mathbb{R}^n \setminus \{0\}$.

A typical example of the kernel K is given by $K(x) := |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n.$$

Aim of this paper is to get the existence of weak solutions for problem (P_K^f). By a *weak solution* for (P_K^f), we mean a function $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in X_0$ and

$$(a + b\|u\|_{X_0}^2) \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dxdy$$
$$= \int_\Omega f(x, u(x))\varphi(x)\, dx \quad \forall\, \varphi \in X_0$$

Here and in the sequel we set

$$X_0 := \big\{u \in X : u = 0 \text{ a.e. in } \mathcal{C}\Omega\big\},$$

where the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and

$$((x,y) \mapsto (u(x) - u(y))\sqrt{K(x-y)}) \in L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big),$$

with $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$.

Setting

$$F(x,t) := \int_0^t f(x,\tau)\, d\tau, \quad \text{and} \quad G(x,t) := f(x,t)t - 4F(x,t), \quad \forall\, (x,t) \in \Omega \times \mathbb{R}$$

the main result reads as follows.

Theorem 1.1. *Let Ω be a bounded domain in $(\mathbb{R}^n, |\cdot|)$, where $2s < n < 4s$ and $s \in (0,1)$, with continuous boundary $\partial\Omega$. Further, let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying hypotheses* (k_1) *and* (k_2)*. Finally, let $f \in C^0(\Omega \times \mathbb{R})$ such that the following conditions hold:*

(f_0) *There exists a positive constant C such that*

$$|f(x,t)| \leq C(1 + |t|^{q-1}), \quad \forall\, (x,t) \in \Omega \times \mathbb{R}$$

for some $q \in \big(4, \frac{2n}{n-2s}\big)$;

(f_1) $tf(x,t) \geq 0$ *in* $\Omega \times \mathbb{R}$;

(f_2) *For some* $\sigma \geq 1$, *one has*

$$G(x,t) \geq \frac{G(x,\zeta t)}{\sigma}, \quad \forall\,(x,t) \in \Omega \times \mathbb{R}$$

and every $\zeta \in [0,1]$;

(f_3) *There is* $\delta > 0$ *such that*

$$F(x,t) \leq a\frac{\lambda_1}{2}t^2,$$

for every $x \in \Omega$ *and* $t \in (-\delta,\delta)$, *where* λ_1 *is the first eigenvalue of* $-\mathcal{L}_K$ *in* X_0;

(f_4) $\lim_{|t|\to+\infty} \frac{f(x,t)}{t^3} = +\infty$, *uniformly in* $x \in \Omega$.

Then, problem (P_K^f) *has at least one non-trivial weak solution.*

The above result represents a non-local version of an existence result obtained by Sun and Tang for Kirchhoff-type equations defined on bounded domains of the n-dimensional Euclidean space, with $n < 4$ (see [34, Theorem 1]). This dimensional restriction is replaced in the fractional setting by $2s < n < 4s$.

This assumption is essential in our technical approach in order to guarantee the embedding of the working space X_0 in the Lebesgue space $L^q(\mathbb{R}^n)$, where

$$4 < q < \frac{2n}{n-2s}.$$

The embedding seems to be crucial in the proof of the main result (see condition (f_0)). Note that this bound on the dimension n, previously used in literature by Servadei and Valdinoci studying fractional critical problems (see [31]), implies that $s > \frac{1}{4}$. See also the recent papers [1, 13, 22].

A simple model case of nonlinearity f that verifies all the assumptions of Theorem 1.1 can be exhibit considering the real function

$$f(t) := t^3 \log(1+|t|), \quad \forall\, t \in \mathbb{R},$$

see Example 4.4 for details.

The analogous and classical counterpart of our problem models several interesting phenomena studied in mathematical physics, even in the one-dimensional case. Its origins, as well known, date back to 1883 when G. Kirchhoff proposed his celebrated equation

$$\rho\partial_{tt}^2 u - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\partial_x u(x)|^2\,dx\right)\partial_{xx}^2 u = 0 \tag{1}$$

as a nonlinear extension of D'Alambert's wave equation for free vibrations of elastic strings, where the above constants have the following meaning:

$u = u(x,t)$ is the transverse string displacement at the space coordinate x and time t, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension (see [15]).

In the very recent paper [14], Fiscella and Valdinoci proposed an interesting and fascinating physical interpretation of Kirchhoff's equation in the fractional scenario. In their correction of the early (one-dimensional) model, the tension of the string, which has classically a "nonlocal" nature arising from the average of the kinetic energy $\frac{1}{2}|\partial_x u|^2$ on $[0,L]$, possesses a further nonlocal behavior provided by the H^s-norm (or other more general fractional norms) of u.

For completeness, in the vast literature on this subject, we refer the reader to some interesting recent results (in the non-fractional setting) obtained in [1–4] studying Kirchhoff equations by using different approaches.

In our setting problem (P_K^f) is highly nonlocal due to the presence of the fractional operator $\mathcal{L}_K$ as well as in the term

$$a + b\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dxdy.$$

Moving along this direction, in [6] the authors studied the existence and multiplicity of solutions for elliptic equations in $\mathbb{R}^n$, driven by the nonlocal integro-differential operator $\mathcal{L}_K$ (this work is related to the results on general quasilinear elliptic problems given in [5]).

Here, motivated by this increasing interest in the current literature, we seek conditions on the datum f for which problem (P_K^f) possesses at least one non-trivial weak solutions. It is worth pointing out that the variational approach to attack such problems is not often easy to perform. Fortunately, our approach here is realizable by checking that the associated energy functional (see Section 3) given by

$$\mathcal{J}_K(u) := \frac{a}{2}\|u\|_{X_0}^2 + \frac{b}{4}\|u\|_{X_0}^4 - \int_\Omega F(x,u(x))\,dx, \quad \forall\, u \in X_0 \tag{2}$$

satisfies all the assumptions requested by a suitable version of the celebrated mountain pass theorem which can be found in [26]. See also the comprehensive survey [24].Usually, the famous Ambrosetti-Rabinowitz condition plays a crucial role in proving that every Palais-Smale sequence is bounded, as well as that the so called "mountain pass geometry" is satisfied.However, even dealing with different problems than ours, several authors studied more general or different assumptions that still allow to apply min-max methods in order to assure the existence of critical points.

With respect to the compactness condition, we prove, by using a a monotonicity trick, that the energy functional $\mathcal{J}_K$, defined in (2), satisfies the Cerami condition when the nonlinearity verifies superlinear conditions (see Lemma 3.1).

More precisely, we need to introduce the technical condition (f_2) in order to ensure compactness of critical sequences.

Moreover, the nonlocal analysis (see Section 2) that we perform here in order to use Theorem 2.1 is quite general and has been successfully exploited for other goals in several recent contributions; see [16, 19–21, 28–30, 33] and [12] for an elementary introduction to this topic and for a list of related references.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework and preliminaries. Successively, Section 3 is devoted to some basic propositions which will be used in the proof of our main result. Finally, in Section 4 we prove Theorem 1.1 and a concrete example of an application is presented in Example 4.4.

We cite the monographs [18, 25] for related topics and variational methods adopted in this paper and [8–11] for nice results in the fractional setting.

2. Variational framework

In this section we briefly recall the definition of the functional space X_0, firstly introduced in [28, 29]. The reader familiar with this topic may skip this section and go directly to the next one. The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$((x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)}) \in L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. We denote by X_0 the following linear subspace of X

$$X_0 := \big\{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

We remark that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [29, Lemma 5.1].

Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left(\int_Q |g(x)-g(y)|^2 K(x-y)\, dxdy\right)^{\frac{1}{2}},$$

where $Q := (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} := \mathcal{C}\Omega \times \mathcal{C}\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. It is easily seen that $\|\cdot\|_X$ is a norm on X; see [28].

By [28, Lemmas 6 and 7] in the sequel we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} := \left(\int_Q |v(x)-v(y)|^2 K(x-y)\, dxdy\right)^{\frac{1}{2}} \tag{3}$$

as a norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u, v\rangle_{X_0} := \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) K(x-y)\, dxdy,$$

see [28, Lemma 7].

Note that in (3) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

While for a general kernel K satisfying conditions (k_1) and (k_2) we have that $X_0 \subset H^s(\mathbb{R}^n)$, in the model case $K(x) := |x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω; see [33, Lemma 7].

Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}}\, dxdy\right)^{\frac{1}{2}}.$$

Before concluding this subsection, we recall the embedding properties of X_0 into the usual Lebesgue spaces; see [28, Lemma 8]. The embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*)$, where $2^* := \frac{2n}{n-2s}$ denotes the *fractional critical Sobolev exponent.*

For further details on the fractional Sobolev spaces we refer to [12] and to the references therein, while for other details on X and X_0 we refer to [29], where these functional spaces were introduced, and also to [17, 27, 28, 30, 33], where various properties of these spaces were proved.

Finally, our abstract tool for proving the main result of the present paper is the following mountain pass theorem (see [26]) that we recall here for reader's convenience.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a real Banach space. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{J(0), J(u_1)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} J(u)$$

for some $\alpha < \beta$, $\rho > 0$ and $u_1 \in E$ with $\|u_1\| > \rho$. Let

$$\Gamma := \{\gamma \in C^0([0,1], E) : \gamma(0) = 0, \gamma(1) = u_1\}.$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} J(\gamma(\tau)).$$

Then $c \geq \beta > 0$ and there exists a sequence $\{u_j\}_{j \in \mathbb{N}} \subset E$ such that

$$J(u_j) \to c, \quad \text{and} \quad (1 + \|u_j\|)J'(u_j) \to 0.$$

Moreover, if J satisfies the $(\mathrm{C})_\mathrm{c}$ *condition, then c is a critical value of J.*

For the sake of completeness, we also recall that the C^1-functional $J\colon E \to \mathbb{R}$ satisfies the *Cerami condition* at level $c \in \mathbb{R}$ (briefly $(\mathrm{C})_c$ condition) when

(C)$_c$ *Every sequence* $\{u_j\}_{j\in\mathbb{N}} \subset E$ *such that* $J(u_j) \to c$ *and*

$$(1+\|u_j\|)\sup\Big\{|\langle J'(u_j),\varphi\rangle| : \varphi\in E,\ \|\varphi\|=1\Big\} \to 0$$

as $j\to\infty$*, possesses a convergent subsequence in* E*.*

Such a sequence is then called a *Cerami sequence* of the functional J. Finally, J satisfies the compactness Cerami condition ((C) condition for short) if (C)$_c$ holds for every $c \in \mathbb{R}$.

3. Technical results

Among others, two notions of fractional operators are well-known and widely studied in the literature in connection with elliptic problems of fractional type, namely the *integral* one (which reduces to the classical fractional Laplacian), and the *spectral* one (that is sometimes called the local, fractional Laplacian).

We would like to note that, as pointed out in [32], these two fractional operators are different. Indeed, the spectral operator depends on the domain Ω considered (since its eigenfunctions and eigenvalues depend on Ω), while the integral one $(-\Delta)^s$ evaluated at some point is independent on the domain in which the equation is set.

Further, it is easily seen that the eigenvalues of the spectral Laplacian are the s-th power of the eigenvalues of the classical Laplacian. On the contrary, our abstract framework is more delicate and our approach is based on a careful analysis of the linear problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ \quad\ \ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{4}$$

related to the operator $-\mathcal{L}_K$. A spectral theory for general integrodifferential nonlocal operators was developed in [30, Proposition 9 and Appendix A]. See also [27] for further properties of the spectrum of $-\mathcal{L}_K$ and of its eigenfunctions.

With respect to the eigenvalue problem (4), we recall that it possesses a divergent sequence of eigenvalues

$$0<\lambda_1<\lambda_2\le\cdots\le\lambda_k\le\lambda_{k+1}\le\cdots.$$

To avoid possible confusions, we stress the fact that the eigenvalues that we consider, even in the model case of the fractional Laplacian, are not the s-th power of the eigenvalues of the standard Laplacian.

As usual, we denote by e_k the eigenfunction related to the eigenvalue λ_k, $k\in\mathbb{N}$. From [30, Proposition 9], we know that we can choose $\{e_k\}_{k\in\mathbb{N}}$ normalized in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0, so that for any $k, i\in\mathbb{N}$ with $k\neq i$

$$\langle e_k, e_i\rangle_{X_0} = \int_\Omega e_k(x)e_i(x)\,dx = 0 \quad\text{and}\quad \|e_k\|^2_{X_0} = \lambda_k\|e_k\|^2_{L^2(\Omega)} = \lambda_k.$$

Furthermore, by [30, Proposition 9 and Appendix A], we have the following characterization of the eigenvalue λ_1:

$$\lambda_1 = \min_{u\in X_0\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dxdy}{\displaystyle\int_\Omega |u(x)|^2\,dx}.$$

Finally, the first eigenfunction $e_1 \in X_0$ is non-negative in Ω, see [30, Proposition 9 and Appendix A].

Denote by $\mathcal{A}$ the class of all continuous functions $f : \Omega\times\mathbb{R}\to\mathbb{R}$ such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1+|t|^{q-1}} < +\infty, \quad \text{for some } q \in (4, 2^*).$$

For the proof of our result, we observe that problem (P_K^f) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_K : X_0 \to \mathbb{R}$ defined in (2).

Note that the functional $\mathcal{J}_K$ is Fréchet differentiable in $u \in X_0$ and one has

$$\begin{aligned}\langle \mathcal{J}_K'(u), \varphi\rangle = (a+b\|u\|_{X_0}^2) &\int_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dxdy \\ &- \int_\Omega f(x,u(x))\varphi(x)\,dx,\end{aligned}$$

for every $\varphi \in X_0$. Thus, critical points of $\mathcal{J}_K$ are solutions to problem (P_K^f).

Lemma 3.1. *Let $f \in \mathcal{A}$ and assume that conditions* (f_1), (f_2) *are verified in addition to* (f_4). *Then, every Cerami sequence $\{u_j\}_{j\in\mathbb{N}} \subset X_0$ of the functional $\mathcal{J}_K$ is bounded in X_0.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a Cerami sequence, i.e. for some $c \in \mathbb{R}$, one has

$$\mathcal{J}_K(u_j) = \frac{a}{2}\|u_j\|_{X_0}^2 + \frac{b}{4}\|u_j\|_{X_0}^4 - \int_\Omega F(x,u_j(x))\,dx \to c,$$

and

$$(1+\|u_j\|_{X_0}) \sup\Big\{|\langle \mathcal{J}_K'(u_j),\varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0}=1\Big\} \to 0,$$

as $j\to\infty$. Hence

$$c = \mathcal{J}_K(u_j) + o(1), \tag{5}$$

and, since $|\langle \mathcal{J}_K'(u_j), u_j\rangle| \le \|u_j\|_{X_0} \sup\Big\{|\langle \mathcal{J}_K'(u_j),\varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0}=1\Big\}$, we also have that

$$\langle \mathcal{J}_K'(u_j), u_j\rangle = o(1), \tag{6}$$

where $o(1) \to 0$, as $j \to \infty$.Thus, from (5) and (6), for j large enough, it follows that

$$\begin{aligned} 1 + c &\geq \mathcal{J}_K(u_j) - \frac{1}{4}\langle \mathcal{J}_K'(u_j), u_j\rangle \\ &= \frac{a}{4}\|u_j\|^2 + \int_\Omega \left(\frac{1}{4} f(x, u_j(x))u_j(x) - F(x, u_j(x))\right) dx. \end{aligned} \tag{7}$$

We claim that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0. If the assertion were false, up to a subsequence, we could suppose $\|u_j\|_{X_0} \to \infty$, as $j \to \infty$. Set

$$w_j := \frac{u_j}{\|u_j\|_{X_0}}, \quad \forall\, j \in \mathbb{N}.$$

Clearly $\|w_j\|_{X_0} = 1$, so that $\{w_j\}_{j\in\mathbb{N}}$ is bounded. Hence, since X_0 is a reflexive space, bearing in mind that for every $r \in [1, 2^*)$ the embedding $X_0 \hookrightarrow L^r(\Omega)$ is compact, we may assume (up to a subsequence) that

$w_j \rightharpoonup w$ in X_0,
$w_j \to w$ in $L^r(\Omega)$, for every $1 \leq r < 2^*$,
$w_j(x) \to w(x)$ a.e. $x \in \Omega$,

for some $w \in X_0$. Now, we divide the proof in two cases.

Case 1. If $w \equiv 0$, we choose a sequence $\{t_j\}_{j\in\mathbb{N}} \subset [0, 1]$ such that

$$\mathcal{J}_K(t_j u_j) = \max_{t\in[0,1]} \mathcal{J}_K(tu_j).$$

Now, for any $m > 0$, let

$$v_{j,m} := \sqrt[4]{\frac{8m}{b}} w_j, \quad (\text{note that}\, b > 0) \text{ for every } j \in \mathbb{N}.$$

At this point, owing to $v_{j,m} \to 0$ in $L^q(\Omega)$, by (f_0) one has that

$$\int_\Omega |F(x, v_{j,m}(x))|\, dx \leq C_1 \left(\|v_{j,m}\|_{L^1(\Omega)} + \|v_{j,m}\|^q_{L^q(\Omega)}\right) \to 0, \quad (C_1 > 0)$$

as $j \to \infty$. Thus

$$\lim_{j\to\infty} \left|\int_\Omega F(x, v_{j,m}(x))\, dx\right| = 0.$$

So, for j sufficiently large, $\frac{\sqrt[4]{\frac{8m}{b}}}{\|u_j\|_{X_0}} \in (0, 1)$, and

$$\|v_{j,m}\|^2_{X_0} = \left\|\sqrt[4]{\frac{8m}{b}} w_j\right\|^2_{X_0} = \left\|\sqrt[4]{\frac{8m}{b}} \frac{u_j}{\|u_j\|_{X_0}}\right\|^2_{X_0} = \sqrt{\frac{8m}{b}},$$

as well as $\|v_{j,m}\|^4_{X_0} = \frac{8m}{b}$. Hence, there exists $j_0 \in \mathbb{N}$ such that

$$\mathcal{J}_K(t_j u_j) \geq \mathcal{J}_K(v_{j,m}) \geq 2m - \left|\int_\Omega F(x, v_{j,m}(x))\, dx\right| \geq m,$$

for every $j \geq j_0$. Then, we have

$$\mathcal{J}_K(t_j u_j) \to +\infty, \quad \text{as } j \to \infty. \tag{8}$$

Now, since $\mathcal{J}_K(0) = 0$ and $\mathcal{J}_K(u_j) \to c$, we deduce that $t_j \in (0,1)$ and

$$\begin{aligned}
&(a + b\|t_j u_j\|_{X_0}^2)\|t_j u_j\|_{X_0}^2 - \int_\Omega f(x, t_j u_j(x)) t_j u_j(x)\, dx \\
&= \langle \mathcal{J}_K'(t_j u_j), t_j u_j \rangle \\
&= t_j \frac{d\mathcal{J}_K(t w_j)}{dt}\Big|_{t=t_j} \\
&= 0.
\end{aligned} \tag{9}$$

Therefore, by using (f$_2$), it follows that,

$$G(x, t_j u_j(x)) \leq \sigma G(x, u_j(x)), \quad \forall\, x \in \Omega \tag{10}$$

and $t_j \in (0,1)$. Hence, by (9) and (10) one has

$$\begin{aligned}
4\mathcal{J}_K(t_j u_j) &= 4\mathcal{J}_K(t_j u_j) - \langle \mathcal{J}_K'(t_j u_j), t_j u_j \rangle \\
&= a\|t_j u_j\|_{X_0}^2 + \int_\Omega G(x, t_j u_j(x))\, dx \\
&\leq a\|t_j u_j\|_{X_0}^2 + \sigma \int_\Omega G(x, u_j(x))\, dx.
\end{aligned} \tag{11}$$

Moreover, since

$$a\|t_j u_j\|_{X_0}^2 \leq a\sigma \|u_j\|_{X_0}^2 \quad (\sigma \geq 1),$$

$$\int_\Omega f(x, u_j(x)) u_j(x)\, dx = a\|u_j\|_{X_0}^2 + b\|u_j\|_{X_0}^4 + o(1),$$

$$\int_\Omega F(x, u_j(x))\, dx = \frac{a}{2}\|u_j\|_{X_0}^2 + \frac{b}{4}\|u_j\|_{X_0}^4 - c + o(1),$$

by (11), it follows that

$$\mathcal{J}_K(t_j u_j) \leq c\sigma + o(1),$$

which contradicts (7).

Case 2. The function $w \in X_0$ is not identically zero in Ω. Hence, let us denote

$$\Omega_1 := \{x \in \Omega : w(x) \neq 0\}, \quad \text{and} \quad \Omega_2 := \{x \in \Omega : w(x) = 0\}.$$

Clearly, one has that

$$|\Omega_1| > 0, \quad \Omega = \Omega_1 \cup \Omega_2, \quad \text{and} \quad \Omega_1 \cap \Omega_2 = \emptyset.$$

Since $|u_j(x)| = |w_j(x)|\|u_j\|_{X_0}$, we have that $|u_j(x)| \to \infty$ for every $x \in \Omega_1$, and, thanks to (f_4), we also have

$$\lim_{j\to\infty} \frac{f(x, u_j(x))}{u_j(x)^3} = +\infty,$$

uniformly in Ω_1. Hence, the Fatou's Lemma, implies that

$$\lim_{j\to\infty} \int_{\Omega_1} \frac{f(x, u_j(x))}{u_j(x)^3} |w_j(x)|^4 \, dx \to +\infty, \quad \text{as } j \to \infty. \tag{12}$$

On the other hand, taking into account that f is a continuous function, it is easy to see that

$$\int_{\Omega_2} \frac{f(x, u_j(x))}{u_j(x)^3} |w_j(x)|^4 \, dx \geq -\frac{C_2}{\|u_j\|^4_{X_0}} |\Omega_2|, \tag{13}$$

for some constant $C_2 > 0$. Then, relations (12) and (13) imply that

$$\lim_{j\to\infty} \int_{\Omega} \frac{f(x, u_j(x))u_j(x)}{\|u_j\|^4_{X_0}} \, dx = +\infty. \tag{14}$$

Now, by (6) it follows that $\int_\Omega \frac{f(x,u_j(x))u_j(x)}{\|u_j\|^4_{X_0}} \, dx = b + \frac{a}{\|u_j\|^2_{X_0}} - \frac{o(1)}{\|u_j\|^4_{X_0}}$. Consequently

$$\liminf_{j\to\infty} \int_{\Omega} \frac{f(x, u_j(x))u_j(x)}{\|u_j\|^4_{X_0}} \, dx = b, \tag{15}$$

that contradicts (14). In conclusion, in any case, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0. □

Lemma 3.2. *Let $f \in \mathcal{A}$ and assume that conditions* (f_1), (f_2) *are verified in addition to* (f_4). *Then, the functional $\mathcal{J}_K$ satisfies the* (C) *compactness condition.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset X_0$ be a Cerami sequence. By Lemma 3.1, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is necessarily bounded in X_0. Since X_0 is reflexive, we can extract a subsequence which for simplicity we shall call again $\{u_j\}_{j\in\mathbb{N}}$, such that $u_j \rightharpoonup u_\infty$ in X_0. This means that

$$\begin{aligned}&\int_Q \big(u_j(x) - u_j(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dxdy \\ &\to \int_Q \big(u_\infty(x) - u_\infty(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dxdy,\end{aligned} \tag{16}$$

for any $\varphi \in X_0$, as $j \to \infty$.

We will prove that $\{u_j\}_{j\in\mathbb{N}}$ strongly converges to $u_\infty \in X_0$. Exploiting the derivative $\mathcal{J}_K'(u_j)(u_j - u_\infty)$, we obtain

$$\langle a(u_j), u_j - u_\infty\rangle = \langle \mathcal{J}_K'(u_j), u_j - u_\infty\rangle + \int_\Omega f(x, u_j(x))(u_j - u_\infty)(x)\,dx, \tag{17}$$

where we set

$$\begin{aligned}\langle a(u_j), u_j - u_\infty\rangle := \Big(&\int_Q |u_j(x) - u_j(y)|^2 K(x-y)\,dxdy\\ &- \int_Q \big(u_j(x) - u_j(y)\big)\big(u_\infty(x) - u_\infty(y)\big)K(x-y)\,dxdy\Big)\\ &\times \left(a + b\int_Q |u_j(x) - u_j(y)|^2 K(x-y)\,dxdy\right).\end{aligned}$$

Since $(1 + \|u_j\|_{X_0})\sup\Big\{|\langle \mathcal{J}_K'(u_j), \varphi\rangle| : \varphi \in X_0,\ \|\varphi\|_{X_0} = 1\Big\} \to 0$, and taking into account that the sequence $\{u_j - u_\infty\}_{j\in\mathbb{N}}$ is bounded in X_0, one gets

$$\langle \mathcal{J}_K'(u_j), u_j - u_\infty\rangle \to 0, \quad \text{as } j \to \infty. \tag{18}$$

Since the embedding $X_0 \hookrightarrow L^q(\Omega)$ is compact, clearly $u_j \to u_\infty$ strongly in $L^q(\Omega)$. So, by condition (f_0), standard computations ensure that

$$\int_\Omega |f(x, u_j(x))||u_j(x) - u_\infty(x)|\,dx \to 0. \tag{19}$$

By (17) relations (18) and (19) yield

$$\langle a(u_j), u_j - u_\infty\rangle \to 0, \quad \text{as } j \to \infty. \tag{20}$$

Now, observe that

$$0 < a \le a + b\int_Q |u_j(x) - u_j(y)|^2 K(x-y)\,dxdy, \tag{21}$$

for every $j \in \mathbb{N}$. Hence by (21) and (20) we can write

$$\begin{aligned}&\int_Q |u_j(x) - u_j(y)|^2 K(x-y)\,dxdy\\ &- \int_Q \big(u_j(x) - u_j(y)\big)\big(u_\infty(x) - u_\infty(y)\big)K(x-y)\,dxdy\\ &\to 0, \quad \text{as } j \to \infty.\end{aligned} \tag{22}$$

Thus, by (22) and (16) it follows that

$$\limsup_{j\to\infty} \int_Q |u_j(x) - u_j(y)|^2 K(x-y)\,dxdy = \int_Q |u_\infty(x) - u_\infty(y)|^2 K(x-y)\,dxdy.$$

In conclusion, thanks to [7, Proposition III.30], $u_j \to u_\infty$ in X_0. The proof is complete. $\square$

4. Proof of Theorem 1.1

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying conditions (f_0)–(f_4). The following preliminary results on the geometry of the functional $\mathcal{J}_K$, should be proved.

Lemma 4.1. *There exist two constants $\rho, \beta > 0$, such that $\mathcal{J}_K(u) \geq \beta$ for every $u \in X_0$ with $\|u\|_{X_0} = \rho$.*

Proof. By (f_0) and (f_3), there exists $C_1 > 0$ such that

$$F(x,t) \leq \frac{a}{2}\lambda_1 t^2 + C_1|t|^q, \tag{23}$$

for every $(x,t) \in \Omega \times \mathbb{R}$. Thus, by (23), we have

$$\begin{aligned}\mathcal{J}_K(u) &= \frac{a}{2}\,\|u\|^2_{X_0} + \frac{b}{4}\|u\|^4_{X_0} - \int_\Omega F(x,u(x))\,dx \\ &\geq \frac{a}{2}\|u\|^2_{X_0} + \frac{b}{4}\|u\|^4_{X_0} - \frac{a}{2}\lambda_1 \int_\Omega |u(x)|^2\,dx - C_1\int_\Omega |u(x)|^q\,dx,\end{aligned}$$

for every $u \in X_0$. Now, since $X_0 \hookrightarrow L^q(\Omega)$ continuously, the above inequality becomes $\mathcal{J}_K(u) \geq \frac{b}{4}\|u\|^4_{X_0} - C_2\|u\|^q_{X_0}$, where $C_2 := C_1 c_q^q$. Since $q > 4$, choosing

$$\rho < \left(\frac{b}{4C_2}\right)^{\frac{1}{q-4}},$$

one has

$$\mathcal{J}_K(u) \geq \beta := \frac{b}{4}\rho^4 - C_2\rho^q > 0,$$

for every $u \in X_0$ with $\|u\|_{X_0} = \rho$. This concludes the proof. □

Lemma 4.2. *There exists $e \in X_0$ with $\|e\|_{X_0} > \rho$ such that $\mathcal{J}_K(e) < 0$.*

Proof. By (f_4), for every $x \in \Omega$, one has

$$\lim_{t\to+\infty} \frac{f(x,t)}{t^3} = +\infty.$$

Then, for any $M > 0$, there exists $\delta_M > 0$ such that

$$\frac{f(x,t)}{t^3} \geq \frac{1}{M},$$

for every $t > \delta_M$ and $x \in \Omega$. Setting $c_M := \frac{\sqrt[3]{\delta_M}}{\varepsilon}$, it follows that $f(x,t) \geq \frac{t^3}{\varepsilon} - c_M$, for every $t \geq 0$ and $x \in \Omega$. Then

$$f(x,st)t \geq \frac{z^3 t^4}{M} - c_M t, \tag{24}$$

for every $(x,t) \in \Omega \times [0,+\infty)$, and $z \in [0,1]$. Integrating both sides of the inequality (24) on $[0,1]$ with respect to z, we obtain

$$F(x,t) \geq \frac{t^4}{4M} - c_M t, \tag{25}$$

for every $(x,t) \in \Omega \times [0,+\infty)$. Now, since the first eigenfunction e_1 of the operator $-\mathcal{L}_K$ in X_0, is not negative in Ω (see Section 3), by (25) it follows that

$$F(x, te_1(x)) \geq \frac{t^4 e_1(x)^4}{4M} - c_M t e_1(x), \tag{26}$$

for every $(x,t) \in \Omega \times [0,+\infty)$. Hence, for every $j \in \mathbb{N}$, one has

$$\frac{F(x, je_1(x))}{j^4} \geq \frac{e_1(x)^4}{4M} - \frac{c_M e_1(x)}{j^3},$$

for every $x \in \Omega$. Consequently

$$\int_\Omega \frac{F(x, je_1(x))}{j^4}\, dx \geq \int_\Omega \left(\frac{e_1(x)^4}{4M} - \frac{c_M e_1(x)}{j^3} \right) dx. \tag{27}$$

By (27) the Fatou's lemma, immediately yields

$$\liminf_{j\to\infty} \int_\Omega \frac{F(x, je_1(x))}{j^4}\, dx \geq \frac{\|e_1\|^4_{L^4(\Omega)}}{4M},$$

for every $M > 0$. Hence, passing to the limit for $M \to 0$, one has

$$\lim_{j\to\infty} \int_\Omega \frac{F(x, je_1(x))}{j^4}\, dx = +\infty.$$

Thus

$$\frac{\mathcal{J}_K(je_1)}{j^4} = \frac{a}{2j^2} \|e_1\|^2_{X_0} + \frac{b}{4} \|e_1\|^4_{X_0} - \int_\Omega \frac{F(x, je_1(x))}{j^4}\, dx \to -\infty,$$

as $j \to \infty$. Finally, the above relation ensures that there exists $\nu_0 \in \mathbb{N}$ such that, setting $e := \nu_0 e_1 \in X_0$, it follows that $\|e\|_{X_0} > \rho$ and $\mathcal{J}_K(e) < 0$. The conclusion is achieved. □

Proof of Theorem 1.1. Take $E = X_0$ and the functional $J = \mathcal{J}_K$ in Proposition 2.1. Let us define

$$\Gamma := \{\gamma \in C^0([0,1], X_0) : \gamma(0) = 0, \gamma(1) = e\}, \quad \text{and} \quad c := \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} \mathcal{J}_K(\gamma(\tau)),$$

where $e \in X_0$ is given in Lemma 4.2. Now, since $\mathcal{J}_K(0) = 0$, by Lemma 4.2 one has $\max\{\mathcal{J}_K(0), \mathcal{J}_K(e)\} = 0$. Moreover, Lemmas 4.1 and 4.2 ensure that

$$0 < \beta \leq \inf_{\|u\|_{X_0}=\rho} \mathcal{J}_K(u),$$

for some $\beta, \rho > 0$ and $e \in X_0$, with $\|e\|_{X_0} > \rho$. Then, by Proposition 2.1, it follows that $c \geq \beta > 0$. Finally, by Lemma 3.1, since the Cerami compactness condition holds at level c, there exists $u_0 \in X_0$ such that $\mathcal{J}'_K(u_0) = 0$ and $\mathcal{J}_K(u_0) = c \geq \beta > 0$. Thus $u_0 \in X_0$ is a non-trivial critical point of $\mathcal{J}_K$. This completes the proof of Theorem 1.1. $\square$

Remark 4.3. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that

(f'_3) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$, uniformly in Ω.

Assumption (f'_3) yields

$$\lim_{t\to 0} \frac{F(x,t)}{t^2} = 0,$$

uniformly in Ω. Consequently, condition (f_3) immediately holds.

In conclusion, we present a direct application of Theorem 1.1 and Remark 4.3.

Example 4.4. Let $s \in (\frac{3}{4}, 1)$ and let Ω be an open bounded set of $\mathbb{R}^3$ with continuous boundary $\partial\Omega$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$f(t) := t^3 \log(1 + |t|), \quad \forall\, t \in \mathbb{R}.$$

Simple and direct computations ensure that

$$F(t) = \frac{(t^2+3)}{12}|t| + \frac{(t^4-1)}{4}\log(|t|+1) - \frac{(t^2+2)}{16}t^2,$$

and

$$G(t) := f(t)t - 4F(t) = -\frac{(t^2+3)}{3}|t| + \log(|t|+1) + \frac{(t^2+2)}{4}t^2,$$

for every $t \in \mathbb{R}$. Further, one has $|f(t)| \leq 1 + |t|^4$, as well as

$$\lim_{t\to 0} \frac{f(t)}{t} = 0, \quad \text{and} \quad \lim_{|t|\to\infty} \frac{f(t)}{t^3} = +\infty.$$

Finally

$$G(t) \geq G(st), \quad \forall\, (t,s) \in \mathbb{R} \times [0,1].$$

Then, owing to Theorem 1.1, the following problem

$$\begin{cases} \left(a + b\displaystyle\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)(-\Delta)^s u = u^3\log(1+|u|) & \text{in } \Omega \\ u|_{\mathbb{R}^3\setminus\Omega} = 0, & \end{cases}$$

admits one non-trivial weak solution in the fractional Sobolev space

$$\mathbb{H}_0 := \left\{u \in H^s(\mathbb{R}^3) : u = 0 \text{ a.e. in } \mathcal{C}\Omega\right\},$$

for every real constants $a, b > 0$.

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